A Nonlinear Correction FV Scheme for Near-Well Regions

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Abstract We present a finite volume method with improved well modelling for the subsurface flow simulation. The method is based on the nonlinear monotone finite volume scheme developed for diffusion, advection-diffusion and multiphase flow model equations with full anisotropic discontinuous permeability tensors on conformal polyhedral meshes. The new method uses the nonlinear (e.g. logarithmic) correction for the flux approximation in the near-well regions to utilize the singularity of the well-driven flow solution and improve accuracy of the pressure and the flux calculation. The method is applicable for anisotropic media, polyhedral grids, and different well cases including slanted, partially perforated or shifted from the grid cell center. Numerical experiments show the significant reduction of numerical errors compared to the original monotone nonlinear FV scheme with the conventional Peaceman well model or with the given analytical well rate.

Keywords Finite volume method · Improved well modelling · Nonlinear correction

MSC (2010): 35Q86 · 65M08 · 65N08

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C. Cancès and P. Omnes (eds.), *Finite Volumes for Complex Applications VIII—Hyperbolic, Elliptic and Parabolic Problems*, Springer Proceedings in Mathematics & Statistics 200, DOI 10.1007/978-3-319-57394-6_53

1 Introduction

Cell-centered finite volume methods with nonlinear flux discretization on cell faces have proven to be an effective instrument for multiphase flow modelling and attract growing attention [5]. A monotone second order method with nonlinear two-point discretization of the diffusion and convection fluxes that preserves the non-negativity of the discrete solution was presented in [2]. The method was implemented for the two- and three-phase black oil models [11] on conformal hexahedral meshes, polyhedral meshes based on dynamic octrees [14] or dynamic octrees with cut cells. The scheme was later modified [1, 9] to a nonlinear multi-point scheme which satisfies the Discrete Maximum Principle (DMP). Benefits of using the DMP scheme for two-phase flows were discussed in [10].

The latest enhancement of the nonlinear method aims to incorporate well modelling into the finite volume framework. The well model is the sensitive part of the black-oil simulator and has the largest impact on all calculated well rates and breakthrough times. The solution in the near-well region is highly influenced by the singularity (e.g. logarithmic) of the well. The idea to use the solution singularity in the FV schemes was suggested in [3]. Later this approach was combined with the nonlinear FV method for the well-oriented prismatic grids with isotropic homogeneous and heterogeneous media [4]. Our new method generalizes these ideas for anisotropic media, arbitrary polyhedral grids and arbitrary wells adjusted neither with cells centers nor with edges [8].

The central idea of the method is to use a nonlinear correction for the reconstructed solution inside the nonlinear flux discretization scheme in the near-well region. For the isotropic case the linear-logarithmic reconstruction is used. The resulting method is exact on both linear and logarithmic solutions by construction and is generalized for the anisotropic case and for slanted wells. Numerical experiments show the significant reduction of the numerical errors compared to the original nonlinear FV scheme with the conventional Peaceman well model [12] or with the given analytical well rate.

2 Original FV Method

First we consider the stationary diffusion equation in order to introduce the numerical scheme and remind the basic ideas of the FV schemes construction.

Let Ω be a three-dimensional polyhedral domain with the Lipschitz boundary $\Gamma = \Gamma_N \cup \Gamma_D$. The diffusion equation for unknown pressure p with the Dirichlet or Neumann boundary conditions is written in the mixed form:

$$\begin{aligned} \mathbf{q} = -\mathbb{K}\nabla p, \qquad \operatorname{div} \mathbf{q} &= g \quad \text{in } \Omega, \\ p &= g_D \quad \text{on } \Gamma_D \\ \mathbf{q}\cdot\mathbf{n} &= 0 \quad \text{on } \Gamma_N. \end{aligned} \tag{1}$$

Here $\mathbb{K}(\mathbf{x})$ is a symmetric positive definite (possibly anisotropic) diffusion tensor, $g(\mathbf{x})$ is a source term, $g_D(\mathbf{x})$ is a given value on the Dirichlet part of the boundary Γ_D.

The cell-centered FV scheme uses one degree of freedom per cell T, p_T, collocated at cell barycenter $\mathbf{x}_T$. Integrating the mass balance Eq. (1) over T and using the divergence theorem, we obtain:

$$\sum_{f \in \partial T} \sigma_{T,f}\, q_f |f| = \int_T g \, \mathrm{d}x, \qquad q_f = \frac{1}{|f|} \int_f \mathbf{q} \cdot \mathbf{n}_f \, \mathrm{d}s, \tag{2}$$

where $q_f|f|$ is the normal flux across the face, $|f|$ is the area of face f, and $\sigma_{T,f}$ is either 1 or -1 depending on the mutual orientation of the unit normal vectors $\mathbf{n}_f$ and $\mathbf{n}_T$ ($\mathbf{n}_T$ denotes the outward normal vector for T).

