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ON ELLIPTICITY OF HYPERELASTIC MODELS RESTORED BY EXPERIMENTAL DATA

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Abstract. The condition of ellipticity of the equilibrium equation plays an important role for the correct description of the mechanical behavior of materials and is a necessary condition for new constitutive relations. Earlier, new deformation measures were proposed to remove correlations between the terms, which dramatically simplifies restoration of constitutive relations from experimental data. One of these new deformation measures is based on the QR-expansion of deformation gradient. In this paper, we study the strong ellipticity condition for hyperelastic material using the QR-expansion of deformation gradient.

CONTENTS

Introduction

Solving problems of the mechanics of deformable solid bodies based on principles of continuum mechanics, one has to give the constitutive relation setting the dependence of stresses on the deformation (see [19]). The constitutive relation entirely determines the mechanical behavior of the investigated material; it is an equation closing the system of equations describing the motion of a deformable body. Formulations of new biomedical problems led to the interest in constructing and investigating various kinds of constitutive relations for soft human tissues.

Approaches to the nonlinear elasticity theory are used to describe the mechanical behavior of soft tissues (see [9]). Usually, a model of hyperelastic materials is used, assuming the existence of an elastic potential entirely determining the kind of the constitutive relation (see [9, 14]). A large amount of papers are devoted to the constructing of constitutive relations for soft tissues. However, there is no unified rule to select the correct constitutive relation. The standard approach to the construction of constitutive relations is as follows: the kind of constitutive relation is selected (from a pool of already known models) a priori, while the parameters of the model are found by means of the fitting method based on the experimental data available for the investigated material.

The kind of constitutive relations depends mainly on the definition of the deformation measure. Currently, several deformation measures are proposed (see [13]). The Cauchy–Green deformation measure is generally accepted for the description of soft tissues. In [2], it is shown that the use of invariants of this deformation measure for the setting of the constitutive relation leads to the correlation

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of terms of the constitutive relation; in the framework of the standard approach to the construction of the constitutive relation, this causes problems in the processing of results of experiments for the determination of the parameters of the model. There are papers [3, 12, 18] where other deformation measures were used for the construction of constitutive relations to avoid such correlations. The absence of correlations allows one to find the so-called *response functions*, which are the corresponding derivatives of the elastic potential, directly by the experimental data; then the kind of constitutive relations is not set a priori, but is restored via the response functions. For the first time, the approach using response functions was proposed in [16]. It was developed by Humphrey [10] for the case of biomembranes. The use of constitutive relations with noncorrelating terms in the framework of the approach with response functions is a promising research direction for the description of soft-tissue mechanics. The said approach was successfully applied to describe the mechanical behavior of blood vessels (see [12]) and the myocardium (see [4]). However, once new deformation measures are used in the framework of the said approach, theoretical issues related to restrictions of the form of the elastic potential (and, therefore, of response functions) remain unclear.

To construct constitutive relations, one has to satisfy a number of conditions ensuring that the problem is well posed (see [1, 13]). In particular, the strong ellipticity condition for the elastic potential coincides with the ellipticity condition for the system of equilibrium differential equations, is equivalent to the condition that the propagation velocities of low-amplitude waves in the elastic medium are real (see [13]), and is a necessary condition of stability for the balanced elastic deformation (see [11, 15]). For a given potential, it is possible that there exist values of the deformation gradient such that it is strongly elliptic for these values, but is not strongly elliptic for others. The fulfillment of the ellipticity conditions is important for the correct description of the mechanical behavior of the material (see [8], [1, p. 282]). Thus, creating new hyperelastic models, one has to verify the ellipticity condition. In [17], for the case of new deformation measures leading to constitutive relations with noncorrelating terms, the fulfillment of the strong ellipticity condition is investigated for constitutive relations represented in the invariants proposed in [3].

The aim of our paper is to study the fulfillment of the strong ellipticity condition for constitutive relations based on the QR-expansion of the gradient of deformations (see [18]). As was noted above, for the case of constitutive relations with noncorrelating terms, one can restore the kind of elastic potential directly by the experimental data. The ellipticity conditions obtained in the present paper can be used for the investigation of well-posedness of restored hyperelastic models.

1. Kinematics

Consider a domain $\Omega_t \subset \mathbb{R}^3$ occupied by an elastic body at moment t (the actual configuration). Denote this domain at the initial time by Ω_0 (the initial configuration). The location of a point at its reference configuration is denoted by $\mathbf{X} = (X_1, X_2, X_3)$ (the Lagrangian coordinates); its location at its actual configuration is denoted by $\mathbf{x} = (x_1, x_2, x_3)$ (the Eulerian coordinates). The following relations hold with respect to the Cartesian base $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ related to the initial configuration Ω_0 and the Cartesian base $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ related to the actual configuration Ω_t:

$$\mathbf{X} = X_I \mathbf{E}_I \quad \text{and} \quad \mathbf{x} = x_i \mathbf{e}_i \tag{1.1}$$

(in the sequel, we assume summation from 1 to 3 with respect to repeating indices, omitting the sign of the sum).

