

# A quasi-Lagrangian finite element method for the Navier–Stokes equations in a time-dependent domain<sup>☆</sup>

Alexander Lozovskiy<sup>a</sup>, Maxim A. Olshanskii<sup>b,\*</sup>, Yuri V. Vassilevski<sup>c</sup>

<sup>a</sup> *Institute of Numerical Mathematics RAS, Russian Federation*

<sup>b</sup> *Department of Mathematics, University of Houston, USA*

<sup>c</sup> *Institute of Numerical Mathematics RAS, Moscow Institute of Physics and Technology, Sechenov University, Russian Federation*

Received 17 July 2017; received in revised form 12 January 2018; accepted 17 January 2018

## Abstract

The paper develops a finite element method for the Navier–Stokes equations of incompressible viscous fluid in a time-dependent domain. The method builds on a quasi-Lagrangian formulation of the problem. The paper provides stability and convergence analysis of the fully discrete (finite-difference in time and finite-element in space) method. The analysis does not assume any CFL time-step restriction, it rather needs mild conditions of the form  $\Delta t \leq C$ , where  $C$  depends only on problem data, and  $h^{2m_u+2} \leq c \Delta t$ ,  $m_u$  is polynomial degree of velocity finite element space. Both conditions result from a numerical treatment of practically important non-homogeneous boundary conditions. The theoretically predicted convergence rate is confirmed by a set of numerical experiments. Further we apply the method to simulate a flow in a simplified model of the left ventricle of a human heart, where the ventricle wall dynamics is reconstructed from a sequence of contrast enhanced computed tomography images.

© 2018 Elsevier B.V. All rights reserved.

**Keywords:** Incompressible Navier–Stokes; Moving domain; Finite elements; Error analysis

## 1. Introduction

Fluid flows in time-dependent domains are ubiquitous in nature and engineering. In many cases, finding the domain evolution is part of the problem and the mathematical model couples fluid and structure dynamics. Examples include fluid–structure interaction problems for blood flow in compliant vessels, flows around turbine blades or fish locomotion. In other situations, one may assume that the motion of the domain is given and one has to recover the induced fluid flow. One example of a problem, which often assumes *a priori* information about the flow domains evolution, is the blood flow simulation in a human heart when the (patient-specific) motion of the heart walls is

<sup>☆</sup> This work has been supported by the Russian Science Foundation (RSF) grant 14-31-00024.

\* Corresponding author.

E-mail addresses: [saiya-jin@yandex.ru](mailto:saiya-jin@yandex.ru) (A. Lozovskiy), [molshan@math.uh.edu](mailto:molshan@math.uh.edu) (M.A. Olshanskii), [yuri.vassilevski@gmail.com](mailto:yuri.vassilevski@gmail.com) (Y.V. Vassilevski).

recovered from a sequence of medical images [1–8]. Nowadays numerical simulations are commonly used to understand fluid dynamics and predict statistics of practical interest in this and other applications. In the present paper, we develop a finite element (FE) method for a quasi-Lagrangian formulation of the incompressible Navier–Stokes equations in a moving domain. We consider an implicit–explicit method, i.e. an implicit method with advection field in the inertia term lagged in time. For the spatial discretization we employ inf–sup stable pressure–velocity elements.

Several techniques have been introduced in the literature to overcome numerical difficulties due to the evolution of the domain. This includes space–time finite element formulations, immersed boundary methods, level-set method, fictitious domain method, unfitted finite elements, and arbitrary Lagrangian–Eulerian (ALE) formulation, see, e.g., [9–17]. In this paper we analyze a finite element method based on a quasi-Lagrangian formulation of the equations in the reference domain. Related analysis of finite element methods for parabolic or fluid equations in moving domains can be found in several places in the literature. We note that well-posedness of space–time weak saddle-point formulations of the (Navier–)Stokes equations is a subtle question, see the recent treatment in [18] for the case of a steady domain. A rigorous stability and convergence analysis of space–time (FE) methods for fluid problems seems to be largely lacking. Scalar problems have been understood much better; for example, a space–time discontinuous FE method for advection–diffusion problems on time-dependent domains was analyzed in [19]. ALE and Lagrangian finite element methods are more amenable to analysis. The stability of ALE finite element methods for parabolic evolution problems was treated in [16]. The authors of [20] analyzed the convergence of a finite element ALE method for the Stokes equations in a time-dependent domain when the motion of the domain is given. The analysis [20] imposes time step restriction, assumes zero velocity boundary condition and certain smoothness assumptions for the finite element displacement field. A closely related method to the one studied here was considered in [21]. However, that paper introduced an assumption that a divergence free extension of a boundary condition function to the computational domain is given. This assumption is not always practical and the present paper avoids it. Moreover, this paper develops error analysis, while the thrust of [21] was the stability analysis and the numerical recovery procedure of the domain motion from medical images.

In the present paper, we analyze a quasi-Lagrangian FE formulation that is closely related to an ALE formulation, although they are not equivalent. In the present approach we discretize equations in a reference time-independent domain. The geometry evolution is accounted in time-dependent coefficients. Inertia terms are further linearized so that only a system of linear algebraic equations is solved in each time step. We consider practically relevant boundary conditions, which result in non-homogeneous velocity on the boundary. For this method we prove numerical stability and optimal order error estimates in the energy norm without a CFL condition in the time step. Divergence-free condition enforced in the reference domain leads to time dependent functional spaces; this and handling non-homogeneous boundary conditions are two main difficulties that we overcome in the analysis. For the numerical stability bound we shall need the condition on the space mesh size and time step of the form  $h^{2m_u+2} \leq c \Delta t$ , where  $m_u \geq 1$  is polynomial degree of velocity finite element space and  $c$  is a constant. We note that if one assumes zero boundary conditions for velocity, which is a standard assumption for FE stability bounds in steady domains, then the results of the paper hold without the above conditions on  $h$  and  $\Delta t$ . In our opinion, homogeneous boundary conditions are not a suitable assumption for the FE analysis in evolving domains, see discussion in Section 3.

Thus the paper advances the known analysis by including inertia effect, removing CFL time-step restriction, handling physically meaningful boundary conditions, and making no further auxiliary assumptions except the following one: the domain evolution is given *a priori* by a smooth mapping from a reference domain to a physical domain and exact quadrature rules are applied in the reference domain, i.e. we do not analyze possible errors due to inexact numerical integration. The mapping is *not* necessarily Lagrangian in the internal points, but it has to be Lagrangian for those parts of the boundary, where correct tangential velocity boundary values are important. Theoretical results are illustrated numerically for an example with a synthetic known solution. We further illustrate the performance of the numerical method by applying it to blood flow simulation in a simplified model of the human left ventricle. The domain motion in this example is reconstructed from a sequence of CECT images of a real patient heart over one cardiac cycle. The reconstruction procedure is described in detail in our preceding paper [21].

The remainder of this paper is organized as follows. In Section 2 we review the mathematical model, including governing equations and boundary conditions, and some useful results for this model found in the literature. We recall the energy balance satisfied by smooth solutions. A suitable weak formulation is introduced. Based on the weak formulation, in Section 3 we introduce the finite element method. Non-homogeneous boundary conditions are interpolated numerically. Energy stability estimate for the finite element method is shown in Section 4. Optimal order error bound for the method is demonstrated in Section 5. Section 6 collects results of numerical experiments. Some closing remarks can be found in the summary and outlook Section 7.

## 2. Mathematical model

Consider a time-dependent domain  $\Omega(t) \subset \mathbb{R}^d$ ,  $d = 2, 3$ , occupied by fluid. To formulate a flow problem, we introduce the reference domain  $\Omega_0 = \Omega(0)$  and a mapping from the space–time cylinder  $Q := \Omega_0 \times [0, T]$  to the physical domain,

$$\xi : Q \rightarrow Q^{\text{phys}} := \bigcup_{t \in [0, T]} \Omega(t) \times \{t\}.$$

The mapping is assumed to be level-preserving, i.e.  $\xi(\Omega_0 \times \{t\}) = \Omega(t)$  for all  $t \in [0, T]$ . We assume also that the evolution of  $\Omega(t)$  is sufficiently smooth such that  $\xi \in C^3(Q)^d$ . Denote the spatial gradient matrix of  $\xi$  by  $\mathbf{F} = \nabla_{\mathbf{x}} \xi$ , and  $J := \det(\mathbf{F})$ . Furthermore, we assume that there exist such positive reals  $C_F, c_J$  that

$$\inf_Q J \geq c_J > 0, \quad \sup_Q (\|\mathbf{F}\|_F + \|\mathbf{F}^{-1}\|_F) \leq C_F, \quad \text{with } \|\mathbf{F}\|_F := \text{tr}(\mathbf{F}\mathbf{F}^T)^{\frac{1}{2}}. \quad (1)$$

The dynamics of incompressible Newtonian fluid can be described in terms of the velocity vector field  $\hat{\mathbf{u}}(\mathbf{x}, t)$  and the pressure function  $\hat{p}(\mathbf{x}, t)$  defined in  $\Omega(t)$  for  $t \in [0, T]$ . This paper studies a finite element method for fluid equations formulated in the reference domain. For  $\mathbf{u} = \hat{\mathbf{u}} \circ \xi$ ,  $p = \hat{p} \circ \xi$  defined in  $Q$ , the fluid dynamics is given by the following set of equations:

$$\begin{cases} \mathbf{u}_t - J^{-1} \text{div}(J(\hat{\sigma} \circ \xi)\mathbf{F}^{-T}) + (\nabla \mathbf{u})(\mathbf{F}^{-1}(\mathbf{u} - \xi_t)) = \mathbf{f} \\ \text{div}(J\mathbf{F}^{-1}\mathbf{u}) = 0 \end{cases} \quad \text{in } Q, \quad (2)$$

with body forces  $\mathbf{f} = \hat{\mathbf{f}} \circ \xi$  and the initial condition  $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$  in  $\Omega_0$ . We assume the fluid to be Newtonian, with the kinematic viscosity parameter  $\nu$ . The constitutive relation in the reference domain reads

$$\hat{\sigma} \circ \xi = -p\mathbf{I} + \nu(\nabla \mathbf{u}\mathbf{F}^{-1} + \mathbf{F}^{-T}(\nabla \mathbf{u})^T) \quad \text{in } Q. \quad (3)$$

### 2.1. Boundary conditions

We distinguish between the no-slip  $\partial\Omega^{ns}(t)$ , Dirichlet  $\partial\Omega^D(t)$  and outflow  $\partial\Omega^N(t)$  parts of the boundary, and  $\partial\Omega(t) = \partial\Omega^{ns}(t) \cup \partial\Omega^D(t) \cup \partial\Omega^N(t)$ . On  $\partial\Omega^{ns}(t)$  we impose no-penetration no-slip boundary condition, i.e. the fluid velocity on  $\partial\Omega(t)$  is equal to the material velocity of the boundary (see the discussion below),

$$\hat{\mathbf{u}} = \xi_t \circ \xi^{-1} \quad \text{on } \partial\Omega^{ns}(t), \quad (4)$$

while on  $\partial\Omega^D(t)$  and  $\partial\Omega^N(t)$  we prescribe Dirichlet and Neumann conditions,

$$\hat{\mathbf{u}} = \hat{\mathbf{u}}_D \quad \text{on } \partial\Omega^D(t), \quad \hat{\sigma}\hat{\mathbf{n}} = \hat{\mathbf{g}} \quad \text{on } \partial\Omega^N(t). \quad (5)$$

Here  $\hat{\mathbf{u}}_D$  is a given velocity and  $\hat{\mathbf{n}}$  is the exterior unit normal vector on  $\partial\Omega(t)$ . If  $\partial\Omega^N(t) = \emptyset$  for some  $t \in [0, T]$  we assume  $\int_{\partial\Omega^{ns}(t)} \hat{\mathbf{n}} \cdot \xi_t \circ \xi^{-1} \, ds + \int_{\partial\Omega^D(t)} \hat{\mathbf{n}} \cdot \hat{\mathbf{u}}^D \, ds = 0$ .

In the reference domain, define  $\partial\Omega_0^D = \xi^{-1}\partial\Omega^D(t)$ ,  $\partial\Omega_0^N = \xi^{-1}\partial\Omega^N(t)$ ,  $\partial\Omega_0^{ns} = \xi^{-1}\partial\Omega^{ns}(t)$ . We assume that  $\partial\Omega_0^D, \partial\Omega_0^N, \partial\Omega_0^{ns}$  are independent of  $t$ .

**Remark 1.** The normal velocity of the boundary  $\partial\Omega(t)$  is  $v_\Gamma = \hat{\mathbf{n}} \cdot (\xi_t \circ \xi^{-1})$ . However, the material tangential velocity of the boundary is defined by the tangential part of  $\xi_t$  only if  $\xi$  is the Lagrangian mapping, i.e.  $\xi(\mathbf{x}, t)$ ,  $t \in [0, T]$ , defines the material trajectory for  $\mathbf{x} \in \Omega_0$  (or at least for  $\mathbf{x} \in \partial\Omega_0$ ). In some applications such Lagrangian mapping is not available, and in this case (4) may produce spurious tangential velocities on the boundary. For example, this may happen if  $\xi$  is reconstructed from medical images. Thus, in practice one may or may not amend (4) based on any additional information about the tangential motions for a better model.