Possible approaches for the flux (2) discretization include the nonlinear monotone two-point scheme [2] and the nonlinear DMP preserving compact multi-point scheme [1, 9]. In the next chapter we present a multi-point scheme designed for the near-well regions.

3 Near-Well Correction Scheme

Consider an isolated well which generates pressure singularity (see Fig. 1). The central idea of the nonlinear correction finite volume (NCFV) method is to select some region around the well and modify the FV scheme (following [3, 4]) to utilize the singularity and take into account the nonlinear component of the solution. In contrast to [4], our method is designed for anisotropic media, arbitrary polyhedral cells and arbitrary well location.

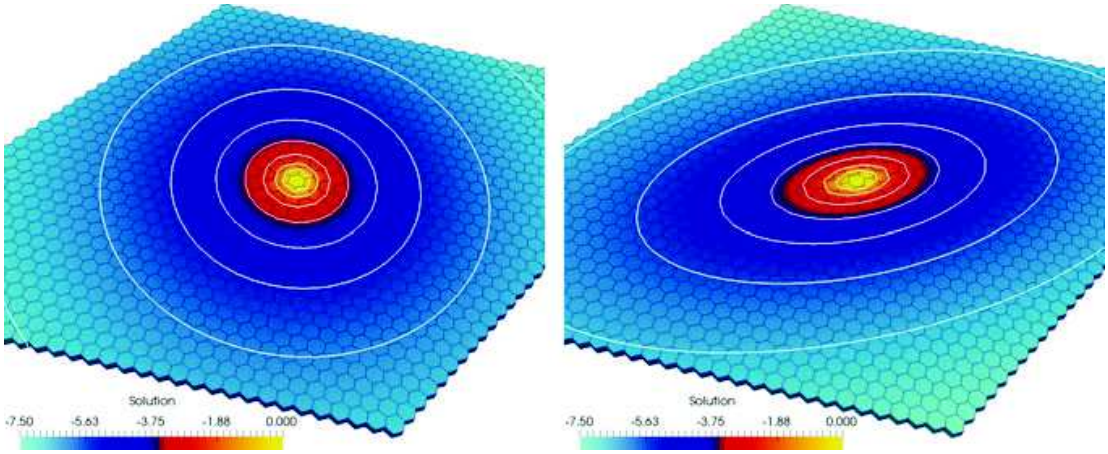

Fig. 1 Example of singularity in the near-well region: isotropic (*left*) and anisotropic (*right*) media

The original nonlinear FV method uses the piecewise linear reconstruction of the unknown field for flux calculation. The NCFV method takes into account the nonlinear component of the solution near the specific objects such as wells or large fractures.

We consider the pressure field to be the sum of the linear and nonlinear functions for each cell in a near-well region:

$$p_T = \underbrace{a\,x + b\,y + c\,z + d}_{p_{lin}} + \underbrace{e\,F(x, y, z)}_{p_F}, \tag{3}$$

where $F(x, y, z)$ is a function representing the singularity.

The finite volume discretization requires the mean value of the normal component of the flux $\mathbf{q} = -\mathbb{K}\nabla p$ to be calculated for each face f of T:

$$\int_f \mathbf{q} \cdot \mathbf{n}_f dS = -\int_f (\mathbb{K}\nabla p_T) \cdot \mathbf{n}_f dS = -\int_f (\mathbb{K}\nabla p_{lin}) \cdot \mathbf{n}_f dS - \int_f (\mathbb{K}\nabla p_F) \cdot \mathbf{n}_f dS. \tag{4}$$

Since the method is derived for arbitrary grid cells and well direction, we consider the diagonal permeability tensor $\mathbb{K} = diag\left(k_x, k_y, k_z\right)$ for clarity. Using (3) for the integral (4) gives:

$$\begin{aligned} q_f = -\frac{1}{|f|}\int_f \mathbf{q} \cdot \mathbf{n}_f dS = a k_x S_{fx} + b k_y S_{fy} + c k_z S_{fz} + e \int_f (\mathbb{K}\nabla F(x, y, z)) \cdot \mathbf{n}_f dS \\ = a\boldsymbol{\ell}_1 + b\boldsymbol{\ell}_2 + c\boldsymbol{\ell}_3 + e\boldsymbol{\ell}_4. \end{aligned} \tag{5}$$

The coefficients $\boldsymbol{\ell}_i$ depend solely on the mesh and problem data and are calculated explicitly, while the coefficients (a, b, c, e) are recovered from the solution in a set of neighboring cells.

