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NUMERICAL METHODS

Finite Element Models of Hyperelastic Materials Based on a New Strain Measure

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Abstract—To construct constitutive equations for hyperelastic materials, one increasingly often proposes new strain measures, which result in significant simplifications and error reduction in experimental data processing. One such strain measure is based on the upper triangular (QR) decomposition of the deformation gradient. We describe a finite element method for solving nonlinear elasticity problems in the framework of finite strains for the case in which the constitutive equations are written with the use of the QR-decomposition of the deformation gradient. The method permits developing an efficient, easy-to-implement tool for modeling the stress–strain state of any hyperelastic material.

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1. INTRODUCTION

In mechanics of deformable solids, *constitutive equations* (equations of state) are relations expressing the dependence of the stress tensor on the variables characterizing the motion of medium particles [1, p. 80]. This dependence completely determines the mechanical behavior of the material under study and plays a key role in solving applied problems. The form of a constitutive equation significantly depends on the chosen strain measure, and the constitutive equation must satisfy several conditions [1, p. 386; 2, p. 187 of the Russian translation]. For example, the constitutive equations must reflect the symmetry of physical properties of the material in question and be independent of the reference system.

The model of a hyperelastic material is widely used in studies of the stress–strain state of soft biological tissues. In the framework of this model, it is postulated that there exists an elastic potential completely determining the constitutive equations [1, p. 103; 2, p. 171 of the Russian translation]. The right Cauchy–Green strain tensor is the most common strain measure used when dealing with hyperelastic materials. This is due to the relatively simple form of constitutive equations and the existence of well-developed techniques for solving nonlinear elasticity problems in the case of this measure. This strain measure also has a drawback: determining the form of constitutive equations from experimental data encounters difficulties due to the correlation between the terms in these equations [3]. In this connection, one proposes new strain measures leading to constitutive equations with uncorrelated terms (the orthogonality property) [4–7]. One such measure is based on the upper triangular (QR) decomposition of the deformation gradient. The orthogonality property allows one to obtain a simple description of the mechanical behavior of soft tissues without using any a priori given constitutive equation and minimizes errors in experimental data processing.

In the present paper, we describe an approach to modeling the deformation of hyperelastic materials with the use of a strain measure based on the QR-decomposition of the deformation gradient. In the framework of this approach, we suggest to use the interpolation properties of barycentric coordinates and the principle of minimum potential energy, which, in the case of linear triangular finite elements, permits obtaining all necessary formulas in the analytical and concise representation. The concept was first proposed in [8] for a Saint-Venant–Kirchhoff material and was further described for the entire class of isotropic hyperelastic materials in [9].

The paper is organized as follows. In Section 2, we introduce basic notation and definitions. Further, we use the new strain characteristics to pose the problem on the equilibrium of an elastic body (Section 3) and its finite element discretization (Section 4). The results of numerical experiments are described in Section 5. All problems considered in this paper are posed in the two-dimensional statement, but our approach can be used in a similar way in the three-dimensional case.

2. CONSTITUTIVE EQUATIONS FOR SOFT TISSUES

2.1 Kinematics

Consider the domain $\Omega_t \subset \mathbb{R}^2$ occupied by an elastic body at time t. Let Ω_0 be the domain at the initial time. The position of a point at the initial time is denoted by $\mathbf{X} = (X_1, X_2)$, and the position of the point at time t, by $\mathbf{x} = (x_1, x_2)$. For the Cartesian frame $\{\mathbf{E}_1, \mathbf{E}_2\}$ fixed to the initial configuration Ω_0 and the Cartesian frame $\{\mathbf{e}_1, \mathbf{e}_2\}$ fixed to the actual configuration Ω_t, we can write

$$\mathbf{X} = X_1\mathbf{E}_1 + X_2\mathbf{E}_2, \qquad \mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2.$$

The deformation $\mathbf{x} = \phi(\mathbf{X}, t)$ of the elastic body is defined as a one-to-one mapping

$$\phi : \Omega_0 \times [0, t] \to \Omega_t,$$

and the corresponding displacements have the form $\mathbf{u}(\mathbf{X}, t) := \phi(\mathbf{X}, t) - \mathbf{X}$.