### 2.2. An extension result

The solvability of the problem (2)–(3) and the existence of its weak solutions is treated, for example, in [22]. Moreover, it is shown in [22] that for smoothly evolving  $\Omega(t)$  the mapping  $\xi$  can be chosen in such a way that  $J$

depends only on  $t$ . From numerical viewpoint, such a mapping  $\xi$  may not be practically available, and so we allow  $J$  to vary in time and space. However, the following corollary of this result is important for us, see Theorem 4.4. in [22]: Assume  $|\Omega(t)| = |\Omega(0)|$ , then there exists  $\hat{\mathbf{v}}_1 \in C^2(\overline{Q^{\text{phys}}})^d$  such that  $\hat{\mathbf{v}}_1 = \xi_t \circ \xi^{-1}$  on  $\partial\Omega(t)$  and  $\text{div } \hat{\mathbf{v}}_1 = 0$  in  $\Omega(t)$  for  $t \in [0, T]$ . The condition  $|\Omega(t)| = |\Omega(0)|$  is satisfied in the case of  $\partial\Omega(t) = \partial\Omega^{ns}(t)$  for all  $t \in [0, T]$ . Indeed, the Reynolds transport theorem and the incompressibility assumption for fluid imply

$$\frac{d}{dt}|\Omega(t)| = \frac{d}{dt} \int_{\Omega(t)} d\mathbf{x} = \int_{\partial\Omega(t)} v_\Gamma ds = \int_{\partial\Omega(t)} \hat{\mathbf{n}} \cdot \hat{\mathbf{u}} ds = \int_{\Omega(t)} \text{div } \hat{\mathbf{u}} d\mathbf{x} = 0.$$

For the finite element analysis in this paper we *assume* that  $\hat{\mathbf{v}}_1$  can be taken  $C^3$ -smooth. We define smooth function  $\mathbf{v}_1 = \hat{\mathbf{v}}_1 \circ \xi$  that satisfies

$$\mathbf{v}_1 \in C^3(Q)^d, \quad \text{div}(J\mathbf{F}^{-1}\mathbf{v}_1) = 0 \text{ in } \Omega_0, \quad \mathbf{v}_1 = \xi_t \text{ on } \partial\Omega_0. \quad (6)$$

We stress that we need the result about existence of  $\mathbf{v}_1$  for the finite element *analysis*, but one never needs to know or compute  $\mathbf{v}_1$  for the implementation of the FE method.

### 2.3. Energy equality

In this section, we assume no-penetration no-slip boundary condition (4) imposed on the whole boundary, i.e.  $\partial\Omega(t) = \partial\Omega^{ns}(t)$ . By  $(\cdot, \cdot)$  we denote the  $L^2(\Omega_0)$  scalar product, and  $\|\cdot\|$  denotes the  $L^2(\Omega_0)$  norm. For vector fields  $\mathbf{v}, \mathbf{u} : \Omega_0 \rightarrow \mathbb{R}^d$  and tensor fields  $\mathbf{A}, \mathbf{B} : \Omega_0 \rightarrow \mathbb{R}^{d \times d}$ , we use the same notation to denote  $(\mathbf{u}, \mathbf{v}) = \int_{\Omega_0} \mathbf{u}^T \mathbf{v} d\mathbf{x}$  and  $(\mathbf{A}, \mathbf{B}) = \int_{\Omega_0} \text{tr}(\mathbf{A}\mathbf{B}^T) d\mathbf{x}$ , and obviously  $\|\mathbf{u}\| := (\mathbf{u}, \mathbf{u})^{\frac{1}{2}}$ ,  $\|\mathbf{A}\| := (\mathbf{A}, \mathbf{A})^{\frac{1}{2}}$ . We shall also make use of the identity for all  $u, v \in H^1(\Omega)$ ,  $\mathbf{w} \in H^1(\Omega)^d$ :

$$(\mathbf{w} \cdot \nabla u, v) + \frac{1}{2}((\text{div } \mathbf{w})u, v) = \frac{1}{2}((\mathbf{w} \cdot \nabla u, v) - (\mathbf{w} \cdot \nabla v, u)) + \frac{1}{2} \int_{\partial\Omega_0} (\mathbf{n} \cdot \mathbf{w})uv ds. \quad (7)$$

We multiply the first equality in (2) by  $J\mathbf{u}$ , integrate it over the reference domain, and employ (7) for integration by parts. We get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|J^{\frac{1}{2}} \mathbf{u}\|^2 - \frac{1}{2} (J_t \mathbf{u}, \mathbf{u}) + (J(\hat{\sigma} \circ \xi) \mathbf{F}^{-T}, \nabla \mathbf{u}) - \int_{\partial\Omega_0} (J(\hat{\sigma} \circ \xi) \mathbf{F}^{-T} \mathbf{n}) \cdot \xi_t ds \\ + \frac{1}{2} (\text{div}(J\mathbf{F}^{-1}(\mathbf{u} - \xi_t)) \mathbf{u}, \mathbf{u}) = (J\mathbf{f}, \mathbf{u}), \end{aligned}$$

here  $\mathbf{n}$  is the exterior unit normal vector on  $\partial\Omega_0$ . The mass balance yields the equality

$$J_t + \text{div}(J\mathbf{F}^{-1}(\mathbf{u} - \xi_t)) = 0 \quad \text{in } Q. \quad (8)$$

This identity leads to some cancellations and we get

$$\frac{1}{2} \frac{d}{dt} \|J^{\frac{1}{2}} \mathbf{u}\|^2 + (J(\hat{\sigma} \circ \xi) \mathbf{F}^{-T}, \nabla \mathbf{u}) - \int_{\partial\Omega_0} (J(\hat{\sigma} \circ \xi) \mathbf{F}^{-T} \mathbf{n}) \cdot \xi_t ds = (J\mathbf{f}, \mathbf{u}).$$

The Piola identity,  $\text{div}(J\mathbf{F}^{-1}) = 0$ , implies the following equality

$$\text{div}(J\mathbf{F}^{-1}\mathbf{u}) = J(\nabla \mathbf{u}) : \mathbf{F}^{-T} \quad \text{in } Q, \quad (9)$$

where  $\mathbf{A} : \mathbf{B} := \text{tr}(\mathbf{A}\mathbf{B}^T)$ . Using the notation  $\mathbf{D}_\xi(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} \mathbf{F}^{-1} + \mathbf{F}^{-T}(\nabla \mathbf{u})^T)$  for the rate of deformation tensor in the reference coordinates, we get with the help of (9) and the second equation in (2)

$$\begin{aligned} (J(\hat{\sigma} \circ \xi) \mathbf{F}^{-T}, \nabla \mathbf{u}) &= (J(-p\mathbf{I} + \nu(\nabla \mathbf{u} \mathbf{F}^{-1} + \mathbf{F}^{-T}(\nabla \mathbf{u})^T)) \mathbf{F}^{-T}, \nabla \mathbf{u}) = 2\nu(J\mathbf{D}_\xi(\mathbf{u}) \mathbf{F}^{-T}, \nabla \mathbf{u}) \\ &= 2\nu(J\mathbf{D}_\xi(\mathbf{u}), \nabla \mathbf{u} \mathbf{F}^{-1}) = 2\nu(J\mathbf{D}_\xi(\mathbf{u}), \mathbf{D}_\xi(\mathbf{u})). \end{aligned}$$

In the last equality we used that for any symmetric tensor  $\mathbf{A}$  and any tensor  $\mathbf{B}$ , it holds  $\mathbf{A} : \mathbf{B} = \frac{1}{2} \mathbf{A} : (\mathbf{B} + \mathbf{B}^T)$ . Therefore, the energy balance equality in reference coordinates takes the form

$$\frac{1}{2} \frac{d}{dt} \|J^{\frac{1}{2}} \mathbf{u}\|^2 + 2\nu \|J^{\frac{1}{2}} \mathbf{D}_\xi(\mathbf{u})\|^2 - \int_{\partial\Omega_0} (J(\hat{\sigma} \circ \xi) \mathbf{F}^{-T} \mathbf{n}) \cdot \xi_t ds = (J\mathbf{f}, \mathbf{u}). \quad (10)$$

The mechanical interpretation of (10) is the following one: the work of external forces (right-hand side) is balanced by the change of kinetic energy (the first term), viscous dissipation of energy (the second term), and flow intensification due to the boundary condition (the third term).

#### 2.4. Weak formulation

For  $t \in [0, T]$  we introduce the following time-dependent trilinear and bilinear forms:

$$\begin{aligned} c(\xi; \mathbf{w}, \mathbf{u}, \psi) &= \int_{\Omega_0} J((\nabla \mathbf{u}) \mathbf{F}^{-1} \mathbf{w}) \cdot \psi \, d\mathbf{x}, \quad \mathbf{w}, \mathbf{u}, \psi \in H^1(\Omega_0)^d, \\ a(\xi; \mathbf{u}, \psi) &= \int_{\Omega_0} 2\nu J \mathbf{D}_\xi \mathbf{u} : \mathbf{D}_\xi \psi \, d\mathbf{x}, \quad \mathbf{u}, \psi \in H^1(\Omega_0)^d, \\ b(\xi; p, \psi) &= \int_{\Omega_0} p J \mathbf{F}^{-T} : \nabla \psi \, d\mathbf{x}, \quad p \in L^2(\Omega_0), \psi \in H^1(\Omega_0)^d, \end{aligned}$$

with  $J = J(\xi)$ ,  $\mathbf{F} = \mathbf{F}(\xi)$ .

The weak formulation of (2)–(3) reads: Find  $\{\mathbf{u}, p\} \in L^2(0, T; H^1(\Omega_0)^d) \cap L^\infty(0, T; L^2(\Omega_0)^d) \times L^2(Q)$  satisfying  $\mathbf{u} = \xi_t$  on  $\partial\Omega_0^{ns}$ ,  $\mathbf{u} = \mathbf{u}_D$  on  $\partial\Omega_0^D$  and

$$(J \mathbf{u}_t, \psi) + c(\xi; \mathbf{u} - \xi_t, \mathbf{v}, \psi) + a(\xi; \mathbf{u}, \psi) - b(\xi; p, \psi) + b(\xi; q, \mathbf{u}) = (J \mathbf{f}, \psi) + \int_{\partial\Omega_0^N} J \mathbf{g} \cdot \psi \, ds \quad (11)$$

for all  $\psi \in H^1(\Omega_0)^d$ ,  $\psi = 0$  on  $\partial\Omega_0^{ns} \cup \partial\Omega_0^D$ ,  $q \in L^2(\Omega_0)$  for all  $t \in [0, T]$ .

#### 3. Discretization method

In this section we introduce both time and space discretizations of the formulation (2) in the reference domain. Treating the flow problem in reference coordinates allows us to avoid triangulations and finite element function spaces dependent on time. In this paper, we assume that the mapping  $\xi$  is given explicitly and used in the finite element formulation without any further numerical approximation apart from the boundary condition.

Let a collection of simplices  $\mathcal{T}_h$  (triangles for  $d = 2$  and tetrahedra for  $d = 3$ ) form a consistent regular triangulation  $\mathcal{T}_h$  of the reference domain  $\bar{\Omega}_0$ . We let  $h = \max_{T \in \mathcal{T}_h} \text{diam}(T)$ . Consider conforming FE spaces  $\mathbb{V}_h \subset H^1(\Omega_0)^d$  and  $\mathbb{Q}_h \subset L^2(\Omega_0)$ ;  $\mathbb{V}_h^0$  is a subspace of  $\mathbb{V}_h$  of functions vanishing on  $\partial\Omega_0^{ns} \cup \partial\Omega_0^D$ . We assume that  $\mathbb{V}_h^0$  and  $\mathbb{Q}_h$  form the LBB-stable finite element pair: There exists a mesh-independent constant  $c_0$ , such that

$$\inf_{q_h \in \mathbb{Q}_h} \sup_{\mathbf{v}_h \in \mathbb{V}_h^0} \frac{(q_h, \text{div } \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\| \|q_h\|} \geq c_0 > 0. \quad (12)$$

As an example of admissible discretization, we consider the generalized Taylor–Hood finite element spaces,

$$\begin{aligned} \mathbb{V}_h &= \{\mathbf{u}_h \in C(\Omega_0)^d : \mathbf{u}_h|_T \in [P^{m+1}(T)]^d, \forall T \in \mathcal{T}_h\}, \\ \mathbb{Q}_h &= \{q_h \in C(\Omega_0) : q_h|_T \in P^m(T), \forall T \in \mathcal{T}_h\}, \end{aligned} \quad (13)$$

where integer  $m \geq 1$  is polynomial degree.

Assuming a constant time step  $\Delta t = \frac{T}{N}$ , we use the notation  $\mathbf{u}^k(\mathbf{x}) := \mathbf{u}(k\Delta t, \mathbf{x})$ , and similar for  $p$  and  $\xi$ . To emphasize the dependence on  $k$ , denote  $\mathbf{F}_k := \nabla \xi^k$ ,  $J_k := \det(\mathbf{F}_k)$ ,  $\mathbf{D}_k(\mathbf{v}) := \mathbf{D}_{\xi^k}(\mathbf{v})$ .

For given spatial functions  $f^i$ ,  $i = 0, \dots, k$ ,  $[f]_t^k := \frac{f^k - f^{k-1}}{\Delta t}$  denotes the backward finite difference at  $t_k = k\Delta t$ . For a sufficiently smooth vector function  $\mathbf{v}$ , denote by  $I_h(\mathbf{v}) \in \mathbb{V}_h$  its *nodal Lagrange interpolant*.

Let  $\mathbf{u}_h^0 = I_h(\mathbf{u}(t_0))$ . The finite element discretization of (11) reads: For  $k = 1, 2, \dots$ , find  $\{\mathbf{u}_h^k, p_h^k\} \in \mathbb{V}_h \times \mathbb{Q}_h$  satisfying  $\mathbf{u}_h^k = I_h(\xi_t^k)$  on  $\partial\Omega_0^{ns}$ ,  $\mathbf{u}_h^k = I_h(\mathbf{u}_D^k)$  on  $\partial\Omega_0^D$  and the following equations

$$\begin{aligned} & (J_{k-1}[\mathbf{u}_h]_t^k, \psi_h) + \left( \frac{1}{2} [J]_t^k \mathbf{u}_h^k, \psi_h \right) + \frac{1}{2} (\text{div} (J_k \mathbf{F}_k^{-1} \mathbf{w}_h^k) \mathbf{u}_h^k, \psi_h) \\ & + c(\xi^k; \mathbf{w}_h^k, \mathbf{u}_h^k, \psi_h) + a(\xi^k; \mathbf{u}_h^k, \psi_h) - b(\xi^k; p_h^k, \psi_h) + b(\xi^k; q_h, \mathbf{u}_h^k) = (J_k \mathbf{f}^k, \psi_h) + \int_{\partial\Omega_0^N} J_k \mathbf{g}^k \cdot \psi \, ds \end{aligned} \quad (14)$$

for all  $\psi_h \in \mathbb{V}_h^0$ ,  $q_h \in \mathbb{Q}_h$  with advection velocity  $\mathbf{w}_h^k := (\mathbf{u}_h^{k-1} - \xi_t^k)$ .

