A multiscale method for linear heterogeneous poroelasticity

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Outline

- Linear Poroelasticity
- Classical FEM
- Multiscale Methods
- Practical Aspects
- Numerical Example
- Outlook: Semi-Explicit Time Discretization
- Conclusion

Linear Poroelasticity

- Deformation of porous media (heterogeneous) saturated by an incompressible viscous fluid
- Couples fluid flow with behavior of surrounding solid
- Quasi-static: internal equilibrium preserved at any time
- Applications:
  - geomechanical modeling (reservoir engineering)
  - human anatomy for medical applications
  - similar to linear thermoelasticity
- Variables:
  - displacement u, pressure p
  - both averaged across (infinitesimal) cubic elements
  - both treated as variables on the entire domain
    $\rightarrow$ 3 scales!

Linear Poroelasticity

- On a Lipschitz domain $D \subset \mathbb{R}^d$,

$$
\begin{aligned}
-\nabla \cdot \big(\sigma(u)\big) + \nabla(\alpha p) &= 0, \\
\partial_t\Big(\alpha \nabla \cdot u + \frac{1}{M} p\Big) - \nabla \cdot \Big(\frac{\kappa}{\nu} \nabla p\Big) &= f
\end{aligned}
$$

- Initial condition for p, boundary conditions
- Heterogeneous media $\rightarrow$ oscillatory μ, λ, κ, α

Linear Poroelasticity

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- Initial condition for p, boundary conditions
- Heterogeneous media $\rightarrow$ oscillatory μ, λ, κ, α
- Variational form:

$$\begin{aligned} a(u, v) - d(v, p) &= 0, \\ d(\partial_t u, q) + c(\partial_t p, q) + b(p, q) &= (f, q), \end{aligned}$$

for all $v \in V := \big[H_0^1(D)\big]^d$, $q \in Q := H_0^1(D)$

Linear Poroelasticity

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- Initial condition for p, boundary conditions
- Heterogeneous media $\to$ oscillatory μ, λ, κ, α
- Bilinear forms $a\colon V \times V \to \mathbb{R}$, $b, c\colon Q \times Q \to \mathbb{R}$, and $d\colon V \times Q \to \mathbb{R}$ defined by

$$a(u,v) := \int_D \sigma(u) : \varepsilon(v)\,\mathrm{d}x, \qquad b(p,q) := \int_D \frac{\kappa}{\nu} \nabla p \cdot \nabla q\,\mathrm{d}x,$$

$$c(p,q) := \int_D \frac{1}{M} pq\,\mathrm{d}x, \qquad d(u,q) := \int_D \alpha\,(\nabla \cdot u) q\,\mathrm{d}x$$

Classical FEM

- V_h, Q_h: piecewise affine finite element spaces on $\mathcal{T}_h$
- Backward Euler in time with uniform time steps τ
- Discrete derivative $D_\tau u_h^n := (u_h^n - u_h^{n-1})/\tau$
- For $n \in \{1, \cdots, N\}$ find $u_h^n \in V_h$ and $p_h^n \in Q_h$ such that

$$\begin{aligned} a(u_h^n, v) - d(v, p_h^n) &= 0, \\ d(D_\tau u_h^n, q) + c(D_\tau p_h^n, q) + b(p_h^n, q) &= (f^n, q) \end{aligned}$$

  for all $v \in V_h$ and $q \in Q_h$
- Properties:
  - unique solution at each time step
  - stability estimates as in [MalP17]

Classical FEM

Theorem ([ErnM09])
For coefficients $\mu, \lambda, \kappa, \alpha \in W^{1,\infty}(D)$ the error of the exact solution (u, p) and the fully discrete solution (u_h^n, p_h^n) is bounded by

$$\|u(t_n) - u_h^n\|_1 + \|p(t_n) - p_h^n\| \leq C_\epsilon h + C\tau,$$

where the constant C_ϵ scales with the maximum of $\|\mu\|_{W^{1,\infty}(D)}$, $\|\lambda\|_{W^{1,\infty}(D)}$, $\|\kappa\|_{W^{1,\infty}(D)}$, and $\|\alpha\|_{W^{1,\infty}(D)}$.