The deformation gradient $\mathbf{F}$, which is a key characteristic of the kinematics, is determined by the relation

$$\mathbf{F} = \mathbf{F}(\mathbf{X}, t) = \frac{\partial \phi}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j,$$

where $\otimes$ is the tensor product, $\mathbf{a} \otimes \mathbf{b} := (a_1, a_2)^{\mathrm{T}}(b_1, b_2)$. The components of the deformation gradient matrix $\mathbf{F}$ have the form

$$[\mathbf{F}]_{ij} = F_{ij} = \frac{\partial x_i}{\partial X_j}.$$

Its determinant is subjected to the constraint $J = \det \mathbf{F} > 0$. Note that the deformation gradient is related to the displacements $\mathbf{u}$ of points of the body as follows:

$$\mathbf{F} = \mathbf{I} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}},$$

where $\mathbf{I}$ is the unit tensor.

The deformation gradient $\mathbf{F}$ contains information about changes in the distance between the points of the body and about rigid rotations of the body. A rigid rotation does not cause additional stresses in the body, and hence one introduces various strain measures eliminating such rotations.

2.2 Strain Measures

The polar decomposition of the deformation gradient $\mathbf{F}$ is widely used in the construction of strain measures [2, p. 127 of the Russian translation].

Theorem 1 (on the polar decomposition of invertible matrices). *Every invertible real matrix* $\mathbf{F}$ *can uniquely be represented in any of the forms* $\mathbf{F} = \mathbf{R}\mathbf{U}$ *and* $\mathbf{F} = \mathbf{V}\mathbf{R}$, *where* $\mathbf{R}$ *is an orthogonal matrix and* $\mathbf{U}$ *and* $\mathbf{V}$ *are symmetric positive definite matrices.*

An application of the polar decomposition theorem to the deformation gradient permits one to single out the rotation tensor $\mathbf{R}$, the right stretch tensor $\mathbf{U}$, and the left stretch tensor $\mathbf{V}$. In other words, the total deformation of a material element can be viewed as the superposition of a rigid rotation and an extension of this element.

Examples of measures based on the polar decomposition are given by the right Cauchy–Green strain tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^2$, the Lagrange strain tensor $\mathbf{E} = (\mathbf{C} - \mathbf{I})/2$, the left Cauchy–Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{V}^2$, and the logarithmic (Hencky) measure $\mathbf{E}_H = \ln \mathbf{B}/2 = \ln(\mathbf{V})$ and $\mathbf{e}_H = \ln \mathbf{C}/2 = \ln(\mathbf{U})$. The tensor $\mathbf{C}$ characterizes the variations in the lengths (distances between points) after a deformation of a solid body; at the same time, the Lagrange strain tensor $\mathbf{E}$ serves as the measure of deviation of a given deformation from a rigid deformation ($\mathbf{C} = \mathbf{I}$).

It was proposed in [6] to use a strain measure based on the QR-decomposition rather than the polar decomposition of the deformation gradient.

Theorem 2 (QR-decomposition [10, p. 168]). *Every nonsingular real matrix* $\mathbf{F}$ *has a decomposition* $\mathbf{F} = \mathbf{Q}\mathbf{R}$, *where* $\mathbf{Q}$ *is an orthogonal matrix and* $\mathbf{R}$ *is an upper triangular matrix with positive diagonal entries.*

Thus, the QR-decomposition of the deformation gradient also permits one to treat the total deformation of a material element as the superposition of a rigid rotation ($\mathbf{Q}$) and a variation ($\mathbf{R}$) in the shape of this element [6, 7].