The second and third terms in (14) are consistent due to the identity (8) and are added in the FE formulation to enforce the conservation property of the discretization. While our computations show that in practice these terms can be skipped, we need these terms for the stability bound in the next section. In the numerical analysis of incompressible Navier–Stokes equations in the Eulerian description, including these terms corresponds to the Temam’s skew-symmetric form of the convective terms [23].

Note that the inertia terms are linearized so that a *linear* algebraic system should be solved at each time step. In the next section we show that the finite element method is energy stable.

Note that if  $\partial\Omega_0^N = \emptyset$ , then the boundary condition should be compatible with divergence constraint. Therefore, in this case we let  $c_\perp^k := \int_{\partial\Omega_0} J_k \mathbf{F}_k^{-T} I_h(\mathbf{u}^k) \cdot \mathbf{n} ds$  and let  $\mathbf{u}_h^k = I_h(\xi_t^k) - c_\perp^k$  on  $\partial\Omega_0^{ns}$  and  $\mathbf{u}_h^k = I_h(\mathbf{u}_D(t_k)) - c_\perp^k$  on  $\partial\Omega_0^D$ .

**Remark 2.** Condition  $\mathbf{u}_h = 0$  on those parts of the boundary, where no-slip and no-penetration take place, of an evolving fluid domain is not physically reasonable. Moreover, it leads to strong simplifications of finite element analysis. Indeed, we shall see that constructing suitable FE extensions of boundary conditions to computational domain is not straightforward.

#### 4. Stability of FEM solution

From now on we assume  $\partial\Omega_0^{ns} = \partial\Omega_0$  for all  $t \in [0, T]$ . To show the stability, we need several preparatory steps which help us to handle non-homogeneous boundary conditions and time-dependent bilinear forms. First we note the Korn’s-type inequality in the reference domain,

$$\|\nabla \mathbf{u}\| \leq C_K \|J^{\frac{1}{2}} \mathbf{D}_\xi(\mathbf{u})\| \quad \forall \mathbf{u} \in H_0^1(\Omega_0)^d, \quad t \in [0, T], \quad (15)$$

with  $C_K$  uniformly bounded with respect to  $t \in [0, T]$ . The estimate (15) easily follows from the standard Korn inequality and assumptions in (1); see [21]. Thanks to (15) the bilinear form  $a(\xi(t); \cdot, \cdot)$  is coercive on  $H_0^1(\Omega_0)^d \times H_0^1(\Omega_0)^d$  uniformly in time. Also due to (1) the bilinear forms  $a(\xi(t); \cdot, \cdot)$  and  $b(\xi(t); \cdot, \cdot)$  are continuous uniformly in time on  $H^1(\Omega_0)^d \times H^1(\Omega_0)^d$  and  $L^2(\Omega_0) \times H^1(\Omega_0)^d$ , respectively.

Unlike the continuous case, the finite element solution  $\{\mathbf{u}_h^k, p_h^k\}$  does not satisfy a strong formulation and so the arguments from Section 2.3 do not apply directly. To show the proper energy balance for the finite element solution, we split it into a part vanishing on the boundary and another *a priori* defined (and so stable) part, which satisfy correct boundary conditions. Therefore, we consider decomposition  $\mathbf{u}_h^k = \mathbf{v}_h^k + \mathbf{v}_{h,1}^k$ , such that

$$\mathbf{v}_{h,1}^k = \mathbf{u}_h^k \text{ on } \partial\Omega_0 \times [0, T], \quad b(\xi^k; q_h, \mathbf{v}_{h,1}^k) = 0 \quad \forall q_h \in \mathbb{Q}_h, \quad k = 1, 2, \dots, N, \quad (16)$$

and

$$\|\mathbf{v}_{h,1}^k\|_{W^{1,\infty}} \leq C, \quad (J_{k-1}(\mathbf{v}_{h,1}^k - \mathbf{v}_{h,1}^{k-1}), \mathbf{v}_h^k) \leq C \Delta t (\|\mathbf{v}_h^k\| + h^{m+2} \|[\mathbf{v}_h]_t^k\|), \quad (17)$$

with some real  $C$  depending only on data and independent of  $k, h$ . The existence of such decomposition will be explicitly demonstrated in the next section.

With the help of this decomposition, the finite element method can be re-formulated as follows: find  $\{\mathbf{v}_h^k, p_h^k\} \in \mathbb{V}_h^0 \times \mathbb{Q}_h$  satisfying for all  $\psi_h \in \mathbb{V}_h^0, q_h \in \mathbb{Q}_h$

$$\begin{aligned} & (J_{k-1}[\mathbf{v}_h]_t^k, \psi_h) + \left( \frac{1}{2} [J]_t^k \mathbf{v}_h^k, \psi_h \right) + \frac{1}{2} (\text{div} (J_k \mathbf{F}_k^{-1} \mathbf{w}_h^k) \mathbf{v}_h^k, \psi_h) \\ & + c(\xi^k; \mathbf{w}_h^k, \mathbf{v}_h^k, \psi_h) + c(\xi^k; \mathbf{v}_h^k, \mathbf{v}_{h,1}^k, \psi_h) + a(\xi^k; \mathbf{v}_h^k, \psi_h) \\ & - b(\xi^k; p_h^k, \psi_h) + b(\xi^k; q_h, \mathbf{v}_h^k) = \langle \tilde{\mathbf{f}}_h^k, \psi_h \rangle \end{aligned} \quad (18)$$

with  $\langle \tilde{\mathbf{f}}_h^k, \psi_h \rangle = (J_k(\mathbf{f} - (\nabla \mathbf{v}_{h,1}^k) \mathbf{F}^{-1}(\mathbf{v}_{h,1}^k - \xi_t^k)), \psi_h) - a(\xi^k; \mathbf{v}_{h,1}^k, \psi_h) - (J_{k-1}[\mathbf{v}_{h,1}]_t^k, \psi_h)$ . We stress that the formulation (18) appears here only for the purpose of analysis. Although equivalent to (14), the formulation (18) is not practical, since it requires an explicit knowledge of  $\mathbf{v}_{1,h}$ . For the analysis, it is sufficient to know that  $\mathbf{v}_{1,h}$  exists.

We test (18) with  $\psi_h = \mathbf{v}_h^k, q_h = p_h^k$ . We handle each resulting term separately and start with the first term in (18):

$$\begin{aligned} (J_{k-1}[\mathbf{v}_h]_t^k, \mathbf{v}_h^k) &= \frac{1}{2\Delta t} \left( \|J_k^{\frac{1}{2}} \mathbf{v}_h^k\|^2 - \|J_{k-1}^{\frac{1}{2}} \mathbf{v}_h^{k-1}\|^2 \right) \\ &- \frac{1}{2} ([J]_t^k \mathbf{v}_h^k, \mathbf{v}_h^k) + \frac{\Delta t}{2} \|J_{k-1}^{\frac{1}{2}} [\mathbf{v}_h]_t^k\|^2. \end{aligned} \quad (19)$$

The term  $-\frac{1}{2}([J]_t^k \mathbf{v}_h^k, \mathbf{v}_h^k)$  in (19) cancels with the second term in (18). Applying (7) to the fourth (inertia) term in (18) and using boundary conditions give

$$(J_k(\nabla \mathbf{v}_h^k \mathbf{F}_k^{-1} \mathbf{w}_h^k), \mathbf{v}_h^k) = -\frac{1}{2}(\operatorname{div} (J_k \mathbf{F}_k^{-1} \mathbf{w}_h^k) \mathbf{v}_h^k, \mathbf{v}_h^k). \quad (20)$$

This cancels with the third term in (18). We keep the fifth term as it is. The sixth term in (18) gives

$$a(\xi^k; \mathbf{v}_h^k, \mathbf{v}_h^k) = 2\nu (J_k \mathbf{D}_k(\mathbf{v}_h^k), \mathbf{D}_k(\mathbf{v}_h^k)) = 2\nu \left\| J_k^{\frac{1}{2}} \mathbf{D}_k(\mathbf{v}_h^k) \right\|^2.$$

The  $b$ -terms cancel out for  $q_h = p_h^k$ . Substituting all equalities back into (18), we obtain the following energy balance for the  $\mathbf{v}_h$ -part of finite element solution  $\mathbf{u}_h$ :

$$\begin{aligned} \frac{1}{2\Delta t} \left( \|J_k^{\frac{1}{2}} \mathbf{v}_h^k\|^2 - \|J_{k-1}^{\frac{1}{2}} \mathbf{v}_h^{k-1}\|^2 \right) + 2\nu \left\| J_k^{\frac{1}{2}} \mathbf{D}_k(\mathbf{v}_h^k) \right\|^2 + \frac{\Delta t}{2} \left\| J_{k-1}^{\frac{1}{2}} [\mathbf{v}_h]_t^k \right\|^2 \\ + (J_k(\nabla \mathbf{v}_{h,1}^k \mathbf{F}_k^{-1}) \mathbf{v}_h^k, \mathbf{v}_h^k) = \langle \tilde{\mathbf{f}}_h^k, \mathbf{v}_h^k \rangle. \end{aligned} \quad (21)$$

We deduce an energy stability estimate for the finite element method from the balance in (21) and *a priori* estimates in (17). For the sake of notation, we introduce  $\|\cdot\|_k := \left( \int_{\Omega_0} J_k |\cdot|^2 d\mathbf{x} \right)^{\frac{1}{2}}$ , which defines a  $k$ -dependent norm uniformly equivalent to the  $L^2$ -norm. Using estimates (17) and the definition of  $\tilde{\mathbf{f}}_h^k$  one shows that the forcing term is bounded,

$$\begin{aligned} \langle \tilde{\mathbf{f}}_h^k, \mathbf{v}_h^k \rangle &\leq C(\|\mathbf{f}\| \|\mathbf{v}_h^k\| + \|\nabla \mathbf{v}_{h,1}^k\| (\|\mathbf{v}_{h,1}^k\|_{L^\infty} + \|\xi_t\|) \|\mathbf{v}_h^k\| + \|\nabla \mathbf{v}_{h,1}^k\| \|\mathbf{D}_k \mathbf{v}_h^k\|) \\ &\quad + C(\|\mathbf{v}_h^k\| + h^{m+2} \|[\mathbf{v}_h]_t^k\|) \\ &\leq C(\|\mathbf{D}_k(\mathbf{v}_h^k)\|_k + (\Delta t)^{-1} h^{2m+4}) + \frac{\Delta t}{2} \|[\mathbf{v}_h]_t^k\|_{k-1}^2 \end{aligned}$$

with a constant  $C$  depending only on problem data. We substitute this in (21) and further use the bound  $C\|\mathbf{D}_k(\mathbf{v}_h^k)\|_k \leq \nu^{-1} C^2 + \nu \|\mathbf{D}_k(\mathbf{v}_h^k)\|_k^2$ . This yields the estimate

$$\frac{1}{2\Delta t} \left( \|\mathbf{v}_h^k\|_k^2 - \|\mathbf{v}_h^{k-1}\|_{k-1}^2 \right) + \nu \|\mathbf{D}_k(\mathbf{v}_h^k)\|_k^2 + (J_k(\nabla \mathbf{v}_{h,1}^k \mathbf{F}_k^{-1}) \mathbf{v}_h^k, \mathbf{v}_h^k) \leq C(\nu^{-1} + (\Delta t)^{-1} h^{2m+4}), \quad (22)$$

where the constant  $C$  depends on the problem data, but not on  $\nu$ ,  $h$  and  $\Delta t$ .

Thanks to Sobolev's embedding inequalities as well as (1) and (15), we bound the fourth term in (22) resulting from the boundary motion in two ways,

$$|(J_k(\nabla \mathbf{v}_{h,1}^k \mathbf{F}_k^{-1}) \mathbf{v}_h^k, \mathbf{v}_h^k)| \leq \begin{cases} C \|\nabla \mathbf{v}_{h,1}^k\| \|\nabla \mathbf{v}_h^k\|^2 \leq C_1 \|\nabla \mathbf{v}_{h,1}^k\| \|\mathbf{D}_k(\mathbf{v}_h^k)\|_k^2, \\ C \|\nabla \mathbf{v}_{h,1}^k\|_{L^\infty(\Omega_0)} \|\mathbf{v}_h^k\|^2 \leq C_2 \|\mathbf{v}_h^k\|_k^2. \end{cases}$$

If the factor  $C_1 \|\nabla \mathbf{v}_{h,1}^k\|$  is not too large such that it holds

$$C_1 \|\nabla \mathbf{v}_{h,1}^k\| \leq \nu/2, \quad (23)$$

then the intensification term can be absorbed by the viscous dissipation term. Note that  $C_1$  depends on  $\Omega_0$  and  $\xi$ . So we obtain from (22)

$$\frac{1}{2} \|\mathbf{v}_h^k\|_k^2 + \nu \Delta t \|\mathbf{D}_k(\mathbf{v}_h^k)\|_k^2 \leq \frac{1}{2} \|\mathbf{v}_h^{k-1}\|_{k-1}^2 + C \Delta t (\nu^{-1} + (\Delta t)^{-1} h^{2m+4}), \quad (24)$$

with some  $C$  depending only on problem data.