The deformation of an elastic body is defined as the following one-to-one correspondence:

$$\phi : \ \Omega_s \mapsto \Omega_t$$

such that at time t we have

$$\phi(\mathbf{X}, t) : \mathbf{X} \mapsto \mathbf{x} = \phi(\mathbf{X}, t), \quad \text{where} \quad x_i = x_i(X_1, X_2, X_3, t).$$

The corresponding movements have the form $\mathbf{u}(\mathbf{X}, t) := \mathbf{x} - \phi(\mathbf{X}, t)$.

An important kinematic characteristic is the following deformation gradient $\mathbf{F}$:

$$\mathbf{F} = \mathbf{F}(\mathbf{X}, t) = \frac{\partial \phi}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial x_i}{\partial X_J}\, \mathbf{e}_i \otimes \mathbf{E}_J, \tag{1.2}$$

where $\otimes$ denotes the tensor product. Components of the deformation gradient $\mathbf{F}$ are represented by the following matrix:

$$F_{ij} = \frac{\partial x_i}{\partial X_J}, \quad [F_{ij}] = \begin{pmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{pmatrix}.$$

The following restriction is imposed on $J = \det \mathbf{F}$:

$$J = \det \mathbf{F} > 0;$$

it guarantees the existence of $\mathbf{F}^{-1}$ and the absence of the self-penetration under deformations.

Note that the deformation gradient is related to movements $\mathbf{u}$ of body points as follows:

$$\mathbf{F} = \mathbf{I} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}, \tag{1.3}$$

where $\mathbf{I}$ is the identity tensor.

The following so-called polar expansion of the deformation gradient $\mathbf{F}$ is broadly applied for the construction of deformation measures (see [1]).

Theorem 1.1 (on polar expansion of invertible matrices)**.** *Any invertible real matrix* $\mathbf{F}$ *can be uniquely represented in the form*

$$\mathbf{F} = \mathbf{R}\mathbf{U} \quad \text{or} \quad \mathbf{F} = \mathbf{V}\mathbf{R}, \tag{1.4}$$

where $\mathbf{R}$ *is an orthogonal matrix, while* $\mathbf{U}$ *and* $\mathbf{V}$ *are symmetric positive definite matrices.*

Applying the polar expansion theorem to the deformation gradient, one can obtain the rotation tensor $\mathbf{R}$, the right-hand extension tensor $\mathbf{U}$, and the left-hand extension tensor $\mathbf{V}$. In other words, the total deformation of a material element can be treated as the superposition of a solid rotation and a dilatation of the said element.

2. Equilibrium Equations of Elastic Bodies

Under the assumption that there are no bulk forces, equilibrium equations for the elastic material have the following form (see [1]):

$$\operatorname{div}(J\mathbf{T}\mathbf{F}^{-T}) = 0 \quad \text{in } \Omega_0, \tag{2.1}$$

$$\mathbf{T}^T = \mathbf{T}, \tag{2.2}$$

where $\mathbf{T}$ is the Cauchy tensor (the true stress tensor) and $J = \det \mathbf{F}$. To close the system of Eqs. (2.1), one has to set a constitutive relation $\mathbf{T} = \mathbf{T}(\mathbf{F}, \mathbf{X})$ and the corresponding boundary conditions.

The constitutive relation characterizes the mechanical behavior of the material. If the material is hyperelastic, then the material state does not depend on the load path. Then there exists an elastic potential W (the potential deformation energy) such that

$$\mathbf{T} = \frac{1}{J}\frac{\partial W}{\partial \mathbf{F}}\mathbf{F}^T. \tag{2.3}$$

The elastic potential is to satisfy the requirement of the material independence of the frame of reference, i.e.,

$$W(\mathbf{F}) = W(\mathbf{Q}\mathbf{F}) \quad \forall\, \mathbf{Q} \in \mathrm{SO}(3), \tag{2.4}$$

where SO(3) is the proper group of rotations of the three-dimensional space. If there exists a symmetry of physical properties of the considered material, then the constitutive relations are to be invariant with respect to all transformations of material coordinates belonging to the group of symmetries for the specified material. In [1, 13, 19], additional restrictions imposed on the form of the energy function of the deformation (elastic potential) are described in detail. One of these restrictions is the Legendre–Hadamard condition; in the next section, we consider it in detail.

2.1. The Legendre–Hadamard condition. Assume that $W(\mathbf{F})$ is a twice continuously differentiable function. Substituting relations (2.3) and (1.3) in the equilibrium equations (2.1), one can represent Eqs. (2.1) as the following equation with respect to the movements $\mathbf{u}(\mathbf{X}, t)$:

$$\mathbb{C}_{ijkl}\frac{\partial^2 u_k}{\partial X_i \partial X_l} = 0. \tag{2.5}$$

Here, $\mathbb{C}_{ijkl}$ are the components of the elasticity tensor $\mathbb{C}(\mathbf{F})$:

$$\mathbb{C}(\mathbf{F}) = \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}}; \quad C_{ijkl} = C_{klij} = \frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}}. \tag{2.6}$$

Note that properties of the elasticity tensor $\mathbb{C}(\mathbf{F})$ determine the type of system of second-order partial differential equations (2.5).