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- Expected convergence rate $h + \tau$
- Involved constant C_ϵ scales with ϵ^{-1} in oscillatory media with period ϵ

  $\rightarrow$ unfeasible approach

Multiscale Spaces and Projections

- Larger mesh size $H > h$, $\mathcal{T}_h$ refinement of $\mathcal{T}_H$
- Defines $V_H \subseteq V_h$ and $Q_H \subseteq Q_h$
- Goals:
  - construct new discrete function space with the same dimension as $V_H \times Q_H$
  - better approximation properties
  - *localized orthogonal decomposition* (LOD) [MaIP14]

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  - *localized orthogonal decomposition* (LOD) [MaIP14]
- Quasi-interpolation operator $I_H \colon V_h \to V_H$

$$H_K^{-1}\|v - I_H v\|_{L^2(K)} + \|\nabla I_H v\|_{L^2(K)} \lesssim \|\nabla v\|_{L^2(\omega_K)}$$

  - example: local L^2-projection + averaging in nodes
  - analogously $I_H \colon Q_h \to Q_H$

Multiscale Spaces and Projections

- Fine scale spaces

$$V_{\mathrm{fs}} := \{v \in V_h : I_H v = 0\}, \quad Q_{\mathrm{fs}} := \{q \in Q_h : I_H q = 0\}$$

- Leads to the decomposition

$$V_h = V_H \oplus V_{\mathrm{fs}}, \qquad Q_h = Q_H \oplus Q_{\mathrm{fs}}$$

- Correctors $\mathcal{C}^1_{\mathrm{fs}} \colon V_h \to V_{\mathrm{fs}}$, $\mathcal{C}^2_{\mathrm{fs}} \colon Q_h \to Q_{\mathrm{fs}}$

$$a(\mathcal{C}^1_{\mathrm{fs}} u, v) = a(u, v), \quad b(\mathcal{C}^2_{\mathrm{fs}} p, q) = b(p, q), \quad v \in V_{\mathrm{fs}},\ q \in Q_{\mathrm{fs}}$$

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- New finite element spaces

$$V_{\mathrm{ms}} := \{v_H - \mathcal{C}^1_{\mathrm{fs}} v_H : v_H \in V_H\},$$
$$Q_{\mathrm{ms}} := \{q_H - \mathcal{C}^2_{\mathrm{fs}} q_H : q_H \in Q_H\}$$

and a/b-orthogonal decompositions

$$V_h = V_{\mathrm{ms}} \oplus V_{\mathrm{fs}}, \qquad Q_h = Q_{\mathrm{ms}} \oplus Q_{\mathrm{fs}}$$

First Multiscale Method

- Multiscale method from thermoelasticity [MaIP17]
- Idea: consider stationary system

  $\rightarrow$ solve time-independent corrector problem
- Problems:
  - two coupled correctors
  - need additional fine scale correction
- Find $\tilde{u}^n_{\mathrm{ms}} = u^n_{\mathrm{ms}} + u^n_{\mathrm{fs}} \in V_{\mathrm{ms}} \oplus V_{\mathrm{fs}}$, $\tilde{p}^n_{\mathrm{ms}} \in Q_{\mathrm{ms}}$ such that

$$\begin{aligned} a(\tilde{u}^n_{\mathrm{ms}}, v) - d(v, \tilde{p}^n_{\mathrm{ms}}) &= 0, \\ d(D_\tau \tilde{u}^n_{\mathrm{ms}}, q) + c(D_\tau \tilde{p}^n_{\mathrm{ms}}, q) + b(\tilde{p}^n_{\mathrm{ms}}, q) &= (f^n, q), \\ a(u^n_{\mathrm{fs}}, w) + d(w, \tilde{p}^n_{\mathrm{ms}}) &= 0 \end{aligned}$$

for $v \in V_{\mathrm{ms}}$, $q \in Q_{\mathrm{ms}}$, and $w \in V_{\mathrm{fs}}$

Alternative Multiscale Method

- New approach: exploit saddle point structure
- Advantages:
  - decoupled corrector problems
  - no additional fine scale correction
  - independent of the bilinear form d (and thus α)
- Find $u_{\mathrm{ms}}^n \in V_{\mathrm{ms}}$ and $p_{\mathrm{ms}}^n \in Q_{\mathrm{ms}}$ such that

$$\begin{aligned} a(u_{\mathrm{ms}}^n, v) - d(v, p_{\mathrm{ms}}^n) &= 0, \\ d(D_\tau u_{\mathrm{ms}}^n, q) + c(D_\tau p_{\mathrm{ms}}^n, q) + b(p_{\mathrm{ms}}^n, q) &= (f^n, q) \end{aligned}$$