By the QR-decomposition theorem, for the deformation gradient $\mathbf{F}$ there exists a matrix $\mathbf{Q} = \mathbf{e}'_i \otimes \mathbf{E}_i$ such that

$$\mathbf{Q}^{\mathrm{T}}\mathbf{F} = \tilde{\mathbf{F}} = \sum_{i\le j}^{i,j=1,2} \tilde{F}_{ij}\mathbf{E}_i \otimes \mathbf{E}_j, \qquad [\tilde{F}_{ij}] := \begin{pmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ 0 & \tilde{F}_{22} \end{pmatrix},$$

where $\mathbf{e}'_i$ is a new orthonormal basis, which can be obtained by the Gram–Schmidt orthogonalization of the vector system $\{\mathbf{F}\mathbf{E}_1, \mathbf{F}\mathbf{E}_2\}$.

Since $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \tilde{\mathbf{F}}^{\mathrm{T}}\tilde{\mathbf{F}}$, it follows that the components of the tensor $\tilde{\mathbf{F}}$ can be obtained by the Cholesky factorization of the Cauchy–Green strain tensor $\mathbf{C}$,

$$\tilde{F}_{11} = \sqrt{C_{11}}, \qquad \tilde{F}_{12} = C_{12}/\tilde{F}_{11}, \qquad \tilde{F}_{22} = \sqrt{C_{22} - \tilde{F}_{12}^2}. \tag{2.1}$$

Like the tensors $\mathbf{U}$ and $\mathbf{V}$, the tensor $\tilde{\mathbf{F}}$ characterizes the deformation of a body as a variation in the distances between points, and all of its components have a physical meaning [6]. For strain measures one takes the following variables ξ_i, $i = 1, 2, 3$:

$$\xi_1 = \ln \tilde{F}_{11}, \qquad \xi_2 = \ln \tilde{F}_{22}, \qquad \xi_3 = \tilde{F}_{12}/\tilde{F}_{11}.$$

2.3 Hyperelastic Material

By the definition of hyperelastic material, there exists an elastic potential $\psi(\mathbf{F})$ such that the Cauchy stress tensor σ has the form [2, p. 171 of the Russian translation]

$$\sigma = \frac{1}{J}\frac{\partial\psi(\mathbf{F})}{\partial\mathbf{F}}\mathbf{F}^{\mathrm{T}}.$$

Here the potential energy U of the elastic body can be expressed via the elastic potential as well,

$$U = \int_{\Omega_0} \psi(\mathbf{F})\, d\Omega = \int_{\Omega_t} J^{-1}\psi(\mathbf{F})\, d\Omega. \tag{2.2}$$

Further, the elastic potential must satisfy the condition of material independence of the reference system; i.e.,

$$\psi(\mathbf{F}) = \psi(\mathbf{Q}\mathbf{F}) \quad \text{for any matrix} \quad \mathbf{Q} \in \mathrm{SO}(2), \tag{2.3}$$

where SO(2) is the group of proper rotations of the two-dimensional space. If there exists a symmetry of the physical properties of the material under study, then the constitutive equations must be invariant under all transformations of material coordinates belonging to the symmetry group of the material. The form of the strain energy function (the elastic potential) is subjected to some more restrictions, which are described in detail in the monographs [1, p. 386; 2, p. 187 of the Russian translation].

As was shown in the monograph [2, p. 176 of the Russian translation], the condition of material independence of the reference system is satisfied for hyperelastic materials if and only if the elastic potential is a function of $\mathbf{F}^{\mathrm{T}}\mathbf{F}$; i.e., $\psi(\mathbf{F}) = \tilde{W}(\mathbf{F}^{\mathrm{T}}\mathbf{F})$. Therefore, one often expresses the elastic potential ψ as a function of the right Cauchy–Green strain tensor $\mathbf{C}$; then

$$\sigma = \frac{2}{J}\mathbf{F}\frac{\partial\psi(\mathbf{C})}{\partial\mathbf{C}}\mathbf{F}^{\mathrm{T}}.$$