Summing up inequalities (24) over  $k = 1, \dots, n$ ,  $n \leq N$ , gives

$$\frac{1}{2} \|\mathbf{v}_h^n\|_n^2 + \nu \sum_{k=1}^n \Delta t \|\mathbf{D}_k(\mathbf{v}_h^k)\|_k^2 \leq \frac{1}{2} \|\mathbf{v}_0\|_0^2 + C \Delta t (\nu^{-1} + (\Delta t)^{-1} h^{2m+4}). \quad (25)$$

Otherwise, if (23) does not hold, we estimate

$$\frac{1}{2} (1 - 2C_2 \Delta t) \|\mathbf{v}_h^k\|_k^2 + \nu \Delta t \|\mathbf{D}_k(\mathbf{v}_h^k)\|_k^2 \leq \frac{1}{2} \|\mathbf{v}_h^{k-1}\|_{k-1}^2 + C \Delta t (\nu^{-1} + (\Delta t)^{-1} h^{2m+4}). \quad (26)$$



Now we assume that  $\Delta t$  is sufficiently small such that  $(1 - 2C_2\Delta t) = \alpha > 0$ . Summing over  $k = 1, \dots, n$ , gives

$$\frac{1}{2} \|\mathbf{v}_h^n\|_n^2 + \nu \sum_{k=1}^n \Delta t \|\mathbf{D}_k(\mathbf{v}_h^k)\|_k^2 \leq C_2 \sum_{k=1}^n \Delta t \|\mathbf{v}_h^k\|_k^2 + \frac{1}{2} \|\mathbf{v}_0\|_0^2 + C(\nu^{-1} + (\Delta t)^{-1} h^{2m+4}). \quad (27)$$

Applying discrete Gronwall's inequality yields

$$\frac{1}{2} \|\mathbf{v}_h^n\|_n^2 + \nu \sum_{k=1}^n \Delta t \|\mathbf{D}_k(\mathbf{v}_h^k)\|_k^2 \leq e^{\frac{2C_2}{\alpha} T} \left( \frac{1}{2} \|\mathbf{v}_0\|_0^2 + C(\nu^{-1} + (\Delta t)^{-1} h^{2m+4}) \right). \quad (28)$$

Constant  $C_2$  depends on  $\xi$ , which defines the evolution of  $\Omega(t)$ , but not on  $\nu$ .

Finally, we apply the triangle inequality and (17) one more time to show the stability bound for the velocity solution  $\mathbf{u}_h$  to (14),

$$\frac{1}{2} \|\mathbf{u}_h^n\|_n^2 + \nu \sum_{k=1}^n \Delta t \|\mathbf{D}_k(\mathbf{u}_h^k)\|_k^2 \leq C(\nu^{-1} + (\Delta t)^{-1} h^{2m+4}) \leq C_{\text{stab}}, \quad \text{if } c\Delta t \geq h^{2m+4}, \quad (29)$$

where constant  $C_{\text{stab}}$  depends on the problem data, i.e.  $\mathbf{f}$ ,  $T$ ,  $\Omega_0$ ,  $\xi$ ,  $\nu$ , but not on  $h$  and  $\Delta t$ . If condition (23) does not hold, then  $\Delta t$  is assumed sufficiently small to satisfy  $1 - 2C_2\Delta t > 0$ .

Note that the restriction  $\Delta t \geq h^{2m+4}$  has resulted from handling non-homogeneous boundary conditions. Indeed, one gets the extra  $h$ -dependent term in (17) while estimating the difference  $\mathbf{v}_{h,1}^k - \mathbf{v}_{h,1}^{k-1}$ . If one defines  $\mathbf{v}_{h,1}^k$  as a nodal interpolant to the smooth divergence-free function  $\mathbf{v}_1$  then the estimate as in (17) follows without this extra term. However, the nodal interpolant does not satisfy the required “divergence-free” condition in (16) and so  $\mathbf{v}_{h,1}^k$  will be defined as a suitable projection of  $\mathbf{v}_1$ .

## 5. Convergence analysis

In this section, we demonstrate optimal order convergence of  $\mathbf{u}_h$  to  $\mathbf{u}$ . From now on, we assume that  $\Omega_0$  is a convex polyhedron (polygon if  $d = 2$ ).

### 5.1. Preliminaries

For the Navier–Stokes equations solution in the reference domain we recall the notations  $\mathbf{u}^k := \mathbf{u}(k\Delta t)$ ,  $p^k := p(k\Delta t)$ . The finite element solution to (14) is  $\{\mathbf{u}_h^k, p_h^k\}$ . Furthermore,

$$\{\mathbf{e}^k, e^k\} := \{\mathbf{u}^k - \mathbf{u}_h^k, p^k - p_h^k\}$$

denotes the finite element error.

For the *fixed* time instance  $t_k = k\Delta t$ , we define the subspace of  $\mathbb{V}_h$ ,

$$\mathbb{X}_h^k := \{\boldsymbol{\psi}_h \in \mathbb{V}_h : b(\boldsymbol{\xi}^k; q_h, \boldsymbol{\psi}_h) = 0 \ \forall q_h \in \mathbb{Q}_h\}.$$

In Lemma 5.1 we prove an important technical result: if the pair of spaces  $\mathbb{V}_h - \mathbb{Q}_h$  is inf–sup stable in the sense of (12), and  $\xi$  satisfies certain assumptions, then the subspace  $\mathbb{X}_h^k$  has full approximation properties. Here and further we assume that for all  $t \in [0, T]$  the domains  $\Omega(t)$  are such that the Stokes problem in the physical domain  $\Omega(t)$  satisfies  $H^2$ -regularity property: for any  $\mathbf{f} \in L^2(\Omega(t))$  the unique solution  $\{\hat{\boldsymbol{\phi}}, \hat{\lambda}\}$  of

$$-\Delta \hat{\boldsymbol{\phi}} + \nabla \hat{\lambda} = \mathbf{f}, \quad \text{div } \hat{\boldsymbol{\phi}} = 0 \text{ in } \Omega(t), \quad \hat{\boldsymbol{\phi}} = 0 \text{ on } \partial\Omega(t) \quad (30)$$

satisfies  $\hat{\boldsymbol{\phi}} \in H^2(\Omega(t))^d$ ,  $\hat{\lambda} \in H^1(\Omega(t))$  and

$$\|\hat{\boldsymbol{\phi}}\|_{H^2(\Omega(t))} + \|\hat{\lambda}\|_{H^1(\Omega(t))} \leq C_R \|\mathbf{f}\|_{L^2(\Omega(t))}, \quad (31)$$

with some  $C_R$  uniformly bounded for  $t \in [0, T]$ .

We recall that  $I_h(\mathbf{v}) \in \mathbb{V}_h$  denotes the nodal interpolant of  $\mathbf{v}$  and  $c_\perp := \int_{\partial\Omega_0} \mathbf{J}\mathbf{F}^{-T} I_h(\mathbf{v}) \cdot \mathbf{n} \, ds$ .

**Lemma 5.1.** Assume (12) and

$$\sup_Q \|\mathbf{I} - \mathbf{F}\|_F \leq \varepsilon, \quad (32)$$



with sufficiently small  $\varepsilon > 0$  and the identity matrix  $\mathbf{I}$ . Then for  $\mathbf{v} \in H^{m+\frac{5}{2}}(\Omega_0)^d$  satisfying  $\operatorname{div}(J_k \mathbf{F}_k^{-1} \mathbf{v}) = 0$ ,  $1 \leq k \leq N$ , we have

$$\inf_{\substack{\mathbf{v}_h \in \mathbb{X}_h^k \\ \mathbf{v}_h = I_h(\mathbf{v}) - c_\perp \text{ on } \partial\Omega_0}} (\|\mathbf{v} - \mathbf{v}_h\| + h \|\nabla(\mathbf{v} - \mathbf{v}_h)\|) \leq c h^{m+2} \|\mathbf{v}\|_{H^{m+\frac{5}{2}}(\Omega_0)}, \quad (33)$$

where  $m$  is the polynomial degree from (13) and  $c$  depends on  $\xi$ , but not on  $k$ ,  $h$  or  $\mathbf{v}$ .

**Proof.** From (32) it follows that

$$\sup_Q \|\mathbf{I} - \mathbf{F}^{-1}\|_F \leq \frac{\varepsilon}{1 - \varepsilon}, \quad \sup_Q |1 - J| \leq c \varepsilon \quad (34)$$

holds with some  $c > 0$  depending only on space dimension  $d = 2, 3$ . The inf-sup condition (12) implies that for a given  $q_h \in \mathbb{Q}_h$  there exists  $\boldsymbol{\psi}_h \in \mathbb{V}_h^0$  such that  $(q_h, \operatorname{div} \boldsymbol{\psi}_h) = \|q_h\|^2$  and  $c_0 \|\nabla \boldsymbol{\psi}_h\| \leq \|q_h\|$ . Thanks to (34) we estimate

$$\begin{aligned} b(\xi^k; q_h, \boldsymbol{\psi}_h) &= (J_k \mathbf{F}_k^{-T} : \nabla \boldsymbol{\psi}_h, q_h) = (\operatorname{div} \boldsymbol{\psi}_h, q_h) + ([J_k \mathbf{F}_k^{-T} - \mathbf{I}] : \nabla \boldsymbol{\psi}_h, q_h) \\ &\geq \|q_h\|^2 - \sup_Q \|\mathbf{I} - \mathbf{F}^{-1}\|_F \|\nabla \boldsymbol{\psi}_h\| \|q_h\| \\ &\geq \|q_h\|^2 - (\sup_Q \|\mathbf{I} - \mathbf{F}^{-1}\|_F + c \varepsilon \sup_Q \|\mathbf{F}^{-1}\|_F) \|\nabla \boldsymbol{\psi}_h\| \|q_h\| \\ &\geq c_0 \|\nabla \boldsymbol{\psi}_h\| \|q_h\| - \tilde{c} \varepsilon \|\nabla \boldsymbol{\psi}_h\| \|q_h\| = (c_0 - \tilde{c} \varepsilon) \|\nabla \boldsymbol{\psi}_h\| \|q_h\|. \end{aligned}$$

Thus we proved that for  $\varepsilon > 0$  small enough, the bilinear form  $b(\xi^k; \cdot, \cdot)$  satisfies the uniform inf-sup condition

$$\inf_{q_h \in \mathbb{Q}_h} \sup_{\boldsymbol{\psi}_h \in \mathbb{V}_h^0} \frac{b(\xi^k; q_h, \boldsymbol{\psi}_h)}{\|\nabla \boldsymbol{\psi}_h\| \|q_h\|} \geq \hat{c}_0 > 0, \quad (35)$$

with  $\hat{c}_0$  independent of  $k$  and  $h$ .

Now consider arbitrary  $\mathbf{v} \in H^{m+\frac{5}{2}}(\overline{\Omega_0})^d$  satisfying  $\operatorname{div}(J \mathbf{F}^{-1} \mathbf{v}) = 0$ , and hence  $b(\xi^k; q, \mathbf{v}) = 0$  for all  $q \in L^2(\Omega_0)$ . The nodal Lagrange interpolant of  $\mathbf{v}$  is well defined. From  $\int_{\partial\Omega_0} J \mathbf{F}^{-1} \mathbf{v} \cdot \mathbf{n} ds = 0$  and approximation properties of the polynomial interpolation, we get

$$\begin{aligned} |c_\perp| &= \left| \int_{\partial\Omega_0} J \mathbf{F}^{-1} (I_h(\mathbf{v}) - \mathbf{v}) \cdot \mathbf{n} ds \right| \leq c \|I_h(\mathbf{v}) - \mathbf{v}\|_{L^2(\partial\Omega_0)} = c \sum_{\Gamma_i \subset \partial\Omega_0} \|I_h(\mathbf{v}) - \mathbf{v}\|_{L^2(\Gamma_i)} \\ &\leq c h^{m+2} \sum_{\Gamma_i \subset \partial\Omega_0} \|\mathbf{v}\|_{H^{m+2}(\Gamma_i)} \leq c h^{m+2} \|\mathbf{v}\|_{H^{m+\frac{5}{2}}(\Omega_0)}, \end{aligned} \quad (36)$$

where we sum over all faces  $\Gamma_i$  of our polyhedral domain  $\Omega_0$ . By the same argument and also using the triangle inequality, we get

$$\|\mathbf{v} - (I_h(\mathbf{v}) - c_\perp)\|_{L^2(\partial\Omega_0)} \leq c h^{m+2} \|\mathbf{v}\|_{H^{m+\frac{5}{2}}(\Omega_0)}. \quad (37)$$

Now we define  $\mathbf{v}_h$  as the quasi-Stokes projection of  $\mathbf{v}$  satisfying suitable boundary condition: One finds  $\mathbf{v}_h \in \mathbb{V}_h$  such that  $\mathbf{v}_h = I_h(\mathbf{v}) - c_\perp$  on  $\partial\Omega_0$

$$a(\xi^k; \mathbf{v} - \mathbf{v}_h, \boldsymbol{\psi}_h) - b(\xi^k; p_h, \boldsymbol{\psi}_h) + b(\xi^k; q_h, \mathbf{v}_h) = 0 \quad \forall \boldsymbol{\psi}_h \in \mathbb{V}_h^0, q_h \in \mathbb{Q}_h.$$

For zero Dirichlet boundary conditions, showing the optimal order estimate of  $\|\nabla(\mathbf{v} - \mathbf{v}_h)\|$  for the Stokes projection is standard, see, e.g., Theorems 5.2.1–5.2.2 in [24] or Section 2.4 in [25]. To handle non-homogeneous boundary conditions, we use Proposition 8 from [26]. The proof of the proposition in [26] is given for the Stokes problem, but the arguments only need the coercivity of the bilinear form  $a(\xi^k; \cdot, \cdot)$  on  $H_0^1(\Omega_0)^d \times H_0^1(\Omega_0)^d$ , the continuity of  $a(\xi^k; \cdot, \cdot)$  and  $b(\xi^k; \cdot, \cdot)$  and the inf-sup property (35). This proposition establishes the estimate

$$\|\nabla(\mathbf{v} - \mathbf{v}_h)\| \leq C \inf_{\substack{\boldsymbol{\psi}_h \in \mathbb{X}_h^k \\ \boldsymbol{\psi}_h = I_h(\mathbf{v}) - c_\perp \text{ on } \partial\Omega_0}} \|\mathbf{v} - \boldsymbol{\psi}_h\|_{H^1(\Omega_0)}.$$