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Theorem

For consistent initial data $u_h^0 \in V_h$, $p_h^0 \in Q_h$ and $u_{ms}^0 \in V_{ms}$, $p_{ms}^0 \in Q_{ms}$, the error of the multiscale solution compared to the fine scale solution satisfies

$$\|u_h^n - u_{ms}^n\|_1 + \|p_h^n - p_{ms}^n\|_1 \lesssim H\,\text{data}^n + t_n^{-1/2} H\,\|p_h^0\|_1$$

Practical Aspects

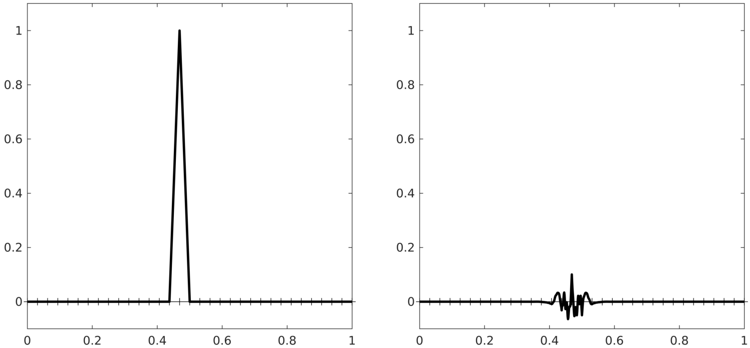
- corrector problems are only computed for basis $\{\Lambda_j\}_{j=1}^M$ of V_H:
$$a(\mathcal{C}_{\mathrm{fs}}^1 \Lambda_j, w) = a(\Lambda_j, w)$$
for all $w \in V_{\mathrm{fs}}$

  $\rightarrow$ support possibly global!

- problems are localized on patches

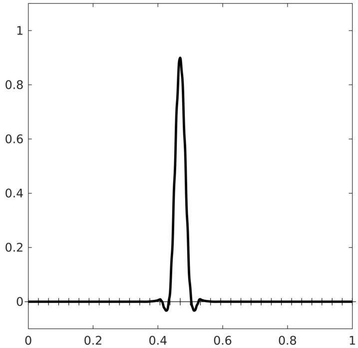
  $\rightarrow$ same error rate with patch size $|H \log H|$

Illustration in $1D$

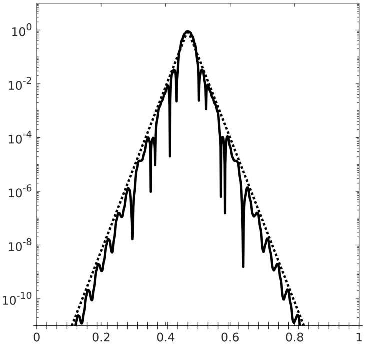


Hat function Λ Correction $\mathcal{C}_{\mathsf{fs}}\Lambda$

Illustration in $1D$



$(1 - \mathcal{C}_{\mathsf{fs}})\Lambda$



$|(1 - \mathcal{C}_{\mathsf{fs}})\Lambda|$

Numerical Example

- Parameters

$$D = (0,1)^2, \quad T = 1, \quad \tau = 0.01, \quad h = \sqrt{2} \cdot 2^{-8} \approx 0.0055$$

- Reference solution on uniform fine grid $\mathcal{T}_h$
- Error measure

$$\|(v,q)\|_{D,N}^2 := \sum_{i=1}^{N} \tau \left(\|\nabla v^i\|^2 + \|\nabla q^i\|^2 \right)$$

- Piecewise constant parameters on $\mathcal{T}_\epsilon$,

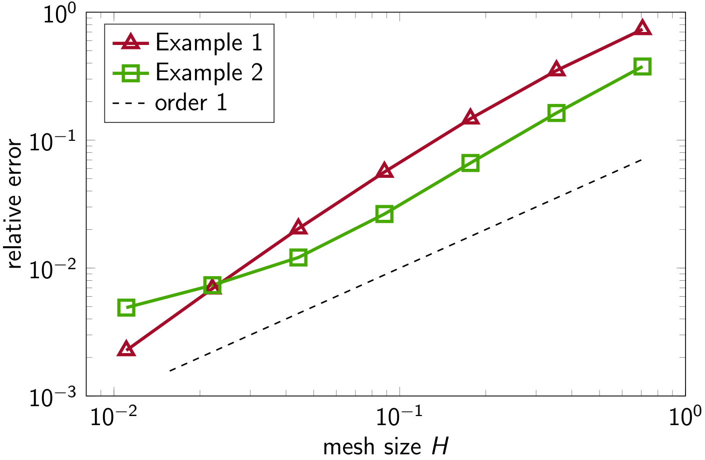
$$\mu \sim U[32.2, 62.2], \qquad \lambda \sim U[41, 61],$$
$$\kappa \sim U[0.1, 0.12], \qquad \alpha \sim U[0.5, 1]$$