Definition 2.1 (*the Legendre–Hadamard condition*)**.** The energy deformation function $W(\mathbf{F})$ leads to an elliptic system of equilibrium equations if the following condition is satisfied:

$$(\mathbf{a} \otimes \mathbf{b}) : \mathbb{C}(\mathbf{F}) : (\mathbf{a} \otimes \mathbf{b}) \geq 0, \ \forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \backslash \mathbf{0}. \tag{2.7}$$

If the inequality is strict, then (2.7) is a condition of the strong ellipticity of the system of equilibrium equations for elastic bodies.

The strong ellipticity condition is satisfied if and only if the propagation velocities of low-amplitude waves in the elastic medium are real (see [13]). On the other hand, if the equilibrium equations (2.5) are not elliptic, then no smoothness of solutions of equilibrium equations of the elastic body is guaranteed (see [11, 13]); this is related to the stability loss for the elastic body (see [11, 15]). Thus, creating new hyperelastic models, one has to verify condition (2.7).

3. Elastic Potentials

3.1. Deformation measures. We have mentioned above the material independence of the frame of reference posed by (2.4) as one of restrictions of the form of elastic potentials. As was shown in [1], this requirement is satisfied for hyperelastic materials if and only if the potential energy function is a function of $\mathbf{F}\mathbf{F}^T$, i.e., $W(\mathbf{F}) = \tilde{W}(\mathbf{F}\mathbf{F}^T)$. In practice, various deformation measures are used to characterize deformations. Examples are the right-hand Cauchy–Green deformation tensor

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2,$$

the Lagrange deformation tensor

$$\mathbf{E} = (\mathbf{C} - \mathbf{I})/2,$$

the left-hand Cauchy–Green deformation tensor

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2,$$

and the logarithmic (Hencky) measure

$$\mathbf{E}_H = \log \mathbf{B}/2 = \log(\mathbf{V}), \quad \mathbf{e}_H = \log \mathbf{C}/2 = \log(\mathbf{U}).$$

where $\mathbf{U}$ and $\mathbf{V}$ are the extension (deformation) tensors of the polar expansion of the deformation gradient (1.4).

In [18], it is proposed to use the deformation measure based on the QR-expansion of the deformation gradient instead of its polar expansion. From the numerical viewpoint for solid-body mechanics, the advantages of the above approach are discussed in [7, 18]. From the viewpoint of constitutive relations, the main advantage is the possibility to construct a dependence with noncorrelating terms.

3.2. The deformation measure based on the QR-expansion of the deformation gradient. In brief, the Srinivasa approach to the constructing of constitutive relations (see [18]) is as follows.

Theorem 3.1 (QR-expansion, see [20]). *For any nondegenerate real matrix* $\mathbf{F}$, *the expansion*

$$\mathbf{F} = \mathbf{QR} \tag{3.1}$$

holds, where $\mathbf{Q}$ *is an orthogonal matrix and* $\mathbf{R}$ *is an upper triangular matrix with positive elements on the diagonal.*

According to the QR-expansion theorem for the deformation gradient $\mathbf{F}$, there exists a matrix $\mathbf{Q} = \mathbf{e}'_i \otimes \mathbf{E}_i$ such that

$$\mathbf{Q}^{\mathbf{T}}\mathbf{F} = \tilde{\mathbf{F}} = \sum_{i \leq j}^{i,j=1,2,3} \tilde{F}_{ij} \mathbf{E}_i \otimes \mathbf{E}_j, \tag{3.2}$$

$$\left[\tilde{F}_{ij}\right] := \begin{pmatrix} \tilde{F}_{11} & \tilde{F}_{12} & \tilde{F}_{13} \\ 0 & \tilde{F}_{22} & \tilde{F}_{23} \\ 0 & 0 & \tilde{F}_{33} \end{pmatrix},$$

where $\mathbf{e}'_i$ is a new orthonormal base, which can be obtained by means of the Gram–Schmidt orthogonalization of the vector system $\{\mathbf{FE}_1, \mathbf{FE}_2, \mathbf{FE}_3\}$.

In the base $\{\mathbf{e}'_i \otimes \mathbf{E}_j\}$, the deformation gradient $\mathbf{F}$ can be expressed as follows:

$$\mathbf{F} = \sum_{i \leq j}^{i,j=1,2,3} \tilde{F}_{ij} \mathbf{e}'_i \otimes \mathbf{E}_j. \tag{3.3}$$

Since $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \tilde{\mathbf{F}}^T\tilde{\mathbf{F}}$, it follows that components of the tensor $\tilde{\mathbf{F}}$ can be obtained by means of the Cholesky factorization of the Cauchy–Green deformation tensor $\mathbf{C}$:

$$\begin{aligned} &\tilde{F}_{11} = \sqrt{C_{11}}, && \tilde{F}_{12} = \frac{C_{12}}{\tilde{F}_{11}}, && \tilde{F}_{13} = \frac{C_{13}}{\tilde{F}_{11}}, \\ &\tilde{F}_{22} = \sqrt{C_{22} - \tilde{F}_{12}^2}, && \tilde{F}_{23} = \frac{C_{23} - \tilde{F}_{12}\tilde{F}_{13}}{\tilde{F}_{22}}, && \tilde{F}_{33} = \sqrt{C_{33} - \tilde{F}_{13}^2 - \tilde{F}_{23}^2}. \end{aligned} \tag{3.4}$$