Letting  $\boldsymbol{\psi}_h = I_h(\mathbf{v}) - c_\perp$  on the right-hand side of this inequality, using (36) and interpolation properties of polynomial functions, we find the bound

$$\|\nabla(\mathbf{v} - \mathbf{v}_h)\| \leq ch^{m+1} \|\mathbf{v}\|_{H^{m+\frac{5}{2}}(\Omega_0)}.$$

In order to show the higher order bound for  $\mathbf{v} - \mathbf{v}_h$  in the  $L^2$  norm, we consider the weak formulation of the quasi-Stokes problem with  $J_k(\mathbf{v} - \mathbf{v}_h)$  on the right-hand side: Find  $\boldsymbol{\phi}, \lambda$  such that

$$a(\boldsymbol{\xi}^k; \boldsymbol{\phi}, \boldsymbol{\psi}) - b(\boldsymbol{\xi}^k; \lambda, \boldsymbol{\psi}) + b(\boldsymbol{\xi}^k; q, \boldsymbol{\phi}) = (J_k(\mathbf{v} - \mathbf{v}_h), \boldsymbol{\psi}) \quad \forall \boldsymbol{\psi} \in H_0^1(\Omega_0)^d, q \in L^2(\Omega_0).$$

Denote  $\hat{\boldsymbol{\phi}} = \boldsymbol{\phi} \circ (\boldsymbol{\xi}^k)^{-1}$ ,  $\hat{\lambda} = \lambda \circ (\boldsymbol{\xi}^k)^{-1}$ , then  $\{\hat{\boldsymbol{\phi}}, \hat{\lambda}\}$  satisfies (30)–(31) with  $t = k\Delta t$ ,  $\mathbf{f} = (\mathbf{v} - \mathbf{v}_h) \circ (\boldsymbol{\xi}^k)^{-1}$ . From this and (1), we derive

$$\|\boldsymbol{\phi}\|_{H^2(\Omega_0)} + \|\lambda\|_{H^1(\Omega_0)} \leq C \|\mathbf{v} - \mathbf{v}_h\|.$$

Now the bound  $\|\mathbf{v} - \mathbf{v}_h\| \leq Ch^{m+2} \|\mathbf{v}\|_{H^{m+\frac{5}{2}}(\Omega_0)}$  follows by the generalized Nitsche argument of Proposition 9 from [26] and (37).  $\square$

Note that the extra  $\frac{1}{2}$  in the Sobolev space order arises in the lemma due to the trace theorem needed to treat non-homogeneous boundary conditions.

**Remark 3.** In the statement of the lemma and in further arguments one may replace  $H^{m+\frac{5}{2}}(\Omega_0)$  by  $C^{m+2}(\overline{\Omega_0})$ . The only change required in the proof is a slightly different argument to show the estimate in (36) and (37). From  $\int_{\partial\Omega_0} J\mathbf{F}^{-1}\mathbf{v} \cdot \mathbf{n} ds = 0$  and approximation properties of polynomial interpolation, we get

$$\begin{aligned} |c_\perp| &= \left| \int_{\partial\Omega_0} J\mathbf{F}^{-1}(I_h(\mathbf{v}) - \mathbf{v}) \cdot \mathbf{n} ds \right| \leq c \|I_h(\mathbf{v}) - \mathbf{v}\|_{L^2(\partial\Omega_0)} \\ &\leq ch^{m+2} \|\mathbf{v}\|_{C^{m+2}(\partial\Omega_0)} \leq ch^{m+2} \|\mathbf{v}\|_{C^{m+2}(\overline{\Omega_0})}. \end{aligned} \quad (38)$$

By  $P_h^k : C(\overline{\Omega_0})^d \rightarrow \mathbb{X}_h^k$  we denote the projection with respect to the scalar product  $(J_k \cdot, \cdot)$ , defined by

$$\|\mathbf{v} - P_h^k(\mathbf{v})\|_{k-1} = \inf_{\substack{\boldsymbol{\psi}_h \in \mathbb{X}_h^k \\ \boldsymbol{\psi}_h = I_h(\mathbf{v}) - c_\perp \text{ on } \partial\Omega_0}} \|\mathbf{v} - \boldsymbol{\psi}_h\|_{k-1}.$$

With the help of the standard variational argument one checks the orthogonality property

$$(J_{k-1}(\mathbf{v} - P_h^k(\mathbf{v})), \boldsymbol{\psi}) = 0 \quad \forall \boldsymbol{\psi} \in \mathbb{X}_h^k \cap H_0^1(\Omega)^d. \quad (39)$$

Due to the equivalence of  $L^2$  and  $\|\cdot\|_{k-1}$  norms, we have

$$\|\mathbf{v} - P_h^k(\mathbf{v})\| \leq c \inf_{\substack{\boldsymbol{\psi}_h \in \mathbb{X}_h^k \\ \boldsymbol{\psi}_h = I_h(\mathbf{v}) - c_\perp \text{ on } \partial\Omega_0}} \|\mathbf{v} - \boldsymbol{\psi}_h\|. \quad (40)$$

On the other hand, due to the finite element inverse inequality it holds

$$\begin{aligned} \|\nabla(\mathbf{v} - P_h^k(\mathbf{v}))\| &\leq \|\nabla(\mathbf{v} - \boldsymbol{\psi}_h)\| + \|\nabla(\boldsymbol{\psi}_h - P_h^k(\mathbf{v}))\| \\ &\leq \|\nabla(\mathbf{v} - \boldsymbol{\psi}_h)\| + h^{-1} \|\boldsymbol{\psi}_h - P_h^k(\mathbf{v})\| \\ &\leq \|\nabla(\mathbf{v} - \boldsymbol{\psi}_h)\| + h^{-1} (\|\boldsymbol{\psi}_h - \mathbf{v}\| + \|\mathbf{v} - P_h^k(\mathbf{v})\|) \end{aligned} \quad (41)$$

for arbitrary  $\boldsymbol{\psi}_h \in \mathbb{X}_h^k$ . Now one applies (33) first in (40) and next in (41) to show the following estimate for a smooth  $\mathbf{v}$  satisfying  $\operatorname{div}(J\mathbf{F}^{-1}\mathbf{v}) = 0$

$$\|\mathbf{v} - P_h^k(\mathbf{v})\| + h \|\nabla(\mathbf{v} - P_h^k(\mathbf{v}))\| \leq ch^{m+2} \|\mathbf{v}\|_{H^{m+\frac{5}{2}}(\Omega_0)}. \quad (42)$$

The orthogonality property (39) and approximation property (42) for the projection  $P_h^k$  are crucial in the proof of an error estimate in the next section.

Before proceeding with the error estimate, we address the question postponed in Section 4. We need to show that there exists a decomposition  $\mathbf{u}_h^k = \mathbf{v}_h^k + \mathbf{v}_{h,1}^k$  satisfying (16) and (17). To this end, we first note that due to the result in (38) one shows the bound similar to (42) as

$$\|\mathbf{v} - P_h^k(\mathbf{v})\| + h \|\nabla(\mathbf{v} - P_h^k(\mathbf{v}))\| \leq c h^{m+2} \|\mathbf{v}\|_{C^{m+2}(\overline{\Omega_0})}. \quad (43)$$

We set  $\mathbf{v}_{h,1}^k = P_h^k(\mathbf{v}_1(t_k))$ , where  $\mathbf{v}_1$  from (6) is the divergence-free function, which satisfies the same boundary condition as the original Navier–Stokes solution. Thus,  $\mathbf{v}_{h,1}^k$  satisfies (16) by construction. The proof of (17) relies on (39), (43), with  $m = 1$ , and the assumption that the norm  $\|\mathbf{v}_1\|_{C^3(Q)}$  is bounded, see Section 2.2. Using FE inverse inequality we now estimate

$$\begin{aligned} \|\nabla(P_h^k(\mathbf{v}_1(t_k)))\|_{L^\infty} &\leq \|\nabla(\mathbf{v}_1(t_k) - I_h(\mathbf{v}_1(t_k)))\|_{L^\infty} + \|I_h(\mathbf{v}_1(t_k)) - P_h^k(\mathbf{v}_1(t_k))\|_{L^\infty} + \|\nabla \mathbf{v}_1(t_k)\|_{L^\infty} \\ &\leq \|I_h(\mathbf{v}_1(t_k)) - P_h^k(\mathbf{v}_1(t_k))\|_{L^\infty} + 2\|\nabla \mathbf{v}_1(t_k)\|_{L^\infty} + \|\nabla I_h(\mathbf{v}_1(t_k))\|_{L^\infty} \\ &\leq c h^{-\frac{d}{2}} \|I_h(\mathbf{v}_1(t_k)) - P_h^k(\mathbf{v}_1(t_k))\| + C \\ &\leq c h^{-\frac{d}{2}} (\|I_h(\mathbf{v}_1(t_k)) - \mathbf{v}_1(t_k)\| + \|\mathbf{v}_1(t_k) - P_h^k(\mathbf{v}_1(t_k))\|) + C \\ &\leq c h^{2-\frac{d}{2}} \|\mathbf{v}_1(t_k)\|_{C^3(\overline{\Omega_0})} + C \leq C. \end{aligned}$$

This proves the first bound in (17). To show the second bound, we note that (39) implies for  $\mathbf{v}_h^k \in \mathbb{X}_h^k \cap H_0^1(\Omega_0)^d$  and  $\mathbf{v}_h^{k-1} \in \mathbb{X}_h^{k-1} \cap H_0^1(\Omega_0)^d$  the following identity

$$\begin{aligned} (J_{k-1}(P_h^k(\mathbf{v}_1(t_k)) - P_h^{k-1}(\mathbf{v}_1(t^{k-1}))), \mathbf{v}_h^k) &= \Delta t \{ (J_{k-1}[\mathbf{v}_1]_t^k, \mathbf{v}_h^k) q + ([J]_t^{k-1}(\mathbf{v}_1(t^{k-1}) - P_h^{k-1}(\mathbf{v}_1(t^{k-1}))), \mathbf{v}_h^k) \\ &\quad + (J_{k-2}(\mathbf{v}_1(t^{k-1}) - P_h^{k-1}(\mathbf{v}_1(t^{k-1}))), [\mathbf{v}_h]_t^k) \}. \end{aligned}$$

Now the desired bound in (17) follows by applying the Cauchy inequality, (43),  $\|P_h^{k-1}(\mathbf{v}_1(t^{k-1}))\| \leq C$  and using the smoothness of  $\xi$  and  $\mathbf{v}_1$  to bound

$$\|[J]_t^{k-1}\| \leq c \|J\|_{C^2(Q)} \leq C \quad \text{and} \quad \|[\mathbf{v}_1]_t^k\| \leq c \|\mathbf{v}_1\|_{C^2(Q)} \leq C.$$

## 5.2. Error estimate

At time  $t = k\Delta t$  the true Navier–Stokes solution  $\{\mathbf{u}, p\}$  satisfies

$$\begin{aligned} (J_{k-1}[\mathbf{u}]_t^k, \psi_h) &+ c(\xi^k; \tilde{\mathbf{w}}^k, \mathbf{u}^k, \psi_h) + a(\xi^k; \mathbf{u}^k, \psi_h) + \frac{1}{2}(\text{div}(J_k \mathbf{F}_k^{-1} \tilde{\mathbf{w}}^k) \mathbf{u}^k, \psi_h) \\ &+ \left( \frac{1}{2} [J]_t^k \mathbf{u}^k, \psi_h \right) - b(\xi^k; p^k, \psi_h) + b(\xi^k; q_h, \mathbf{u}^k) - (J_k \mathbf{f}^k, \psi_h) = \text{Approx}(\psi_h) \end{aligned} \quad (44)$$

for all  $\psi_h \in \mathbb{V}_h^0$ ,  $q_h \in \mathbb{Q}_h$ , with ALE advection velocity  $\tilde{\mathbf{w}}^k := \mathbf{u}^{k-1} - [\xi]_t^k$ , and approximation error term

$$\begin{aligned} \text{Approx}(\psi_h) &= (J_{k-1}[\mathbf{u}]_t^k - J_k(\mathbf{u}_t)^k, \psi_h) + c(\xi^k; \mathbf{w}^k, \mathbf{u}^k, \psi_h) - c(\xi^k; \tilde{\mathbf{w}}^k, \mathbf{u}^k, \psi_h) \\ &\quad + ([J]_t^k \mathbf{u}^k - (J_t)^k \mathbf{u}_k, \psi_h) + \left( \text{div}(J_k \mathbf{F}_k^{-1} \tilde{\mathbf{w}}^k) \mathbf{u}^k - \text{div}(J_k \mathbf{F}_k^{-1} \mathbf{w}^k) \mathbf{u}^k, \psi_h \right). \end{aligned} \quad (45)$$

We estimate this error in the following lemma.

**Lemma 5.2.** Assume the solution  $\mathbf{u}$  of (11) is sufficiently smooth such that  $\mathbf{u}_{tt} \in L^\infty(Q)^d$ , and  $\nabla \mathbf{u} \in L^\infty(Q)^{d \times d}$ . We have

$$|\text{Approx}(\psi_h)| \leq C \Delta t \|\mathbf{D}_k(\psi_h)\|_k, \quad (46)$$

with a finite constant  $C$  depending on  $\xi$  and  $\mathbf{u}$ .