Numerical Example

- Example 1
  - $f = 1$
  - initial data $p^0(x) = (1 - x_1)\, x_1\, (1 - x_2)\, x_2$
  - hom. Dirichlet boundary conditions for p
  - mixed boundary conditions for u

- Example 2
  - random f with values between 0 and 1
  - initial data $p^0(x) = (1 - x_2)\, x_2$
  - hom. Dirichlet boundary conditions on the top,
    hom. Neumann boundary conditions otherwise

Numerical Example



Outlook: Semi-Explicit Scheme

Implicit discretization:

$$\begin{aligned} a(u_h^n, v) - d(v, p_h^{\mathbf{n}}) &= 0, \\ d(D_\tau u_h^n, q) + c(D_\tau p_h^n, q) + b(p_h^n, q) &= (f^n, q) \end{aligned}$$

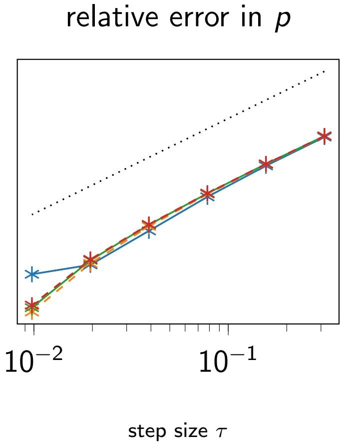
Semi-Explicit discretization under weak coupling [AltMU19]:

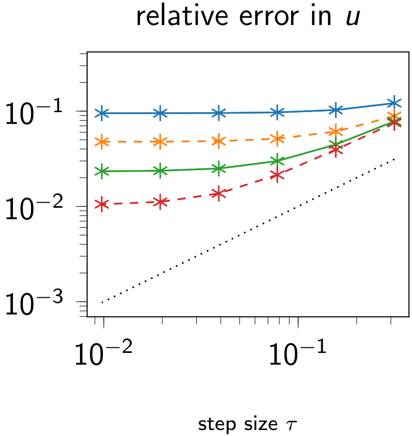
$$\begin{aligned} a(u_h^n, v) - d(v, p_h^{\mathbf{n-1}}) &= 0, \\ d(D_\tau u_h^n, q) + c(D_\tau p_h^n, q) + b(p_h^n, q) &= (f^n, q) \end{aligned}$$

Benefit: decoupled equations, faster computations

$\rightarrow$ combine this with spatial numerical homogenization!

Illustration: Energy Errors





lines $\hat{=}$ semi-explicit, stars $\hat{=}$ implicit scheme, $h = 2^{-\{5,6,7,8\}}$

Conclusion

- Linear heterogeneous poroelasticity: 3 scales
- Standard FEM approach unfeasible
- New multiscale method
- Decoupled corrector problems, independent of α
- Practical implementations include localization
- Optimal first-order convergence theoretically and numerically

Future work: semi-explicit schemes

THANK YOU FOR YOUR ATTENTION!

http://scicomp.math.uni-augsburg.de

References

[AltCM+19] R. Altmann, E. Chung, R. Maier, D. Peterseim, and S.-M. Pun.
Computational multiscale methods for linear heterogeneous poroelasticity.
Accepted for publication in *J. Comput. Math.*, 2019.

[AltMU19] R. Altmann, R. Maier, and B. Unger.
Semi-explicit discretization schemes for weakly-coupled elliptic-parabolic problems.
ArXiv Preprint, 1909.03497, 2019.

[ErnM09] A. Ern and S. Meunier.
A posteriori error analysis of Euler-Galerkin approximations to coupled elliptic-parabolic problems.
M2AN Math. Model. Numer. Anal., 43(2):353–375, 2009.

[MalP17] A. Målqvist and A. Persson.
A generalized finite element method for linear thermoelasticity.
M2AN Math. Model. Numer. Anal., 51(4):1145–1171, 2017.

[MalP14] A. Målqvist and D. Peterseim.
Localization of elliptic multiscale problems.
Math. Comp., 83(290):2583–2603, 2014.