Similarly to the tensors $\mathbf{U}$ and $\mathbf{V}$, the tensor $\tilde{\mathbf{F}}$ characterizes the body deformation as a variation of distances between points; all its components are physically interpreted (see [18]). The deformation measures are $\xi_i, i = 1, \ldots, 6$:

$$\begin{aligned} &\xi_1 = \log \tilde{F}_{11}, && \xi_2 = \log \tilde{F}_{22}, && \xi_3 = \log \tilde{F}_{33}, \\ &\xi_4 = \frac{\tilde{F}_{12}}{\tilde{F}_{11}}, && \xi_5 = \frac{\tilde{F}_{13}}{\tilde{F}_{11}}, && \xi_6 = \frac{\tilde{F}_{23}}{\tilde{F}_{22}}. \end{aligned} \tag{3.5}$$

The elastic potential is a function of ξ_i:

$$W = \psi(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6). \tag{3.6}$$

By virtue of relations (3.4), condition (2.4) of the material independence of the frame of reference is satisfied.

4. Elastic Potentials: Ellipticity Condition in Srinivasa Invariants

Let $\mathbf{a} = a_i \mathbf{e}'_i$ and $\mathbf{b} = b_k \mathbf{E}_k$. Then

$$\mathbf{H} \equiv \mathbf{a} \otimes \mathbf{b} = a_i b_k \mathbf{e}'_i \otimes \mathbf{E}_k. \tag{4.1}$$

Taking into account (3.2), condition (2.7) of the strong ellipticity can be represented as follows:

$$\begin{aligned} &\mathbf{H} : \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}} : \mathbf{H} = (\mathbf{Q}^T \mathbf{H}) : \frac{\partial^2 W}{\partial \tilde{\mathbf{F}} \partial \tilde{\mathbf{F}}} : (\mathbf{Q}^T \mathbf{H}) \\ &= \frac{\partial^2 W}{\partial \xi_n \partial \xi_m} \left(\frac{\partial \xi_n}{\partial \tilde{\mathbf{F}}} : (\mathbf{Q}^T \mathbf{H}) \right) \left(\frac{\partial \xi_m}{\partial \tilde{\mathbf{F}}} : (\mathbf{Q}^T \mathbf{H}) \right) + \frac{\partial W}{\partial \xi_n} \left((\mathbf{Q}^T \mathbf{H}) : \frac{\partial^2 \xi_n}{\partial \tilde{\mathbf{F}} \partial \tilde{\mathbf{F}}} : (\mathbf{Q}^T \mathbf{H}) \right) > 0. \end{aligned} \tag{4.2}$$

Introduce the notation $\bar{H}_{ij} = (\mathbf{e}'_i, \mathbf{H}\mathbf{E}_j) = a_i b_j$. Then condition (4.2) is equivalent to the following positive definiteness condition for a quadratic form:

$$\bar{\mathbf{H}}^T \, \mathbf{\Pi} \, \bar{\mathbf{H}} > 0, \tag{4.3}$$

where $\bar{\mathbf{H}} = (H_{11}, H_{22}, H_{33}, H_{12}, H_{23}, H_{13})^T$ and the matrix $\mathbf{\Pi}$ is defined by (6.1)-(6.2) (see the Appendix). Thus, the following theorem is proved.

Theorem 4.1. *For the given deformation, the condition of strong ellipticity* (4.2) *is equivalent to the positive definiteness of the matrix* $\mathbf{\Pi}$ *defined by* (6.1)-(6.2).

Corollary 4.1. The following restrictions for partial derivatives of the function ψ are necessary conditions of the strong ellipticity:

$$\begin{aligned} \Pi_{11} &= \frac{1}{\tilde{F}_{11}^2} \left(\frac{\partial^2 \psi}{\partial \xi_1^2} + \xi_4^2 \frac{\partial^2 \psi}{\partial \xi_4^2} + \xi_5^2 \frac{\partial^2 \psi}{\partial \xi_5^2} - 2\xi_4 \frac{\partial^2 \psi}{\partial \xi_1 \partial \xi_4} - 2\xi_5 \frac{\partial^2 \psi}{\partial \xi_1 \partial \xi_5} \right. \\ &\quad \left. + 2\xi_4 \xi_5 \frac{\partial^2 \psi}{\partial \xi_4 \partial \xi_5} - \frac{\partial \psi}{\partial \xi_1} + 2\xi_4 \frac{\partial \psi}{\partial \xi_4} + 2\xi_5 \frac{\partial \psi}{\partial \xi_5} \right) > 0, \\ \Pi_{22} &= \frac{1}{\tilde{F}_{22}^2} \left(\frac{\partial^2 \psi}{\partial \xi_2^2} - 2\xi_6 \frac{\partial^2 \psi}{\partial \xi_2 \partial \xi_6} + \xi_6^2 \frac{\partial^2 \psi}{\partial \xi_6^2} - \frac{\partial \psi}{\partial \xi_2} + 2\xi_6 \frac{\partial \psi}{\partial \xi_6} \right) > 0, \\ \Pi_{33} &= \frac{1}{\tilde{F}_{33}^2} \left(\frac{\partial^2 \psi}{\partial \xi_3^2} - \frac{\partial \psi}{\partial \xi_3} \right) > 0, \\ \Pi_{44} &= \frac{1}{\tilde{F}_{11}^2} \frac{\partial^2 \psi}{\partial \xi_4^2} > 0, \\ \Pi_{55} &= \frac{1}{\tilde{F}_{22}^2} \frac{\partial^2 \psi}{\partial \xi_6^2} > 0, \\ \Pi_{66} &= \frac{1}{\tilde{F}_{11}^2} \frac{\partial^2 \psi}{\partial \xi_5^2} > 0. \end{aligned} \tag{4.4}$$