**Proof.** One compares (44) against (11) satisfied at time instance  $k\Delta t$ . The approximation error contains terms with time-derivatives. We handle these terms in a standard way with the help of the Taylor expansion in time. We get

$$\begin{aligned} (J_{k-1}[\mathbf{u}]_t^k - J_k(\mathbf{u}_t)^k, \boldsymbol{\psi}_h) &= (J_k[\mathbf{u}]_t^k + (J_{k-1} - J_k)[\mathbf{u}]_t^k - J_k(\mathbf{u}_t)^k, \boldsymbol{\psi}_h) \\ &= \left( -\frac{1}{\Delta t} J_k \int_{t_{k-1}}^{t_k} \mathbf{u}_{tt}(t - t_{k-1}) dt - \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} J_t dt \cdot \int_{t_{k-1}}^{t_k} \mathbf{u}_t dt, \boldsymbol{\psi}_h \right) \\ &= - \left( J_k \int_{t_{k-1}}^{t_k} \mathbf{u}_{tt} dt + \mathbf{u}_t(t_*^k) \int_{t_{k-1}}^{t_k} J_t dt, \boldsymbol{\psi}_h \right) \\ &\leq C \Delta t (\|\mathbf{u}_{tt}\|_{L^\infty(Q)} + \|\mathbf{u}_t\|_{L^\infty(Q)} \|J_t\|_{L^\infty(Q)}) \|\boldsymbol{\psi}_h\|_{L^1(\Omega_0)} \end{aligned}$$

where  $\{t_1^k, t_*^k\} \in [t_{k-1}, t_k]$ . We note that for  $\boldsymbol{\xi} \in C^2(Q)^d$  it holds  $\|J_t\|_{L^\infty(Q)} < \infty$ . We estimate the  $c$ -term using the same arguments,

$$\begin{aligned} c(\boldsymbol{\xi}^k; \mathbf{w}^k, \mathbf{u}^k, \boldsymbol{\psi}_h) - c(\boldsymbol{\xi}^k; \tilde{\mathbf{w}}^k, \mathbf{u}^k, \boldsymbol{\psi}_h) &= (J_k \nabla \mathbf{u}^k \mathbf{F}_k^{-1} (\mathbf{u}^k - \mathbf{u}^{k-1}), \boldsymbol{\psi}_h) \\ &= \left( J_k \nabla \mathbf{u}^k \mathbf{F}_k^{-1} \int_{t_{k-1}}^{t_k} \mathbf{u}_t dt, \boldsymbol{\psi}_h \right) \leq C \Delta t \|\mathbf{u}_t\|_{L^\infty(Q)} \|\nabla \mathbf{u}^k\| \|\boldsymbol{\psi}_h\|. \end{aligned}$$

Furthermore,

$$([J]_t^k \mathbf{u}^k - (J_t)^k \mathbf{u}^k, \boldsymbol{\psi}_h) = \left( \mathbf{u}^k \int_{t_{k-1}}^{t_k} J_{tt} dt, \boldsymbol{\psi}_h \right) \leq C \Delta t \|J_{tt}\|_{L^\infty(Q)} \|\mathbf{u}^k\| \|\boldsymbol{\psi}_h\|.$$

We note that for  $\boldsymbol{\xi} \in C^2(Q)^d$  it holds  $\|J_{tt}\|_{L^\infty(Q)} < \infty$ . Finally,

$$\begin{aligned} \left( \operatorname{div} (J_k \mathbf{F}_k^{-1} \tilde{\mathbf{w}}^k) \mathbf{u}^k - \operatorname{div} (J_k \mathbf{F}_k^{-1} \mathbf{w}^k) \mathbf{u}^k, \boldsymbol{\psi}_h \right) &= \left( \operatorname{div} \left( J_k \mathbf{F}_k^{-1} \int_{t_{k-1}}^{t_k} \mathbf{u}_t dt \right) \mathbf{u}^k, \boldsymbol{\psi}_h \right) \\ &= - \left( J_k \mathbf{F}_k^{-1} \int_{t_{k-1}}^{t_k} \mathbf{u}_t dt, \nabla (\mathbf{u}^k \cdot \boldsymbol{\psi}_h) \right) \\ &\leq C \Delta t \|\mathbf{u}_t\|_{L^\infty(Q)} \|\mathbf{u}^k\|_{H^1(\Omega_0)} \|\boldsymbol{\psi}_h\|_{H^1(\Omega_0)}. \end{aligned}$$

Collecting all the terms and applying Korn's inequality (15) to the  $\boldsymbol{\psi}_h$ -terms we prove the lemma.  $\square$

The finite element formulation and (44) imply the equations for the error:

$$\begin{aligned} (J_{k-1}[\mathbf{e}]_t^k, \boldsymbol{\psi}_h) + c(\boldsymbol{\xi}^k; \mathbf{w}_h^k, \mathbf{e}^k, \boldsymbol{\psi}_h) + a(\boldsymbol{\xi}^k; \mathbf{e}^k, \boldsymbol{\psi}_h) \\ + \frac{1}{2} (\operatorname{div} (J_k \mathbf{F}_k^{-1} \mathbf{w}_h^k) \mathbf{e}^k, \boldsymbol{\psi}_h) + \left( \frac{1}{2} [J]_t^k \mathbf{e}^k, \boldsymbol{\psi}_h \right) = \operatorname{Approx}(\boldsymbol{\psi}_h) + \operatorname{Consist}(\boldsymbol{\psi}_h) \end{aligned} \quad (47)$$

for all  $\boldsymbol{\psi}_h \in \mathbb{X}_h \cap H_0^1(\Omega)^d$ , where the consistency term on the right-hand side reads

$$\operatorname{Consist}(\boldsymbol{\psi}_h) = b(\boldsymbol{\xi}^k; p^k, \boldsymbol{\psi}_h) + c(\boldsymbol{\xi}^k; \mathbf{w}_h^k, \mathbf{u}^k, \boldsymbol{\psi}_h) - c(\boldsymbol{\xi}^k; \mathbf{u}^k - \boldsymbol{\xi}_t^k, \mathbf{u}^k, \boldsymbol{\psi}_h). \quad (48)$$

Decompose the error

$$\mathbf{e}^k = \mathbf{e}_t^k + \mathbf{e}_h^k = \{\mathbf{u}^k - P_h^k(\mathbf{u}^k)\} + \{P_h^k(\mathbf{u}^k) - \mathbf{u}_h^k\}.$$

We set  $J_{-1} = 1$  to define  $P_h^0$ . Thanks to the definition of  $P_h^k$  and boundary conditions for  $\mathbf{u}_h^k$ , the vector field  $\mathbf{e}_h^k$  vanishes on  $\partial\Omega_0$ . Thus, we can set  $\boldsymbol{\psi}_h = \mathbf{e}_h^k$  in (47) and apply the same arguments that were used to show (21). We obtain the estimate

$$\begin{aligned} \frac{1}{2} \|\mathbf{e}_h^k\|_k^2 + \frac{(\Delta t)^2}{2} \|\mathbf{e}_h\|_t^k^2 + 2\nu \Delta t \|\mathbf{D}_k(\mathbf{e}_h^k)\|_k^2 &\leq \frac{1}{2} \|\mathbf{e}_h^{k-1}\|_{k-1}^2 \\ &+ \Delta t \{ \operatorname{Approx}(\mathbf{e}_h^k) + \operatorname{Consist}(\mathbf{e}_h^k) + \operatorname{Interp}(\mathbf{e}_h^k) \}. \end{aligned} \quad (49)$$

The last term accumulates the integrals with the finite element interpolation error

$$\begin{aligned} \text{Interp}(\mathbf{e}_h^k) &= c(\xi^k; \mathbf{w}_h^k, \mathbf{e}_I^k, \mathbf{e}_h^k) + a(\xi^k; \mathbf{e}_I^k, \mathbf{e}_h^k) \\ &\quad + \left( \frac{1}{2} [J]_I^k \mathbf{e}_I^k, \mathbf{e}_h^k \right) + \left( \frac{1}{2} J_{k-2} \mathbf{e}_I^k, [\mathbf{e}_h]_I^k \right) + \frac{1}{2} (\text{div} (J_k \mathbf{F}_k^{-1} \mathbf{w}^k) \mathbf{e}_I^k, \mathbf{e}_h^k). \end{aligned}$$

Here we exploited the properties of the projections  $P_h^{k-1}$ ,  $P_h^k$ :

$$(J_{k-2} \mathbf{e}_I^{k-1}, \mathbf{e}_h^{k-1}) = 0, \quad (J_{k-1} \mathbf{e}_I^k, \mathbf{e}_h^k) = 0$$

respectively, and so  $(J_{k-1} [\mathbf{e}_I]_I^k, \mathbf{e}_h^k) = -([J]_I^k \mathbf{e}_I^k, \mathbf{e}_h^k) - (J_{k-2} \mathbf{e}_I^k, [\mathbf{e}_h]_I^k)$ .

Now we estimate the consistency and interpolation terms on the right-hand side of (49). First we treat the term  $\text{Consist}(\mathbf{e}_h^k)$ . Denote by  $I(p^k) \in \mathbb{Q}_h$  a suitable interpolant for  $p^k$ . Due to  $\mathbf{e}_h^k \in \mathbb{X}_h^k$ , the polynomial interpolation properties and Korn's inequality, we have

$$\begin{aligned} b(\xi^k; p^k, \mathbf{e}_h^k) &= b(\xi^k; p^k - I(p^k), \mathbf{e}_h^k) \leq c \|p^k - I(p^k)\| \|\nabla \mathbf{e}_h^k\| \leq C h^{m+1} \|p^k\|_{H^{m+1}(\Omega)} \|\mathbf{D}_k(\mathbf{e}_h^k)\|_k \\ &\leq C \left( \delta^{-1} h^{2(m+1)} \|p^k\|_{H^{m+1}(\Omega)}^2 + \delta \|\mathbf{D}_k(\mathbf{e}_h^k)\|_k^2 \right) \end{aligned} \quad (50)$$

for any  $\delta > 0$ .

We apply Hölder's inequality and Sobolev embedding inequalities as well as interpolation properties of the finite element projection in (42) to handle other consistency and interpolation terms. Using  $\|\nabla \mathbf{u}\|_{L^\infty(\Omega)} \leq C$  and (1), we get

$$\begin{aligned} c(\xi^k; \mathbf{w}_h^k, \mathbf{u}^k, \mathbf{e}_h^k) - c(\xi^k; \tilde{\mathbf{w}}^k, \mathbf{u}^k, \mathbf{e}_h^k) &= c(\xi^k; \mathbf{e}^{k-1}, \mathbf{u}^k, \mathbf{e}_h^k) \leq C \|\mathbf{e}^{k-1}\|_k \|\mathbf{e}_h^k\|_k \\ &\leq C (\|\mathbf{e}_h^{k-1}\|_k^2 + \|\mathbf{e}_I^{k-1}\|_k^2 + \|\mathbf{e}_h^k\|_k^2) \\ &\leq C (\|\mathbf{e}_h^{k-1}\|_k^2 + \|\mathbf{e}_h^k\|_k^2 + h^{2(m+1)}). \end{aligned} \quad (51)$$

Now we treat the terms contributing to the interpolation error. Using also stability estimate (29) with  $c\Delta t \geq h^{2m+4}$ , we bound

$$c(\xi^k; \mathbf{w}_h^k, \mathbf{e}_I^k, \mathbf{e}_h^k) \leq C (\|\mathbf{v}_h^k\|_k + 1) \|\mathbf{e}_I^k\| \|\mathbf{e}_h^k\|_k \leq C (\|\mathbf{e}_h^k\|_k^2 + h^{2(m+1)}), \quad (52)$$

$$a(\xi^k; \mathbf{e}_I^k, \mathbf{e}_h^k) \leq C h^{m+1} \|\mathbf{D}_k(\mathbf{e}_h^k)\|_k \leq C \left( \delta^{-1} h^{2(m+1)} + \delta \|\mathbf{D}_k(\mathbf{e}_h^k)\|_k^2 \right). \quad (53)$$

Using  $\xi \in C^2(Q)$  and (42), we get

$$\begin{aligned} ([J]_I^k \mathbf{e}_I^k, \mathbf{e}_h^k) + (J_{k-2} \mathbf{e}_I^k, [\mathbf{e}_h]_I^k) &\leq C (\|J'_I\|_{L^\infty(Q)} + \|J\|_{L^\infty(Q)}) \|\mathbf{e}_I^k\| (\|\mathbf{e}_h^k\|_k + \|[\mathbf{e}_h]_I^k\|) \\ &\leq C (\|\mathbf{e}_h^k\|_k^2 + (\Delta t)^{-1} h^{2(m+2)}) + \frac{\Delta t}{2} \|[\mathbf{e}_h]_I^k\|_k^2. \end{aligned} \quad (54)$$

Summarizing (49)–(54) and using (46), we get for  $\delta$  small enough:

$$\begin{aligned} &\frac{1}{2} \|\mathbf{e}_h^k\|_k^2 + 2\nu \Delta t \|\mathbf{D}_k(\mathbf{e}_h^k)\|_k^2 \\ &\leq \frac{1}{2} \|\mathbf{e}_h^{k-1}\|_{k-1}^2 + \tilde{C} \Delta t \left\{ \|\mathbf{e}_h^k\|_k^2 + \|\mathbf{e}_h^{k-1}\|_k^2 + h^{2(m+1)} + (\Delta t)^2 + (\Delta t)^{-1} h^{2(m+2)} \right\}, \end{aligned} \quad (55)$$

where  $\tilde{C}$  depends only on the data. Now we assume the following:

$$\text{either } \Delta t \text{ is small enough s.t. } \frac{1}{2} - \tilde{C} \Delta t > 0 \quad \text{or} \quad \nu \geq \tilde{C} C_K, \quad (56)$$

where  $C_K$  is Korn's inequality constant from (15). Note that  $\mathbf{e}_h^0 = \mathbf{u}_h^0 - P_h^k(\mathbf{u}(0))$ , where  $\mathbf{u}_h^0$  is the Lagrange interpolant of  $\mathbf{u}(0)$ , and so  $\|\mathbf{e}_h^0\|_0^2 \leq C h^{2(m+2)}$ . We sum (55) over all  $k = 1, \dots, n$ ,  $n \leq N$ , and apply the discrete Gronwall inequality to get

$$\|\mathbf{e}_h^n\|_k^2 + 2\nu \Delta t \sum_{k=1}^n \|\mathbf{D}_k(\mathbf{e}_h^k)\|_k^2 \leq C (h^{2(m+1)} + (\Delta t)^2 + (\Delta t)^{-1} h^{2(m+2)})$$

with some  $C$  independent of  $n, h, \Delta t$ . Applying the triangle inequality and (42) one more time, we obtain the final error bound in the energy norm,

$$\|\mathbf{e}\|_*^2 := \max_{1 \leq k \leq N} \|\mathbf{e}^k\|_k^2 + 2\nu \Delta t \sum_{k=1}^N \|\mathbf{D}_k(\mathbf{e}^k)\|_k^2 \leq C (h^{2(m+1)} + (\Delta t)^2 + (\Delta t)^{-1} h^{2(m+2)}). \quad (57)$$

The main result is summarized in the following theorem.