Let T_+ and T_- be neighboring cells sharing a face f, and $\mathbf{x}_+, \mathbf{x}_-$ denote the centers of these cells. We take four points $\mathbf{x}_i$ ($\mathbf{x}_i \neq \mathbf{x}_+$) that denote centers of the neighboring cells or faces of T_+ and call four vectors $\mathbf{t}_i = \mathbf{x}_i - \mathbf{x}_+$ a *quadruplet*. The points are chosen as described below.

Considering the same representation (3) for vectors of quadruplet gives us:

$$\begin{pmatrix} p_1 - p_+ \\ p_2 - p_+ \\ p_3 - p_+ \\ p_4 - p_+ \end{pmatrix} = \begin{pmatrix} x_1 - x_+ & y_1 - y_+ & z_1 - z_+ & F_1 - F_+ \\ x_2 - x_+ & y_2 - y_+ & z_2 - z_+ & F_2 - F_+ \\ x_3 - x_+ & y_3 - y_+ & z_3 - z_+ & F_3 - F_+ \\ x_4 - x_+ & y_4 - y_+ & z_4 - z_+ & F_4 - F_+ \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix}, \tag{6}$$

where $p_i = p(\mathbf{x}_i)$, $p_+ = p(\mathbf{x}_+)$ and $F_i = F(x_1, y_1, z_1)$.

From the set of admissible quadruplets we choose the one with the largest matrix (6) determinant. Solving it provides the coefficients a_+, b_+, c_+, e_+ for the cell T_+:

$$a_+ = \sum_j (p_j - p_+)\, m_{1,j}, \qquad b_+ = \sum_j (p_j - p_+)\, m_{2,j},$$
$$c_+ = \sum_j (p_j - p_+)\, m_{3,j}, \qquad e_+ = \sum_j (p_j - p_+)\, m_{4,j}, \tag{7}$$

where $m_{i,j}$ are the elements of the inverse matrix from (6). Taking T_- instead of T_+ and considering $-\mathbf{q}\cdot\mathbf{n}_f$ provides us the second flux approximation.

Applying (7) to Eq. (5) gives us:

$$q_+ = -\int_f \mathbf{q}\cdot\mathbf{n}_f dS = \Big[\ell_1 \sum_j (p_j - p_+)\, m^+_{1,j} + \ell_2 \sum_j (p_j - p_+)\, m^+_{2,j} + \ell_3 \sum_j (p_j - p_+)\, m^+_{3,j} + \ell_4 \sum_j (p_j - p_+)\, m^+_{4,j}\Big] = \tag{8}$$
$$\Big[\sum_j p_j \underbrace{\sum_i \ell_i\, m^+_{i,j}}_{k^+_j} - p_+ \sum_j \underbrace{\sum_i \ell_i\, m^+_{i,j}}_{k^+_j}\Big] = \Big(\sum_j k^+_j\ (p_j - p_+)\Big).$$

in similar way we get

$$q_- = -\Big(\sum_j k^+_j\ (p_j - p_-)\Big). \tag{9}$$

The resulting flux approximation is obtained as the weighted sum of q_+ and q_- with coefficients $\mu_+ + \mu_- = 1$. The weights can be chosen to ensure specific features of the solution. In our numerical experiments we considered $\mu_+ = \mu_- = 1/2$ which resulted in the *linear* multi-point flux discretization:

$$q_f = \mu_+ \Big(\sum_j k^+_j\ (p_j - p_+)\Big) + \mu_- \Big(\sum_{j'} k^-_{j'} \cdot (p_{j'} - p_-)\Big). \tag{10}$$

Note: Different cases of anisotropic media and non-trivial wells including slanted or partially perforated are handled by choosing an appropriate singularity function $F(x, y, z)$. For the anisotropic case a special F from [13] can be used, while for more complex cases one can implement techniques presented in [7]. For the wells not passing through the grid cell center we use two collocation points for the well cell (the one in the cell center and an additional point on the well), which provides one additional equation and allows to avoid using the conventional Peaceman formula for the well flux (see [8] for more details).

4 Numerical Experiments

Here we consider three numerical experiments for the near-well nonlinear correction scheme (NCFV) compared with the original monotone nonlinear FV scheme (NFV) with conventional Peaceman well model [12] or direct analytical flux to the well cell. For tests 1 and 3 the permeability tensor is scalar $\mathbb{K} = \mathbb{I}$ and test 2 deals with the anisotropic media. More general cases are presented in [8].