The proof follows from the positive definiteness criterion for the matrix $\mathbf{\Pi}$. According to inequalities (4.4), the exponential growth of the function ψ with respect to ξ_3 and its convexity with respect to ξ_4, ξ_5, ξ_6 are necessary ellipticity conditions for the hyperelastic model in invariants $\xi_i, i = 1, \dots, 6$.

Remark 4.1. Verifying necessary conditions (4.4) for each elastic potential restored by experimental data, one can check whether the strong ellipticity condition is broken for the investigated range of deformations. By virtue of Theorem 4.1, for the specified deformation, sufficient conditions of the positive definiteness of the matrix $\mathbf{\Pi}$ are sufficient for the fulfillment of the strong ellipticity condition as well; in the present paper, we omit their general form because it is too cumbersome.

4.1. The case where $\psi = \psi(\xi_1, \xi_2, \xi_4)$. Consider the fulfillment of the necessary conditions of the ellipticity for the special case of two-dimensional constitutive relations.

For many biomedicine problems, it is very interesting to simulate the work of the heart. The mechanical behavior of various parts of a heart is actively investigated. In particular, this refers to the pericardium (the inner envelope of the heart, also referred to as the pericardial sac). According to experimental data, the pericardium can be treated as an anisotropic orthotropic material, i.e., there exist three mutually orthogonal planes of symmetry of properties. The anisotropy of properties is closely related to the net of elastic fibers located in the pericardium.

In [5, 6], based on experimental data, a constitutive relation was proposed for the pericardium. The deformation measure proposed in [5] coincides with $\xi_1,\ \xi_2,\ \xi_4$, and the relation obtained for the hyperelastic case has the form

$$\psi(\xi_1,\ \xi_2,\ \xi_4) = q_1\xi_1 + g_1\left(\frac{e^{\,\alpha_1\xi_1} - \alpha_1\xi_1 - 1}{\alpha_1^2}\right) + q_2\xi_2 + g_2\left(\frac{e^{\,\alpha_2\xi_2} - \alpha_1\xi_2 - 1}{\alpha_2^2}\right) \\ + \alpha_3\xi_1^2\xi_2^2 + \alpha_4\xi_1^2\xi_2^3 + \xi_4^2 G(\xi_1, \xi_2, \xi_4^2), \tag{4.5}$$

where $q_1 = 1.78$ kPa, $q_2 = 0.7$ kPa, $g_1 = 146$ kPa, $g_2 = 85$ kPa, $\alpha_1 = \alpha_2 = 23.5$, $\alpha_3 = 5550$ kPa, and $\alpha_4 = 26400$ kPa. The function $G(\xi_1, \xi_2, \xi_4^2)$ from Eq. (4.5) cannot be defined on the basis of the collection of experimental data used (the experiment was conducted for $\xi_4 = 0$). For the case of an elastic potential defined by (4.5), the necessary conditions of the strong ellipticity take the form

$$\frac{\partial^2\psi}{\partial\xi_1^2} + \xi_4^2\frac{\partial^2\psi}{\partial\xi_4^2} - 2\xi_4\frac{\partial^2\psi}{\partial\xi_1\partial\xi_4} - \frac{\partial\psi}{\partial\xi_1} + 2\xi_4\frac{\partial\psi}{\partial\xi_4} > 0, \\ \frac{\partial^2\psi}{\partial\xi_2^2} - \frac{\partial\psi}{\partial\xi_2} > 0, \\ \frac{\partial^2\psi}{\partial\xi_4^2} > 0. \tag{4.6}$$

The second inequality of (4.6) is related to the following known fact: the rigidity of soft tissues grows for large extensions (see [9]). This is expressed in the exponential law describing their mechanical behavior. For $\xi_4 = 0$, the elastic potential (4.5) corresponds to the constitutive relation for an isotropic material expressed by a function $\psi_{iso} = \psi_{iso}(\xi_1, \xi_2)$, necessary conditions in the form of the exponential growth of ψ with respect to ξ_1 and ξ_2 are satisfied, and ψ_{iso} satisfies the strong ellipticity condition for all values of ξ_1 and ξ_2.