**Theorem 1.** *For the fluid problem (2)–(4) and the finite element method (14) assume the following:*

- (1) *the domain evolution is given by a smooth mapping which satisfies (1) and (32);*
- (2)  *$\partial\Omega(t) = \partial\Omega^{ns}(t)$  for all  $t \in [0, T]$ ;*
- (3)  *$\xi_t \circ \xi^{-1}$  on  $\bigcup_{t \in [0, T]} \partial\Omega(t) \times \{t\}$  is the trace of some  $\hat{\mathbf{v}}_1 \in C^3(Q^{\text{phys}})$  s.t.  $\text{div } \hat{\mathbf{v}}_1 = 0$ ;*
- (4)  *$\Omega_0$  is a convex polyhedron;*
- (5)  *$\mathbf{u}_{tt} \in L^\infty(\Omega_0)$ ,  $\mathbf{u}(t) \in H^{m+\frac{5}{2}}(\Omega_0)$ ,  $p(t) \in H^{m+1}(\Omega_0)$  for all  $t \in [0, T]$ ;*
- (6)  *$c\Delta t \geq h^{2m+4}$  with some  $c$  independent of  $h, \Delta t$ ;*
- (7) *condition (56) holds.*

*Then the finite element method is stable, see (29), and the error estimate (57) holds.*

For the popular P2–P1 Taylor–Hood element we have  $m = 1$ , and the optimal error bound  $\|\mathbf{e}\|_* \leq C \max\{h^2; \Delta t\}$  follows if  $h^2 \leq c\Delta t$ . The second order in time approximation can be achieved in practice (but not analyzed here) by using BDF2 time stepping instead of backward Euler time stepping.

We note that constant  $C$  in (57) depends on problem data and may blow up for large Reynolds numbers. As it is common for FE methods for fluids, the present method requires stabilization or subgrid scales modeling for higher Re numbers. In Section 6.2 we consider the extension of our quasi-Lagrangian formulation to Smagorinsky eddy viscosity model.

## 6. Numerical experiments

In this section we present numerical results for the finite element method (14) implemented within the open source Ani3D software ([www.sf.net/p/ani3d](http://www.sf.net/p/ani3d)) applied to two flow examples. In the first experiment we demonstrate experimental convergence rates for a given analytical solution and compare those to the established theoretical expectations. The second experiment is concerned with blood flow in a geometrical dynamic model of the human left ventricle. The motion of the ventricle was extracted from a sequence of CECT images of a real patient heart over one cardiac cycle.

### 6.1. Convergence to analytical solution

In cylindrical coordinates  $(r, y, \phi)$ , we define the reference (initial) domain  $\Omega_0$  to be the axisymmetric tube with symmetry axis  $0y$ :

$$\Omega_0 = \{(r, y, \phi) : -4 \leq y \leq 4, r^2 \leq e^{y/4+1}\}.$$

The domain has outflow boundary  $\partial\Omega_0^N = \partial\Omega_0 \cap \{(r, y, \phi) : y = 4\}$  and no-slip no-penetration boundary  $\partial\Omega_0^{ns} = \partial\Omega_0 \setminus \partial\Omega_0^N$ .

The physical time-dependent domain is:

$$\Omega(t) = \{(r, y, \phi) : -4 \leq y \leq 4, r^2 \leq e^{y/4+1}(1 - \frac{1}{4}t)\}, \quad t \in [0, 0.2].$$

The analytical solution  $\{\mathbf{u}, p\}$  to (2)–(5) is given in cylindrical coordinates  $(r, y, \phi)$  by

$$u_r = -\frac{2e^{-\frac{1}{4}(y+4)}r^3}{(4-t)^2}, \quad u_y = \frac{8}{4-t} - \frac{32e^{-\frac{1}{4}(y+4)}r^2}{(4-t)^2}, \quad u_\phi = 0,$$

$$p = 512\nu \frac{e^{-\frac{1}{4}(y+4)}}{(4-t)^2} - 8 \frac{y}{(t-4)^2} + \tilde{p}(t),$$

where  $\tilde{p}(t)$  depends only on  $t$ . To ensure unique pressure, we set  $p = 0$  on  $\partial\Omega^N(t)$ .

**Table 1**

Finite element error for the given analytic solution.

Mesh step size	1.0	$1.0/\sqrt{2}$	0.5	$0.5/\sqrt{2}$	0.25	$0.25/\sqrt{2}$
Number of mesh elements	389	928	2333	5813	16439	46215
Time step size	0.04	0.02	0.01	0.005	0.0025	0.00125
Number of time steps $N$	5	10	20	40	80	160
$\max_{1 \leq k \leq N} \ \mathbf{e}^k\ _k + \sqrt{\sum_{k=1}^N \Delta t \ \mathbf{D}_k(\mathbf{e}^k)\ _k^2}$	0.2652	0.1731	0.0983	0.0534	0.0233	0.0115
Error ratio		1.532	1.761	1.841	2.292	2.034

The external force  $\mathbf{f} = (f_r, f_y, f_\phi)$  is taken such that  $\mathbf{u}$  satisfies (2)–(3) for  $\nu = 0.04$ . To simplify the design of the right-hand side and the boundary conditions, we consider in (3) the full velocity gradient tensor  $\nabla \mathbf{u} \mathbf{F}^{-1}$  instead of its doubled symmetric part. The resulting forcing vector is given by

$$f_r = \nu \frac{e^{-\frac{1}{4}(y+4)}}{(4-t)^2} \left( 16r + \frac{1}{8}r^3 \right) - 4 \frac{e^{-\frac{1}{2}(y+4)}}{(t-4)^4} r^5, \quad f_y = 2\nu \frac{e^{-\frac{1}{4}(y+4)}}{(4-t)^2} r^2 - 128 \frac{e^{-\frac{1}{2}(y+4)}}{(t-4)^4} r^4, \quad f_\phi = 0.$$

The boundary condition on  $\partial \Omega^N(t)$  is non-homogeneous,  $\sigma \mathbf{n} = \nu J \nabla \mathbf{u} \mathbf{F}^{-1} \mathbf{F}^{-T} \mathbf{n}_N$ ,  $\mathbf{n}_N := (0, 1, 0)$ .

We applied the lower degree ( $m = 1$ )  $P_2$ - $P_1$  Taylor–Hood finite elements in (14) with the simplified stress tensor (3). A sequence of six unstructured quasiuniform tetrahedral meshes with mesh sizes  $h_i = h_{i-1}/\sqrt{2}$ ,  $i = 2, \dots, 6$  is used to measure the finite element error. The computed error norms are shown in Table 1. The second order asymptotic convergence rate is consistent with the estimate of Theorem 1.

## 6.2. Blood flow in a geometrical model of the human left ventricle

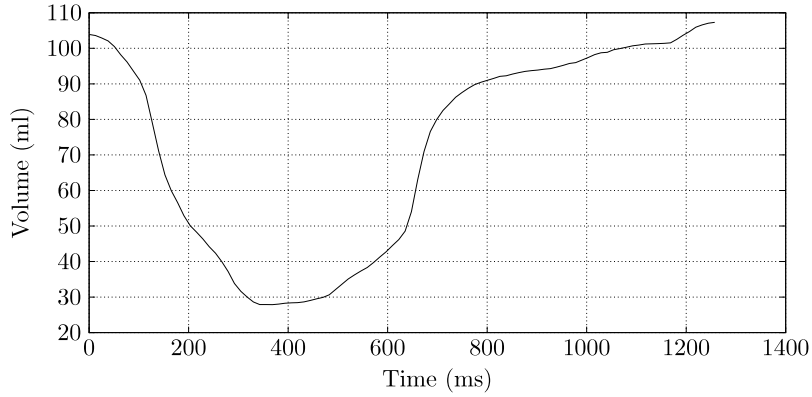
We now illustrate the practical value of the finite element method (14) by applying it to simulation of the blood flow in a simplified model of the human left ventricle. This problem arises in patient-oriented hemodynamic applications. Our simplifications concern anatomical structure and boundary conditions on the valves as well as neglecting ventricle twisting.

The motion of the ventricle  $\Omega(t)$  is recovered from a sequence of CECT images. The input data is a dataset of 100 images with  $512 \times 512 \times 480$  voxels and  $0.625 \times 0.625 \times 0.25$  mm resolution. The images were taken from a chest CECT of a 50 year old female. The mapping  $\xi$  is defined by a dynamic sequence of 1981 meshes for one cardiac cycle. The mesh sequence contains *topologically invariant* tetrahedral meshes with 14 033 nodes, 88 150 edges and 69 257 tetrahedra which differ only in nodes positions. Denote by  $\mathbf{x}$  the spatial coordinate of a node of the reference grid at time  $t = 0$ , and identify the coordinate  $\xi(\mathbf{x}, t)$  of the corresponding node at time  $t = t_k$ . The mapping  $\xi^k$  is defined as the continuous piecewise linear vector function with values  $\xi(\mathbf{x}, t)$  at the reference grid nodes  $\mathbf{x}$ . Of course, the recovered mapping is a piecewise smooth approximation of an unknown smooth mapping  $\xi$ . This introduces additional modeling error which is not treated in our analysis.

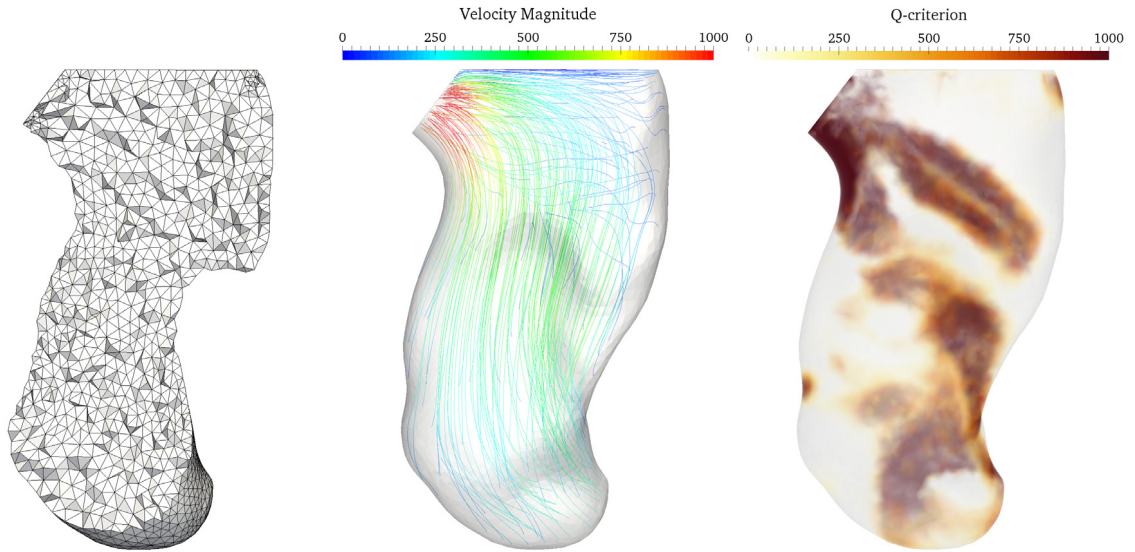
Note that the time between two sequential frames of CECT input data is equal to 12.7 milliseconds. Because of the fast heart walls motion and variation of ventricle volume (see Fig. 1) setting  $\Delta t = 12.7$  ms turned out to be too large to deliver acceptable accuracy. The sequence of 1981 meshes allows us to use 20 times smaller time step  $\Delta t = 0.635$  ms. For the details of generation of this sequence we refer to recent paper [21].

To set up the boundary conditions, we split the left ventricle boundary into aortic valve and mitral valve patches, and the remaining part of the boundary. The ventricle passes through the systole phase approximately until  $t = 355$  ms releasing blood flow through the aortic valve. During this time interval, we set the ‘do-nothing’ boundary condition (5) with  $\hat{\mathbf{g}} = \mathbf{0}$  on the patch associated with the aortic valve. For the remaining time interval ending at  $T = 1.2573$  s, we impose the ‘do-nothing’ boundary condition on the patch associated with the mitral valve. The latter is connected to the atrium and intakes blood during the expansion stage called diastole. On the remaining boundary, including the aortic valve during the diastole phase and the mitral valve during the systole phase, the no-penetration no-slip condition (4) is imposed,  $\mathbf{u} = \xi_t$ . In future, we plan to use more physiologically suitable boundary conditions, see [27], instead of the ‘do-nothing’ boundary conditions.





**Fig. 1.** The ventricle volume change in time.

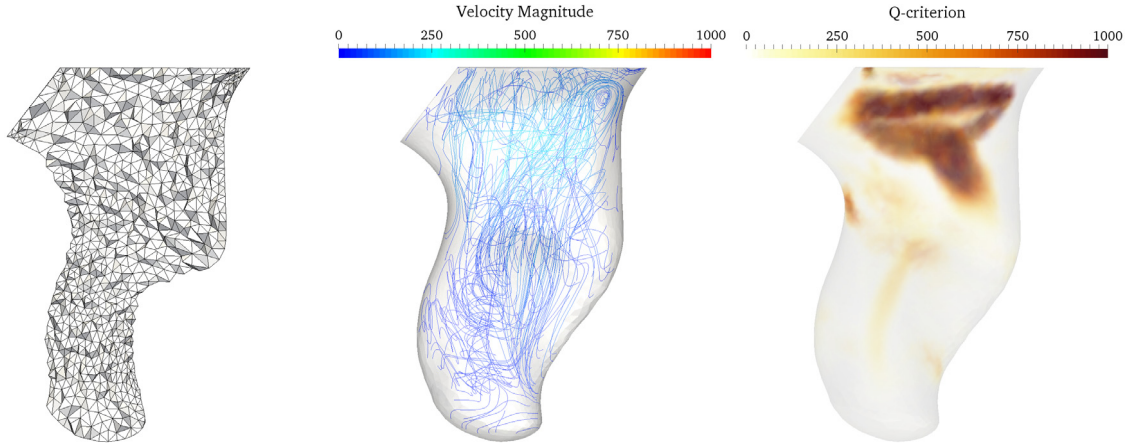


**Fig. 2.** The cutaway of the mesh, velocity streamlines and the Q-criterion field at  $t = 200$  ms, horizontal long axis view.