Defining the well pressure and flux gives us the analytical solution in the domain. For our experiments we put the Dirichlet conditions on the domain boundaries and use either given well pressure or given well flux from the analytical solution.

If the analytical rate for the well cell is given, we can compare the NFV scheme and the NCFV scheme without the influence of the well cell model. In this case we compute relative L^2-norms for the numerical pressure field errors of the NFV and NCFV schemes compared to known analytical solution: $err(p)_{NFV,anl}$ and $err(p)_{NCFV,anl}$, respectively.

If the well pressure is given, we use the numerical model for the well cell. Peaceman formula is applicable only for the cubic grids and is used with the NFV scheme, while the NCFV scheme is used for all experiments. In this case we compute relative L^2-norm for the pressure error for the NFV scheme + Peaceman ($err(p)_{NFV,pcm}$) and for the NCFV scheme ($err(p)_{NCFV}$), and the errors between the numerical well rate of the NFV and NCFV schemes and the analytical rate ($err(q)_{NFV}$ and $err(q)_{NCFV}$, respectively).

4.1 Test 1: Single Shifted Well, Hexagonal Prismatic Grid

For the first experiment we use one layer of the regular hexagonal prismatic grid. The well is shifted from the well cell centroid along the vector $v = (1, 1, 0)$ by the value $\alpha \cdot d/2$, where d is the cell diagonal length.

Table 1 shows the relative L^2-norms of pressure error for the NFV scheme with the analytical well cell rate and for the NCFV scheme.

Table 1 Solution relative errors for the NFV scheme and the near-well correction method for shifted well on hexagonal prismatic grid

α	$err(p)_{NFV,anl}$	$err(p)_{NCFV}$
0	1.1e-4	8.2e-11
0.1	1.4e-3	2.0e-11
0.3	4.2e-3	1.0e-11
0.5	7.1e-3	1.1e-11

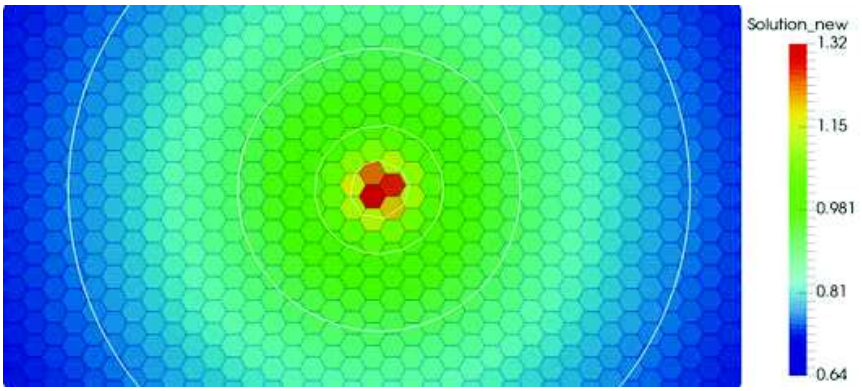

Fig. 2 Solution for the NCFV scheme for shifted well on hexagonal prismatic grid, $\alpha = 0.5$

Table 2 Solution error for the NFV and the NCFV, and the flux error for the NCFV scheme. 3D anisotropic case with the 60° slanted well

$err(p)_{NFV,anl}$	$err(p)_{NCFV}$	$err2(p)_{NFV,anl}$	$err2(p)_{NCFV}$	$err(q)_{NCFV}$
3.8e-6	2.5e-10	1.9e-2	1.2e-6	2.9e-5

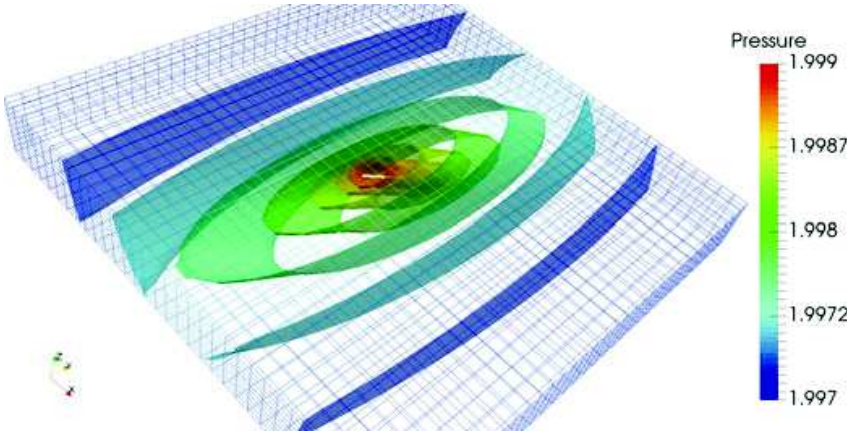

Fig. 3 Analytical solution for 3D anisotropic case with the 60° slanted well

Any well index based method incorporating the well within a single cell, will not provide the non-symmetric solution by construction. In contrast, the NCFV scheme can reproduce a non-symmetric solution (see Fig. 2).