If the strain measure based on the QR-decomposition is used, then the elastic potential is a function of the ξ_i,

$$\psi = \psi_{QR}(\xi_1, \xi_2, \xi_3).$$

By relations (2.1) between the components of the tensors $\tilde{\mathbf{F}}$ and $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, the condition of independence of the reference system (2.3) is satisfied. If such a strain measure is used, then the constitutive equations become

$$\sigma = \frac{1}{J}\sum_{i=1}^{3}\frac{\partial\psi_{QR}}{\partial\xi_i}\mathbf{A}_i,$$

$$\mathbf{A}_1 = \mathbf{e}_1' \otimes \mathbf{e}_1', \qquad \mathbf{A}_2 = \mathbf{e}_2' \otimes \mathbf{e}_2', \qquad \mathbf{A}_3 = \frac{\tilde{F}_{22}}{2\tilde{F}_{11}}(\mathbf{e}_1' \otimes \mathbf{e}_2' + \mathbf{e}_2' \otimes \mathbf{e}_1').$$

One advantage of this measure from the viewpoint of determining the constitutive equations is the possibility of constructing a dependence with uncorrelated terms; i.e., $\mathbf{A}_i : \mathbf{A}_j = \operatorname{tr}(\mathbf{A}_i^{\mathrm{T}}\mathbf{A}_j) = 0$ for $i \neq j$. This property permits reconstructing the functions $\partial\psi_{QR}/\partial\xi_i$ directly from experimental data (stress–strain curves), because

$$\frac{1}{J}\frac{\partial\psi_{QR}}{\partial\xi_i} = \sigma : \mathbf{A}_i. \tag{2.4}$$

If the orthogonality condition is not satisfied (for example, when the invariants of the Cauchy–Green tensor are used), then the corresponding partial derivatives of the elastic potential depend on each other, which complicates their determination from experimental data, and the errors in the results are larger in this case [3]. The use of the orthogonality property and formulas (2.4) allows one to diminish the errors when determining the functions $\partial\psi_{QR}/\partial\xi_i$.

3. EQUILIBRIUM EQUATIONS

The equilibrium equations of an elastic body in differential statement become

$$\operatorname{div}\sigma + \mathbf{b} = 0 \quad \text{in the domain} \quad \Omega_t, \tag{3.1}$$

where $\mathbf{b}$ is the bulk force density.

Let $\partial\Omega_t$ be the boundary of the domain Ω_t, and let $\partial\Omega_t = \Gamma_u(t) \cup \Gamma_\sigma(t)$, where $\Gamma_u(t) = \overline{\Gamma}_u(t)$. Consider the mixed boundary conditions

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \Gamma_u(t), \qquad \sigma\mathbf{n} = \mathbf{t_0} \quad \text{on} \quad \Gamma_\sigma(t), \tag{3.2}$$

where $\mathbf{n}$ is the outward normal on $\partial\Omega_t$ and $\bar{\mathbf{u}}$ and $\mathbf{t_0}$ are given displacements and forces on the boundaries $\Gamma_u(t)$ and $\Gamma_\sigma(t)$, respectively.

Since there exists an elastic potential for hyperelastic materials, it follows that the finite-element approach to the approximate solution of Eqs. (3.1), (3.2) can be based on the virtual work principle [11, p. 177 of the Russian translation] : find a function $\mathbf{u} \in \tilde{H}^1(\Omega_t)$ such that

$$\delta W - \delta U = 0, \tag{3.3}$$

where the internal energy variation

$$\delta U = \int\limits_{\Omega_t} \sigma : \nabla \delta \mathbf{u} \, dS$$

is due to the work

$$\delta W = \int\limits_{\Gamma_\sigma(t)} \mathbf{t_0} \cdot \delta \mathbf{u} \, dS + \int\limits_{\Omega_t} \mathbf{b} \cdot \delta \mathbf{u} \, d\Omega$$

of the applied bulk and surface forces and the set $\tilde{H}^1(\Omega_t)$ of functions is defined as

$$\tilde{H}^1(\Omega_t) = \{\mathbf{v} \in H^1(\Omega_t) : \mathbf{v} = \bar{\mathbf{u}} \;\; \text{on} \;\; \Gamma_u(t)\}.$$