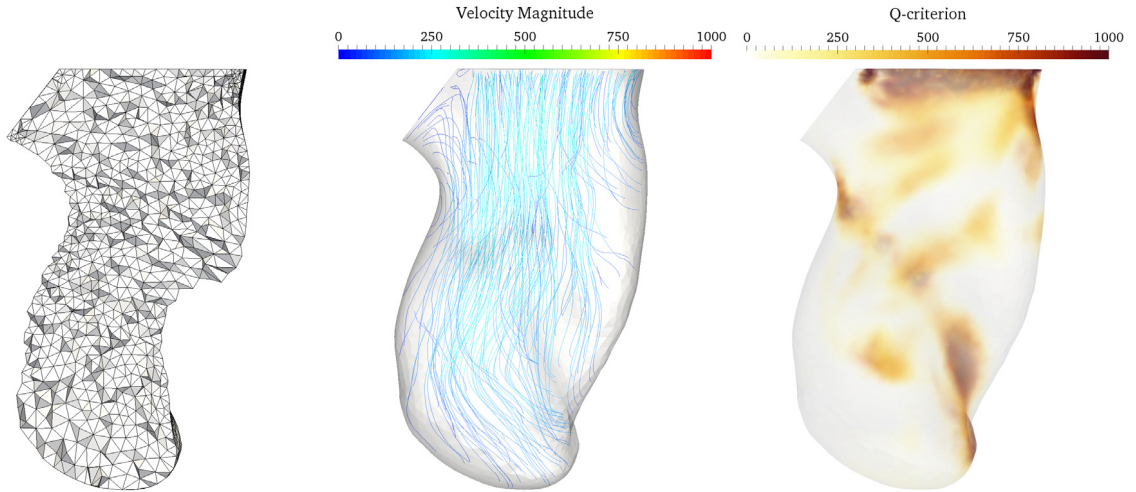
At the initial time moment, the beginning of the systole stage, we assume the system is at rest with zero pressure (this is, of course, an idealized initial condition). Similarly to the previous test case, we use the lower degree Taylor–Hood finite element spaces (13) with  $m = 1$ . At every time step one has to solve a linear system with 320 582 unknowns, comprised of 14 033 nodal pressure degrees of freedom and  $14\,033 + 88\,150$  degrees of freedom residing at the mesh vertices and edge centers, for each velocity component.

The hemodynamics of the heart is characterized by transitional or even turbulent flow regimes (see, e.g., [28,29,6]). Our mesh is not sufficiently fine to resolve all scales in the flow, and hence a subgrid model has to be employed. For the purposes of this paper, we use the simplest approach and replace  $\nu$  with the Smagorinsky turbulent viscosity coefficient  $\nu_\tau$ :

$$\nu_\tau = \nu + (C_s h_T)^2 \sqrt{2\mathbf{D}_k(\mathbf{w}_h^k) : \mathbf{D}_k(\mathbf{w}_h^k)},$$



**Fig. 3.** The cutaway of the mesh, velocity streamlines and the Q-criterion field at  $t = 400$  ms, horizontal long axis view.



**Fig. 4.** The cutaway of the mesh, velocity streamlines and the Q-criterion field at  $t = 600$  ms, horizontal long axis view.

where  $C_s = 0.2$  and  $h_T = \text{diam}(T)$ , for any tetrahedral cell  $T \in \mathcal{T}_h$ . The finite element method takes the form: Find velocity  $\mathbf{u}^k \in \mathbb{V}_h$  and pressure  $p^k \in \mathbb{Q}_h$  satisfying equation

$$\begin{aligned} & \int_{\Omega_0} J_k \frac{\mathbf{u}^k - \mathbf{u}^{k-1}}{\Delta t} \cdot \boldsymbol{\psi} \, d\mathbf{x} + \int_{\Omega_0} J_k \nabla \mathbf{u}^k \mathbf{F}_k^{-1} \left( \mathbf{u}^{k-1} - \frac{\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1}}{\Delta t} \right) \cdot \boldsymbol{\psi} \, d\mathbf{x} - \int_{\Omega_0} J_k p^k \mathbf{F}_k^{-T} : \nabla \boldsymbol{\psi} \, d\mathbf{x} \\ & + \int_{\Omega_0} J_k q \mathbf{F}_k^{-T} : \nabla \mathbf{u}^k \, d\mathbf{x} + \int_{\Omega_0} \nu_\tau J_k (\nabla \mathbf{u}^k \mathbf{F}_k^{-1} \mathbf{F}_k^{-T} + \mathbf{F}_k^{-T} (\nabla \mathbf{u}^k)^T \mathbf{F}_k^{-T}) : \nabla \boldsymbol{\psi} \, d\mathbf{x} = 0 \end{aligned} \quad (58)$$

and the no-penetration no-slip  $\mathbf{u}^k = (\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1})/\Delta t$  or the ‘do-nothing’ boundary conditions, for all  $\boldsymbol{\psi}$  and  $q$  from the corresponding FE spaces.

The cutaway of the mesh and computed velocity streamlines and the Q-criterion field at 200, 400, 600 ms are shown in Figs. 2, 3, 4, respectively. The velocity magnitude is given in mm/s. The viscosity of blood was set  $\nu = 4 \text{ mm}^2/\text{s}$ . These instances are chosen to demonstrate blood flow features in the middle of the systole phase, the very beginning of the diastole phase, and the beginning of the second quarter of the diastole phase (Fig. 1). The velocity streamlines and the Q-criterion are shown in the entire 3D domain, whereas the mesh is shown partly: only cells lying beyond the cross-section plane are visible.

## 7. Conclusions

The paper shows that the quasi-Lagrangian description of the flow in a time-dependent domain is suitable for a practically efficient finite element method, which is amenable to mathematically rigorous numerical analysis. Major challenges in analysis are dealing with the time-dependent divergence-free constraint in finite element spaces and handling non-homogeneous boundary conditions. This paper proves the optimal error estimate in the energy norm under fairly practical assumptions. In a relatively straightforward way, implementation of the method builds on standard Navier–Stokes finite element solvers in a fixed triangulated domain. The example of the flow in the simplified model of the human left ventricle demonstrated the practical value of the method for certain medical applications.

## Acknowledgment

We would like to thank Dr. Alexander Danilov for providing the sequence of meshes and postprocessing the solution for the experiment with the flow in the simplified model of the human left ventricle.

## References

- [1] N.R. Saber, N.B. Wood, A. Gosman, R.D. Merrifield, G.-Z. Yang, C.L. Charrier, P.D. Gatehouse, D.N. Firmin, Progress towards patient-specific computational flow modeling of the left heart via combination of magnetic resonance imaging with computational fluid dynamics, *Ann. Biomed. Eng.* 31 (1) (2003) 42–52.
- [2] Q. Long, R. Merrifield, X. Xu, P. Kilner, D. Firmin, G. Yang, Subject-specific computational simulation of left ventricular flow based on magnetic resonance imaging, *Proc. Inst. Mech. Eng.* 222 (4) (2008) 475–485.
- [3] T. Schenkel, M. Malve, M. Reik, M. Markl, B. Jung, H. Oertel, MRI-based CFD analysis of flow in a human left ventricle: methodology and application to a healthy heart, *Ann. Biomed. Eng.* 37 (3) (2009) 503–515.
- [4] V. Mihalef, R.I. Ionasec, P. Sharma, B. Georgescu, I. Voigt, M. Suehling, D. Comaniciu, Patient-specific modelling of whole heart anatomy, dynamics and haemodynamics from four-dimensional cardiac CT images, *Interface Focus* 1 (3) (2011) 286–296.
- [5] T. Doenst, K. Spiegel, M. Reik, M. Markl, J. Hennig, S. Nitzsche, F. Beyersdorf, H. Oertel, Fluid-dynamic modeling of the human left ventricle: methodology and application to surgical ventricular reconstruction, *Ann. Thorac. Surg.* 87 (4) (2009) 1187–1195.
- [6] C. Chnafa, S. Mendez, F. Nicoud, Image-based large-eddy simulation in a realistic left heart, *Comput. & Fluids* 94 (2014) 173–187.
- [7] R. Mittal, J.H. Seo, V. Vedula, Y.J. Choi, H. Liu, H.H. Huang, S. Jain, L. Younes, T. Abraham, R.T. George, Computational modeling of cardiac hemodynamics: current status and future outlook, *J. Comput. Phys.* 305 (2016) 1065–1082.
- [8] B. Su, R. San Tan, J. Le Tan, K.W.Q. Guo, J.M. Zhang, S. Leng, X. Zhao, J.C. Allen, L. Zhong, Cardiac MRI based numerical modeling of left ventricular fluid dynamics with mitral valve incorporated, *J. Biomech.* 49 (7) (2016) 1199–1205.
- [9] A. Masud, T.J. Hughes, A space-time Galerkin/least-squares finite element formulation of the Navier-Stokes equations for moving domain problems, *Comput. Methods Appl. Mech. Engrg.* 146 (1–2) (1997) 91–126.
- [10] T.E. Tezduyar, M. Behr, S. Mittal, J. Liou, A new strategy for finite element computations involving moving boundaries and interfaces—the deforming-spatial-domain/space-time procedure: II. Computation of free-surface flows, two-liquid flows, and flows with drifting cylinders, *Comput. Methods Appl. Mech. Engrg.* 94 (3) (1992) 353–371.
- [11] S. Osher, R. Fedkiw, *Level Set Methods and Dynamic Implicit Surfaces*, vol. 153, Springer Science & Business Media, 2006.
- [12] R. Glowinski, T.-W. Pan, T.I. Hesla, D.D. Joseph, A distributed Lagrange multiplier/fictitious domain method for particulate flows, *Int. J. Multiph. Flow* 25 (5) (1999) 755–794.
- [13] C.S. Peskin, Numerical analysis of blood flow in the heart, *J. Comput. Phys.* 25 (3) (1977) 220–252.
- [14] S. Groß, A. Reusken, An extended pressure finite element space for two-phase incompressible flows with surface tension, *J. Comput. Phys.* 224 (1) (2007) 40–58.
- [15] C. Hirt, A.A. Amsden, J. Cook, An arbitrary Lagrangian-Eulerian computing method for all flow speeds, *J. Comput. Phys.* 14 (3) (1974) 227–253.
- [16] F. Nobile, L. Formaggia, A stability analysis for the arbitrary Lagrangian-Eulerian formulation with finite elements, *East-West J. Numer. Math.* 7 (2) (1999) 105–132.
- [17] F. Duarte, R. Gormaz, S. Natesan, Arbitrary Lagrangian-Eulerian method for Navier–Stokes equations with moving boundaries, *Comput. Methods Appl. Mech. Engrg.* 193 (45) (2004) 4819–4836.
- [18] R. Guberovic, C. Schwab, R. Stevenson, Space-time variational saddle point formulations of Stokes and Navier–Stokes equations, *ESAIM Math. Model. Numer. Anal.* 48 (3) (2014) 875–894.
- [19] J. Sudirham, J. Van Der Vegt, R. Van Damme, Space-time discontinuous Galerkin method for advection–diffusion problems on time-dependent domains, *Appl. Numer. Math.* 56 (12) (2006) 1491–1518.
- [20] J. Martin, L. Smaranda, T. Takahashi, Convergence of a finite element/ALE method for the Stokes equations in a domain depending on time, *J. Comput. Appl. Math.* 230 (2009) 521–545.
- [21] A. Danilov, A. Lozovskiy, M. Olshanskii, Y. Vassilevski, A finite element method for the Navier–Stokes equations in moving domain with application to hemodynamics of the left ventricle, *Russian J. Numer. Anal. Math. Modelling* 32 (4) (2017) 225–236.
- [22] T. Miyakawa, Y. Teramoto, Existence and periodicity of weak solutions of the Navier-Stokes equations in a time dependent domain, *Hiroshima Math. J.* 12 (3) (1982) 513–528.
- [23] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, Vol. 343, American Mathematical Soc., 2001.

- [24] D. Boffi, F. Brezzi, M. Fortin, et al., *Mixed Finite Element Methods and Applications*, vol. 44, Springer, 2013.
- [25] A. Ern, J.-L. Guermond, *Theory and Practice of Finite Elements*, vol. 159, Springer Science & Business Media, 2013.
- [26] M.D. Gunzburger, S.L. Hou, *Treating inhomogeneous essential boundary conditions in finite element methods and the calculation of boundary stresses*, *SIAM J. Numer. Anal.* 29 (2) (1992) 390–424.
- [27] A. Tagliabue, L. Dede, A. Quarteroni, *Fluid dynamics of an idealized left ventricle: the extended Nitsche’s method for the treatment of heart valves as mixed time varying boundary conditions*, *Internat. J. Numer. Methods Fluids* (2017).
- [28] G. Querzoli, S. Fortini, A. Cenedese, *Effect of the prosthetic mitral valve on vortex dynamics and turbulence of the left ventricular flow*, *Phys. Fluids* 22 (4) (2010) 041901.
- [29] A. Falahatpisheh, A. Kheradvar, *High-speed particle image velocimetry to assess cardiac fluid dynamics in vitro: from performance to validation*, *Eur. J. Mech. B Fluids* 35 (2012) 2–8.