Subsurface flow with a phreatic surface: Simulation in aquifers with the complicated hydrogeological structure

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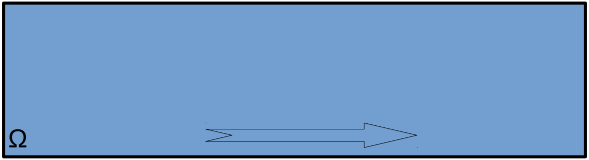
Contents

Model



ω (mass fraction of the salt), p (hydrodynamic pressure).

$$\left.\begin{array}{c} \partial_t(\Phi\rho\omega) + \nabla\cdot(\rho\omega\mathbf{q} - \rho\mathbf{D}\nabla\omega) = 0 \\ \partial_t(\Phi\rho) + \nabla\cdot(\rho\mathbf{q}) = 0. \end{array}\right\} \quad \mathbf{q} = -\frac{\mathbf{K}}{\mu}(\nabla p - \rho\mathbf{g}),$$

where $\rho = \rho(\omega)$, $\mu = \mu(\omega)$.

+ bnd cond. for ω and p + init. cond. for ω

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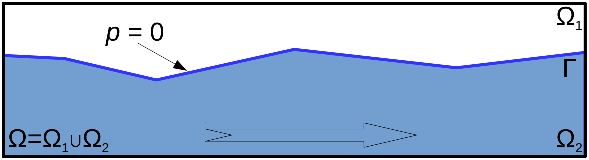

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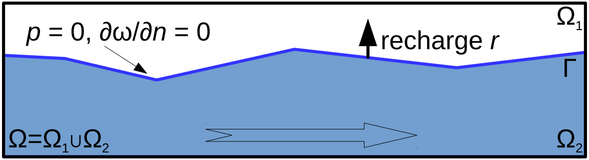
The **phreatic surface** separates
the **saturated** (Ω_2) and **unsaturated** (Ω_1) subdomains

Model



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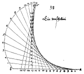
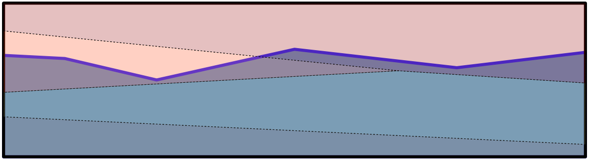
+ velocity of the phreatic surface: $\tilde{\mathbf{q}} := (\mathbf{q} + r\mathbf{e}_z)/\Phi$



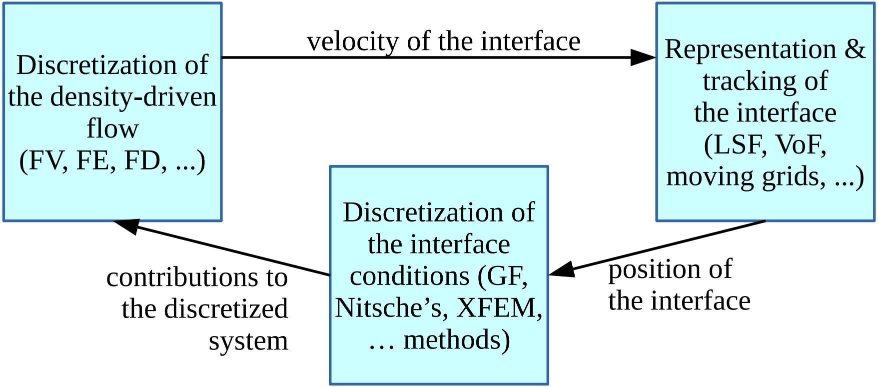
Treatment of the moving boundary



Only one (top) moving boundary, often almost horizontal and near the top of the domain motivates the “moving grids” for the lower subdomain.

but

The layered structure of the domain motivates a fixed grid and a separate treatment of the interface by Level Set or VOF methods.


velocity of the interface
Discretization of the density-driven flow (FV, FE, FD, ...)
Representation & tracking of the interface (LSF, VoF, moving grids, ...)
Discretization of the interface conditions (GF, Nitsche’s, XFEM, … methods)
contributions to the discretized system
position of the interface

Numerical methods

Implementation: Application $\mathbf{d^3f}$ based on the *ug4*-toolbox (https://github.com/UG4/ugcore)

Methods for the groundwater flow:

- Unstructured grid
- Spatial discretization: a vertex-centered collocated FV scheme, linear and multi-linear shape functions.
- Time stepping (in presented examples): implicit Euler scheme
- Discretized non-linear system in time steps: Newton's method
- Linearized systems: BiCGStab, preconditioned with GMG; smoothers: ILU

Interface tracking: a Level-Set method on the same grid

BC at the interface: a Ghost-Fluid method

Level-set representation of the boundary

LSF ψ: $\Gamma = \{x : \psi(x) = 0\}$, $\psi(x) > 0$ in Ω_1, $\psi(x) < 0$ in Ω_2.