In view of (2.2), we can write Eq. (3.3) as

$$\delta W - \frac{\partial}{\partial \mathbf{u}} \Bigg(\int\limits_{\Omega_0} \psi(\mathbf{F}) \, d\Omega \Bigg) \cdot \delta \mathbf{u} = 0. \tag{3.4}$$

4. FINITE-ELEMENT DISCRETIZATION OF THE EQUILIBRIUM EQUATIONS

The approach used in this paper was proposed in [9]. Let us recall its main points.

Given a conformal triangulation of the domain Ω_s, consider the finite element method in which the displacement field is approximated by continuous functions linear on each triangle.

Consider a triangle T_P of computational mesh with vertices $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ which, after the deformation $\phi(\mathbf{X}, t)$, becomes a triangle T_Q with vertices $\mathbf{Q_1}, \mathbf{Q_2}, \mathbf{Q_3}$. We denote the areas of the triangles T_P and T_Q by A_p and A_q, respectively; then $J = A_q/A_p$.

If $\lambda_1(\mathbf{X}), \lambda_2(\mathbf{X}), \lambda_3(\mathbf{X})$ are the barycentric coordinates of a point $\mathbf{X}$, then the coordinates of any point $\mathbf{X} \in T_P$ of the triangle before the deformation and of the corresponding point $\mathbf{x} = \phi(\mathbf{X}) \in T_Q$ of the triangle after the deformation can be written as

$$\mathbf{X} = \sum_{i=1}^{3} \lambda_i(\mathbf{X}) \mathbf{P}_i, \qquad \mathbf{x} = \sum_{i=1}^{3} \lambda_i(\mathbf{X}) \mathbf{Q}_i. \tag{4.1}$$

For the deformation gradient $\mathbf{F} = \partial \mathbf{x}/\partial \mathbf{X}$, with regard to (4.1), we obtain the representation

$$\mathbf{F} = \sum_{i=1}^{3} \mathbf{Q}_i \otimes \mathbf{D}_i, \tag{4.2}$$

where $\mathbf{D}_i := \partial \lambda_i / \partial \mathbf{X}$ and the vectors $\mathbf{D}_i$ are completely determined by the geometry of the triangle T_P,

$$\mathbf{D}_i = \frac{1}{2A_p} (\mathbf{P}_{i+1} - \mathbf{P}_{i+2})^\perp, \qquad i = 1, 2, 3.$$

Here and below, we use the notation $\mathbf{P}_4 := \mathbf{P}_1$, $\mathbf{P}_5 := \mathbf{P}_2$, and $\mathbf{X}^\perp := (X_2, -X_1)^{\mathrm{T}}$ if $\mathbf{X} = (X_1, X_2)^{\mathrm{T}}$.

We use the representation (4.2) to obtain the following expressions for the right Cauchy–Green strain tensor :

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}} \mathbf{F} = \sum_{i=1}^{3} \sum_{j=1}^{3} (\mathbf{Q_i} \cdot \mathbf{Q_j}) \mathbf{D_i} \otimes \mathbf{D_j}.$$

This formula, with (2.1) taken into account, also completely determines the components of the matrix $\tilde{\mathbf{F}}$ and hence the strain characteristics ξ_1, ξ_2, ξ_3.

Since the basis functions are linear, it follows that the value of the elastic potential $\psi(\mathbf{F})$ is constant on the triangle and, by (2.2), the contribution U_p of the triangle T_P to the internal energy is given by $U_p = A_p\psi(\mathbf{G})$, where $\mathbf{G}$ is an arbitrary point of the triangle T_P.