4.2 Test 2: Slanted Well in 3D Anisotropic Media

Now we consider the slanted well in 3D rotated by 60° from the vertical. The tensor is diagonal anisotropic: $\mathbb{K} = diag\{10,\ 100,\ 1\}$. The orthogonal grid has 10 layers and Dirichlet boundary conditions are given for all boundaries (Fig. 3) .

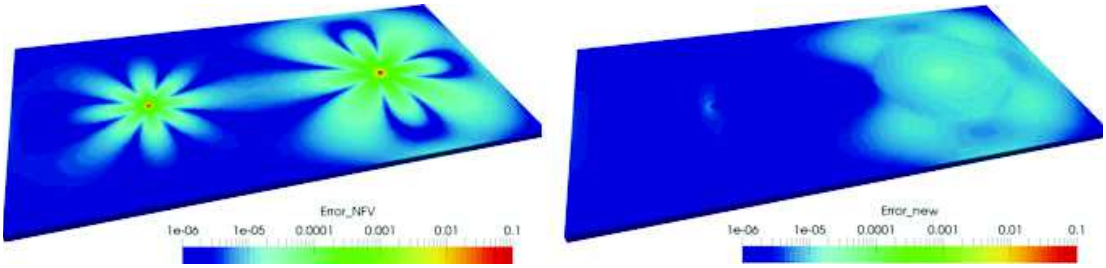

Fig. 4 Relative error for the NFV scheme with Peaceman well model (*left*) and the NCFV scheme (*right*) in the log-scale. Cubic grid $134 \times 67 \times 1$

The pressure and flux errors for this case are presented in Table 2. Due to the high anisotropy the solution variation is very small: $p \in [1.997,\ 1.999]$. To capture the error compared to this variation, we introduce $err2(p)_*$, which is the relative error normalized by $||p_{anl} - p_{anl,min}||$.

4.3 Test 3: Two Vertical Wells, Cubic Grid

The second experiment deals with two wells in the box domain with a cubic grid. The well rates are $q_1 = 1$, $q_2 = 4$ and the analytical solution suggested in [6] is defined by fixing the pressure in the middle point between two wells.

Table 3 shows the relative errors for the NFV and the NCFV scheme for the analytical well rates, relative errors for pressure and well rates (the first and the second well) for the numerical well models: NFV + Peaceman and the NCFV scheme.

Figure 4 presents the error fields for two methods in the log-scale. The NFV scheme reduces to the standard TPFA for this case and the cubic grid is ideal for the Peaceman method. The largest error of the NFV scheme is concentrated in regions around the wells that are covered by the near-well regions of the new method. The NCFV scheme gives considerably smaller errors than the conventional method.

5 Conclusion

We present the near-well nonlinear correction FV scheme applicable for the general case of anisotropic media, polyhedral grids and arbitrarily oriented wells including slanted, shifted and partially perforated cases.

Numerical experiments show the significant reduction of the numerical errors compared to the original monotone nonlinear FV scheme with the conventional Peaceman well model or with the given analytical well rate.

Acknowledgements This work has been supported in part by RFBR grants 15-35-20991, 17-01-00886, Russian President Grant MK-2951.2017.1, and ExxonMobil Upstream Research Company.

Table 3 Solution relative errors and flux errors for q_1 and q_2 for the problem with two wells

$100/h$	$err(p)_{NFV,anl}$	$err(p)_{NCFV,anl}$	$err(p)_{NFV,pcm}$	$err(p)_{NCFV}$	$err(q)_{NFV}$		$err(q)_{NCFV}$	
33	1.2e-2	2.8e-5	1.2e-2	2.8e-5	4.6e-3	1.9e-2	2.1e-5	4.1e-5
67	5.1e-3	7.0e-6	5.2e-3	7.6e-6	4.6e-3	1.9e-2	2.3e-5	5.4e-5
99	3.1e-3	3.2e-6	3.1e-3	4.1e-6	4.6e-3	1.8e-2	2.0e-5	7.0e-5

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