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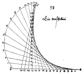
Level-set equation for velocity $\mathbf{u} : \Omega \to \mathbb{R}^3$:

$$\psi_t + \mathbf{u} \cdot \nabla \psi = 0$$

where $\mathbf{u}|_\Gamma = \tilde{\mathbf{q}}|_\Gamma$ should be the velocity of the moving boundary.



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Level-set equation for normal velocity $u_n : \Omega \to \mathbb{R}$:

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Coupling with the flow: $u_n = \tilde{\mathbf{q}} \cdot \frac{\nabla \psi}{|\nabla \psi|}$ on Γ.

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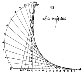
Initial condition for ψ: The **signed distance function**.



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Caution: Inappropriate choice of the extension of u_n causes numerical problems (in $\nabla \psi$, CFL etc.)

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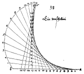
- Compute the **extension** of $u_n = \tilde{\mathbf{q}} \cdot \frac{\nabla \psi_{\mathrm{old}}}{|\nabla \psi_{\mathrm{old}}|}$ from Γ_{old} to Ω.



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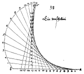
- Solve the level-set equation with u_n: get ψ_{new} from ψ_{old}. This defines Γ_{new} and $\Omega_{2,\mathrm{new}}$.



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- The CFL number of the interface tracking is **determined only by the normal velocity at Γ**.

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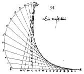

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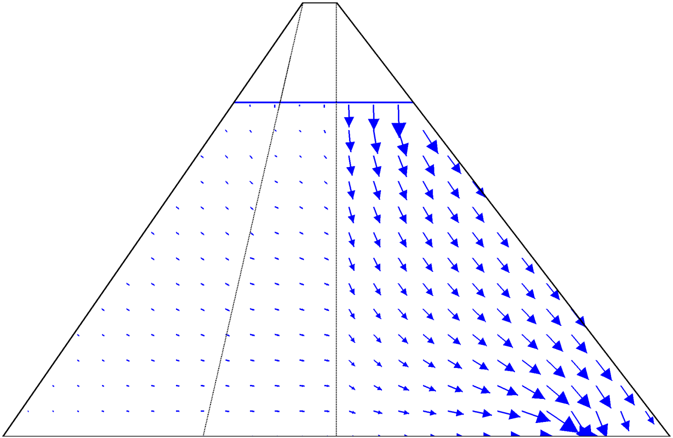
- Advantages: **unstructured** grids, **parallelization**.

Illustrative 2D Example



Phreatic surface and flow in a dam (no salt)

Recomputation of an example from P. Frolkovic, DOI: 10.1016/j.advwatres.2012.06.013

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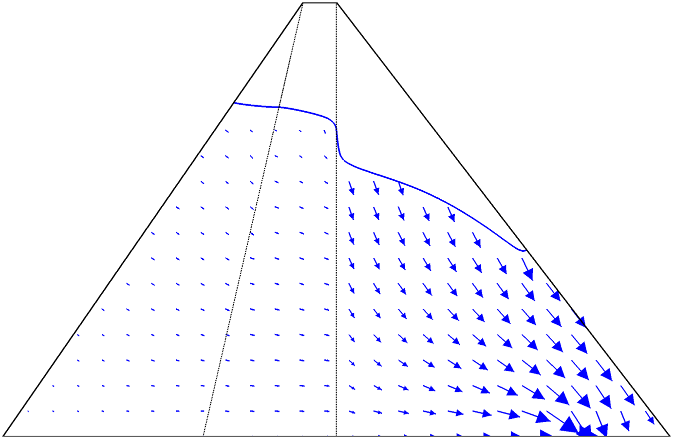