To describe the mechanical behavior of the pericardium for various values of ξ_1, ξ_2, and ξ_4, one has to find a function $G(\xi_1, \xi_2, \xi_4^2)$ from Eq. (4.5) using the experimental data such that $\xi_4 \neq 0$. In this case, conditions (4.6) impose restrictions on the form of the function $G(\xi_1, \xi_2, \xi_4^2)$, guaranteeing the fulfillment of the ellipticity condition for the equilibrium equations. By virtue of Remark 4.1 and the general form of the function $G(\xi_1, \xi_2, \xi_4^2)$, we do not provide the investigation of sufficient ellipticity conditions for the equations.

5. Conclusions

For hyperelastic materials, the fulfillment of strong ellipticity conditions is directly related to the correct description of the mechanical behavior of the material. The specified condition is a necessary condition for the stability of the equilibrium elastic deformation. One of the main corollaries from the strong ellipticity condition is that the propagation velocity of waves in the material is real. In the present paper, we have investigated the strong ellipticity condition for constitutive relations of hyperelastic materials in the case where the deformation measure is based on the QR-expansion of the deformation gradient. We have obtained a matrix such that the strong ellipticity condition is equivalent to its positive definiteness. For the fulfillment of the strong ellipticity condition, we obtain necessary conditions imposing restrictions on the form of the elastic potential in new invariants. In particular, for the known pericardium hyperelastic model restored according to experimental data, we have obtained restrictions on the form of the elastic potential.

6. Appendix

6.1. Derivatives of invariants. Since

$$\xi_1 = \log \tilde{F}_{11}, \quad \xi_2 = \log \tilde{F}_{22}, \quad \xi_3 = \log \tilde{F}_{33}, \quad \xi_4 = \frac{\tilde{F}_{12}}{\tilde{F}_{11}}, \quad \xi_5 = \frac{\tilde{F}_{13}}{\tilde{F}_{11}}, \quad \xi_5 = \frac{\tilde{F}_{23}}{\tilde{F}_{22}},$$

it follows that the corresponding derivatives are expressed in the form

$$\frac{\partial \xi_1}{\partial \tilde{F}_{ij}} = \frac{1}{\tilde{F}_{11}}\delta_{i1}\delta_{j1}, \quad \frac{\partial \xi_2}{\partial \tilde{F}_{ij}} = \frac{1}{\tilde{F}_{22}}\delta_{i2}\delta_{j2}, \quad \frac{\partial \xi_3}{\partial \tilde{F}_{ij}} = \frac{1}{\tilde{F}_{33}}\delta_{i3}\delta_{j3},$$
$$\frac{\partial \xi_4}{\partial \tilde{F}_{ij}} = \frac{1}{\tilde{F}_{11}}\delta_{i1}\delta_{j2} - \frac{\tilde{F}_{12}}{\tilde{F}_{11}^2}\delta_{i1}\delta_{j1},$$
$$\frac{\partial \xi_5}{\partial \tilde{F}_{ij}} = \frac{1}{\tilde{F}_{11}}\delta_{i1}\delta_{j3} - \frac{\tilde{F}_{13}}{\tilde{F}_{11}^2}\delta_{i1}\delta_{j1},$$
$$\frac{\partial \xi_6}{\partial \tilde{F}_{ij}} = \frac{1}{\tilde{F}_{22}}\delta_{i2}\delta_{j3} - \frac{\tilde{F}_{23}}{\tilde{F}_{22}^2}\delta_{i2}\delta_{j2},$$

where δ_{ij} is the Kronecker symbol.

6.2. The matrix $\mathbf{\Pi}$. Let

$$\mathbf{H}_1 = \mathbf{e}'_1 \otimes \mathbf{E}_1, \quad \mathbf{H}_2 = \mathbf{e}'_2 \otimes \mathbf{E}_2, \quad \mathbf{H}_3 = \mathbf{e}'_3 \otimes \mathbf{E}_3,$$
$$\mathbf{H}_4 = \mathbf{e}'_1 \otimes \mathbf{E}_2, \quad \mathbf{H}_5 = \mathbf{e}'_2 \otimes \mathbf{E}_3, \quad \mathbf{H}_6 = \mathbf{e}'_1 \otimes \mathbf{E}_3.$$

Then the matrix $\mathbf{\Pi}$ is defined as follows:

$$\mathbf{\Pi} = \left[(\mathbf{Q}^T \mathbf{H}_\alpha) : \frac{\partial^2 W}{\partial \tilde{\mathbf{F}} \partial \tilde{\mathbf{F}}} : (\mathbf{Q}^T \mathbf{H}_\beta) \right]_{6\times 6}. \tag{6.1}$$

Assume that

$$\frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j} = \frac{\partial^2 \psi}{\partial \xi_j \partial \xi_i}.$$