Now, using Eq. (3.4) for the approximate solution of the problem, we can obtain equations for the new coordinates of the nodes. Consider the contribution of each triangle containing the ith node to the nodal forces. Let $\mathbf{F}_i(T_P)$ be the elastic force, and let $\mathbf{F}_{i,\mathrm{ext}}(T_P)$ be the external force at the ith node of the triangle T_P; then

$$\mathbf{F}_i(T_P) = -\frac{\partial U_p}{\partial \mathbf{Q}_i}, \qquad \mathbf{F}_{i,\mathrm{ext}}(T_P) = \int\limits_{\Gamma_\sigma^e(t)} \mathbf{t_0}\lambda_i\, dS + \int\limits_{T_Q} \mathbf{b}\lambda_i\, d\Omega,$$

where $\Gamma_\sigma^e(t)$ is the triangle edge along which the prescribed forces $\mathbf{t}_0$ are distributed. Assembling over the neighboring triangles, we obtain the static equilibrium equation for the ith node of the triangle in the form

$$\sum_{T_p\in S_i} (\mathbf{F}_i(T_p) + \mathbf{F}_{i,\mathrm{ext}}(T_P)) = 0, \tag{4.3}$$

where S_i is the set of triangles containing the ith node. We have the following assertion.

Theorem 3. *For a hyperelastic material with elastic potential $\psi(\mathbf{G}) = \psi_{QR}(\xi_1,\xi_2,\xi_3)$, the expression for the elastic forces at the ith node of the triangle has the form*

$$\mathbf{F}_i = -\frac{\partial U_p}{\partial \mathbf{Q}_i} = -A_p \sum_{s=1}^{3} \frac{\partial \psi_{QR}}{\partial \xi_s}\frac{\partial \xi_s}{\partial \mathbf{Q}_i}, \tag{4.4}$$

$$\frac{\partial \xi_1}{\partial \mathbf{Q}_i} = \frac{1}{2C_{11}}\frac{\partial C_{11}}{\partial \mathbf{Q}_i}, \tag{4.5}$$

$$\frac{\partial \xi_2}{\partial \mathbf{Q}_i} = \frac{C_{11}}{2(C_{11}C_{22} - C_{12}^2)}\left(\frac{\partial C_{22}}{\partial \mathbf{Q}_i} - 2\frac{C_{12}}{C_{11}}\frac{\partial C_{12}}{\partial \mathbf{Q}_i} + \frac{C_{12}^2}{C_{11}^2}\frac{\partial C_{11}}{\partial \mathbf{Q}_i}\right), \tag{4.6}$$

$$\frac{\partial \xi_3}{\partial \mathbf{Q}_i} = \frac{1}{C_{11}}\left(\frac{\partial C_{12}}{\partial \mathbf{Q}_i} - \frac{C_{12}}{C_{11}}\frac{\partial C_{11}}{\partial \mathbf{Q}_i}\right), \tag{4.7}$$

$$\frac{\partial C_{ij}}{\partial \mathbf{Q}_k} = \sum_{n=1}^{3} \mathbf{Q}_n(\mathbf{D_n}\otimes\mathbf{D_k} + \mathbf{D_k}\otimes\mathbf{D_n})_{ij}. \tag{4.8}$$

The derivatives $\partial\psi_{QR}/\partial\xi_i$ are completely determined by the form of constitutive equations or can be directly reconstructed from experimental data. If necessary, one can obtain analytical formulas for the derivatives $\partial\mathbf{F}_i/\partial\mathbf{Q}_j$.

The solution of the nonlinear system (4.3) can be obtained either by the classical Newton method (the corresponding Jacobian can be written out) or by the Jacobian-free Newton–Krylov method [12, 13].

A distinguishing feature of the proposed approach is its generality; namely, formulas (4.5)–(4.8) hold for any hyperelastic material whose constitutive equation can be written with the use of the QR-decomposition of the deformation gradient. The material behavior is solely determined by the partial derivatives $\partial\psi_{QR}/\partial\xi_i$ of the potential. These derivatives can be given explicitly or obtained from experimental data due to the orthogonality property (2.4). Thus, formulas (4.4)–(4.8) permit developing efficient and easy-to-implement techniques for modeling the stress–strain state of any hyperelastic material.