Taking into account that $\Pi_{ij} = \Pi_{ji}$, we get the following expressions for the elements of the matrix $\mathbf{\Pi}$:

$$\Pi_{11} = \sum_{n=1,4,5} \sum_{m=1,4,5} \frac{\partial^2 \psi}{\partial \xi_n \partial \xi_m} \frac{\partial \xi_n}{\partial \tilde{F}_{11}} \frac{\partial \xi_m}{\partial \tilde{F}_{11}} + \sum_{n=1,4,5} \frac{\partial \psi}{\partial \xi_n} \frac{\partial^2 \xi_n}{\partial \tilde{F}_{11} \partial \tilde{F}_{11}}, \tag{6.2}$$

$$\Pi_{12} = \sum_{n=1,4,5} \sum_{m=2,6} \frac{\partial^2 \psi}{\partial \xi_n \partial \xi_m} \frac{\partial \xi_n}{\partial \tilde{F}_{11}} \frac{\partial \xi_m}{\partial \tilde{F}_{22}},$$

$$\Pi_{13} = \sum_{n=1,4,5} \frac{\partial^2 \psi}{\partial \xi_n \partial \xi_3} \frac{\partial \xi_n}{\partial \tilde{F}_{11}} \frac{\partial \xi_3}{\partial \tilde{F}_{33}} = \frac{1}{\tilde{F}_{33}} \sum_{n=1,4,5} \frac{\partial^2 \psi}{\partial \xi_n \partial \xi_3} \frac{\partial \xi_n}{\partial \tilde{F}_{11}},$$

$$\Pi_{14} = \sum_{n=1,4,5} \frac{\partial^2 \psi}{\partial \xi_n \partial \xi_4} \frac{\partial \xi_n}{\partial \tilde{F}_{11}} \frac{\partial \xi_4}{\partial \tilde{F}_{12}} + \frac{\partial \psi}{\partial \xi_4} \frac{\partial^2 \xi_4}{\partial \tilde{F}_{11} \partial \tilde{F}_{12}} = \frac{1}{\tilde{F}_{11}} \sum_{n=1,4,5} \frac{\partial^2 \psi}{\partial \xi_n \partial \xi_4} \frac{\partial \xi_n}{\partial \tilde{F}_{11}} - \frac{1}{\tilde{F}_{11}^2} \frac{\partial \psi}{\partial \xi_4},$$

$$\Pi_{15} = \sum_{n=1,4,5} \frac{\partial^2 \psi}{\partial \xi_n \partial \xi_6} \frac{\partial \xi_n}{\partial \tilde{F}_{11}} \frac{\partial \xi_6}{\partial \tilde{F}_{23}} = \frac{1}{\tilde{F}_{22}} \sum_{n=1,4,5} \frac{\partial^2 \psi}{\partial \xi_n \partial \xi_6} \frac{\partial \xi_n}{\partial \tilde{F}_{11}},$$

$$\Pi_{16} = \sum_{n=1,4,5} \frac{\partial^2 \psi}{\partial \xi_n \partial \xi_5} \frac{\partial \xi_n}{\partial \tilde{F}_{11}} \frac{\partial \xi_5}{\partial \tilde{F}_{13}} + \frac{\partial \psi}{\partial \xi_5} \frac{\partial^2 \xi_5}{\partial \tilde{F}_{11} \partial \tilde{F}_{13}} = \frac{1}{\tilde{F}_{11}} \sum_{n=1,4,5} \frac{\partial^2 \psi}{\partial \xi_n \partial \xi_5} \frac{\partial \xi_n}{\partial \tilde{F}_{11}} - \frac{1}{\tilde{F}_{11}^2} \frac{\partial \psi}{\partial \xi_5},$$

$$\Pi_{22} = \sum_{n=2,6}\sum_{m=2,6}\frac{\partial^2\psi}{\partial\xi_n\partial\xi_m}\frac{\partial\xi_n}{\partial\tilde{F}_{22}}\frac{\partial\xi_m}{\partial\tilde{F}_{22}} + \sum_{n=2,6}\frac{\partial\psi}{\partial\xi_n}\frac{\partial^2\xi_n}{\partial\tilde{F}_{22}\partial\tilde{F}_{22}},$$

$$\Pi_{23} = \sum_{n=2,6}\frac{\partial^2\psi}{\partial\xi_n\partial\xi_3}\frac{\partial\xi_n}{\partial\tilde{F}_{22}}\frac{\partial\xi_3}{\partial\tilde{F}_{33}} = \frac{1}{\tilde{F}_{33}}\sum_{n=2,6}\frac{\partial^2\psi}{\partial\xi_n\partial\xi_3}\frac{\partial\xi_n}{\partial\tilde{F}_{22}},$$

$$\Pi_{24} = \sum_{n=2,6}\frac{\partial^2\psi}{\partial\xi_n\partial\xi_4}\frac{\partial\xi_n}{\partial\tilde{F}_{22}}\frac{\partial\xi_4}{\partial\tilde{F}_{12}} = \frac{1}{\tilde{F}_{11}}\sum_{n=2,6}\frac{\partial^2\psi}{\partial\xi_n\partial\xi_4}\frac{\partial\xi_n}{\partial\tilde{F}_{22}},$$

$$\Pi_{25} = \sum_{n=2,6}\frac{\partial^2\psi}{\partial\xi_n\partial\xi_6}\frac{\partial\xi_n}{\partial\tilde{F}_{22}}\frac{\partial\xi_6}{\partial\tilde{F}_{23}} + \frac{\partial\psi}{\partial\xi_6}\frac{\partial^2\xi_6}{\partial\tilde{F}_{22}\partial\tilde{F}_{23}} = \frac{1}{\tilde{F}_{22}}\sum_{n=2,6}\frac{\partial^2\psi}{\partial\xi_n\partial\xi_6}\frac{\partial\xi_n}{\partial\tilde{F}_{22}} - \frac{1}{\tilde{F}_{22}^2}\frac{\partial\psi}{\partial\xi_6},$$

$$\Pi_{26} = \sum_{n=2,6}\frac{\partial^2\psi}{\partial\xi_n\partial\xi_5}\frac{\partial\xi_n}{\partial\tilde{F}_{22}}\frac{\partial\xi_5}{\partial\tilde{F}_{13}} = \frac{1}{\tilde{F}_{11}}\sum_{n=2,6}\frac{\partial^2\psi}{\partial\xi_n\partial\xi_5}\frac{\partial\xi_n}{\partial\tilde{F}_{22}},$$

$$\Pi_{33} = \frac{\partial^2\psi}{\partial\xi_3^2}\left(\frac{\partial\xi_3}{\partial\tilde{F}_{33}}\right)^2 + \frac{\partial\psi}{\partial\xi_3}\frac{\partial^2\xi_3}{\partial\tilde{F}_{33}\partial\tilde{F}_{33}} = \frac{1}{\tilde{F}_{33}^2}\frac{\partial^2\psi}{\partial\xi_3^2} - \frac{1}{\tilde{F}_{33}^2}\frac{\partial\psi}{\partial\xi_3},$$

$$\Pi_{34} = \frac{\partial^2\psi}{\partial\xi_3\partial\xi_4}\frac{\partial\xi_3}{\partial\tilde{F}_{33}}\frac{\partial\xi_4}{\partial\tilde{F}_{12}} = \frac{1}{\tilde{F}_{33}}\frac{1}{\tilde{F}_{11}}\frac{\partial^2\psi}{\partial\xi_3\partial\xi_4},$$

$$\Pi_{35} = \frac{\partial^2\psi}{\partial\xi_3\partial\xi_6}\frac{\partial\xi_3}{\partial\tilde{F}_{33}}\frac{\partial\xi_6}{\partial\tilde{F}_{23}} = \frac{1}{\tilde{F}_{33}}\frac{1}{\tilde{F}_{22}}\frac{\partial^2\psi}{\partial\xi_3\partial\xi_6},$$

$$\Pi_{36} = \frac{\partial^2\psi}{\partial\xi_3\partial\xi_5}\frac{\partial\xi_3}{\partial\tilde{F}_{33}}\frac{\partial\xi_5}{\partial\tilde{F}_{13}} = \frac{1}{\tilde{F}_{33}}\frac{1}{\tilde{F}_{11}}\frac{\partial^2\psi}{\partial\xi_3\partial\xi_5},$$

$$\Pi_{44} = \frac{\partial^2\psi}{\partial\xi_4^2}\left(\frac{\partial\xi_4}{\partial\tilde{F}_{12}}\right)^2 = \frac{1}{\tilde{F}_{11}^2}\frac{\partial^2\psi}{\partial\xi_4^2},$$

$$\Pi_{45} = \frac{\partial^2\psi}{\partial\xi_4\partial\xi_6}\frac{\partial\xi_4}{\partial\tilde{F}_{12}}\frac{\partial\xi_6}{\partial\tilde{F}_{23}} = \frac{1}{\tilde{F}_{11}}\frac{1}{\tilde{F}_{22}}\frac{\partial^2\psi}{\partial\xi_4\partial\xi_6},$$

$$\Pi_{46} = \frac{\partial^2\psi}{\partial\xi_4\partial\xi_5}\frac{\partial\xi_4}{\partial\tilde{F}_{12}}\frac{\partial\xi_5}{\partial\tilde{F}_{13}} = \frac{1}{\tilde{F}_{11}^2}\frac{\partial^2\psi}{\partial\xi_4\partial\xi_5},$$

$$\Pi_{55} = \frac{\partial^2\psi}{\partial\xi_6^2}\left(\frac{\partial\xi_6}{\partial\tilde{F}_{23}}\right)^2 = \frac{1}{\tilde{F}_{22}^2}\frac{\partial^2\psi}{\partial\xi_6^2},$$

$$\Pi_{56} = \frac{\partial^2\psi}{\partial\xi_6\partial\xi_5}\frac{\partial\xi_6}{\partial\tilde{F}_{23}}\frac{\partial\xi_5}{\partial\tilde{F}_{13}} = \frac{1}{\tilde{F}_{11}}\frac{1}{\tilde{F}_{22}}\frac{\partial^2\psi}{\partial\xi_6\partial\xi_5},$$

$$\Pi_{66} = \frac{\partial^2\psi}{\partial\xi_5^2}\left(\frac{\partial\xi_5}{\partial\tilde{F}_{13}}\right)^2 = \frac{1}{\tilde{F}_{11}^2}\frac{\partial^2\psi}{\partial\xi_5^2}.$$

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