5. NUMERICAL EXPERIMENTS

Consider the problem of uniaxial extension of a square membrane by a force P. In this case, the deformation is

$$x_1 = \lambda_1 X_1, \qquad x_2 = \lambda_2 X_2. \tag{5.1}$$

The variables λ_1, λ_2 are called the principal elongations, because $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$, $\mathbf{X} = (X_1, X_2)^{\mathrm{T}}$.

The principal elongations λ_1 and λ_2 for various parameters of the neo-Hookean model and the Gent model calculated by formulas (4.3)–(4.8).

P, 10^5 N/cm	d	Gent model		neo-Hookean model	
		λ_1	λ_2	λ_1	λ_2
1	10	0.99246	1.00912	0.99252	1.00920
1	100	0.99178	1.00845	0.99178	1.00845
1	1000	0.99171	1.00837	0.99171	1.00838
5	10	0.96298	1.04578	0.96306	1.04665
5	100	0.95975	1.04273	0.95962	1.04291
5	1000	0.95939	1.04240	0.95927	1.04254

To perform numerical experiments, we consider isotropic models for the membrane, which were previously used in [9], namely, the neo-Hookean model and the Gent model. In the case of uniaxial extension, using the strain measure based on the QR-decomposition, one can write these models as follows: neo-Hookean model,

$$W_{NH} = \frac{\mu}{2}(\exp(2\xi_1) + \exp(2\xi_2) - 2) + \frac{\mu}{2}(d(\exp(2\xi_1 + 2\xi_2) - 1) - 2(d+1)(\exp(\xi_1 + \xi_2) - 1));$$

Gent model,

$$\begin{aligned} W_{\text{Gent}} = {} & -\frac{\mu}{2} J_m \ln\left(1 - \frac{\exp(2\xi_1) + \exp(2\xi_2) - 2}{J_m}\right) \\ & + \frac{\mu}{2}(d(\exp(2\xi_1 + 2\xi_2) - 1) - 2(d+1)(\exp(\xi_1 + \xi_2) - 1)). \end{aligned}$$

Here and below, μ, d, and J_m are material constants.

The following values for a human artery were used as the material parameters [14]: $\mu = 3 \cdot 10^6$ N/cm, $J_m = 2.3$, and $d = 10$, 10^2, 10^3. The membrane dimensions are 1 cm $\times$ 1 cm.

The principal elongations λ_1 and λ_2 obtained by solving system (4.3) for two materials are shown in the table and coincide with the finite-element solutions obtained by the approaches described in [9]. Since the exact solution (5.1) is linear, it follows that the approximation error is zero regardless of the triangulation.

6. CONCLUSION

In the present paper, we describe an approach to calculating the deformation of hyperelastic materials with the use of a strain measure based on the QR-decomposition of the deformation gradient. A distinguishing characteristic of this method is that the equations are analytical and concise, which permits a rather simple implementation of any constitutive equation for a hyperelastic material based on the upper triangle decomposition of the deformation gradient. A restriction of this approach is the use of linear finite elements in it.

Note that, in contrast to [9], where this approach was used only for isotropic materials, there no such restrictions in the present paper, and the formulas obtained here are general and can be used for both isotropic and anisotropic materials. The corresponding material anisotropy is characterized by the form of the corresponding derivatives $\partial\psi_{QR}/\partial\xi_i$ of the elastic potential.

Note that the derivatives $\partial\psi_{QR}/\partial\xi_i$ determine the mechanical behavior of a hyperelastic material and can be prescribed explicitly or obtained from experimental data due to the orthogonality property (2.4). In this case, the general form of formulas (4.4)–(4.8) permits developing efficient and easy-to-implement techniques for modeling the stress–strain state of any hyperelastic material.

In the present paper, we considered only static problems in the two-dimensional setting, but the approach can be used to solve dynamic or three-dimensional problems, which will be the topic of our further research.

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