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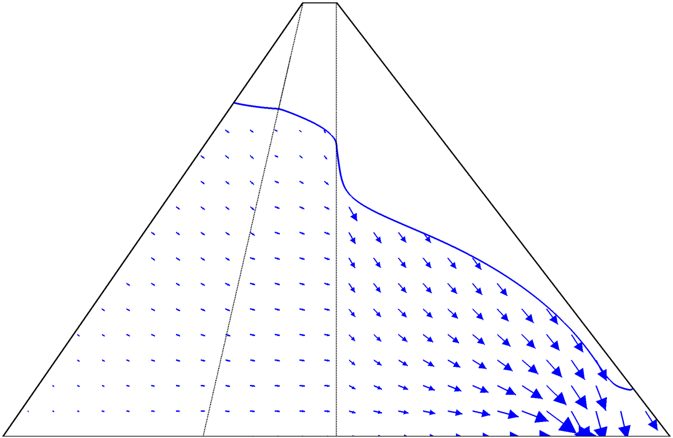

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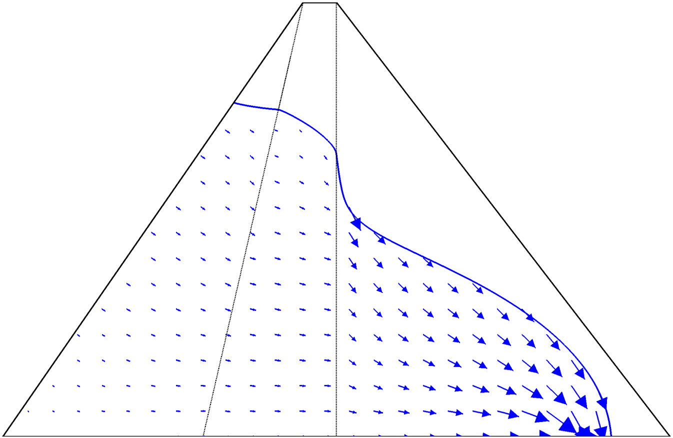

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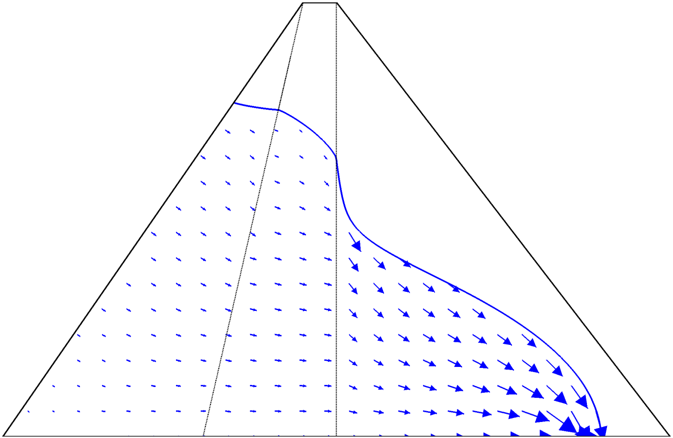

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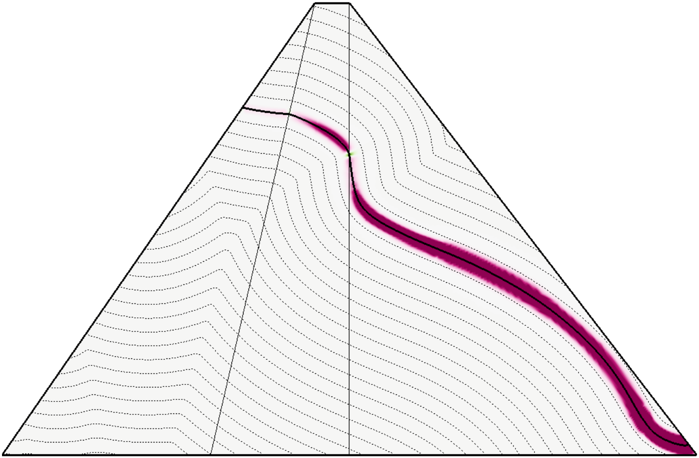
Extended normal velocity



LSF ≈ SDF (dotted lines) and the extended normal velocity (color)

Discretization

These equations have the form

$$\phi_t + \frac{\nabla\hat{\psi}}{|\nabla\hat{\psi}|} \cdot \nabla\phi = Q$$

for $\phi = \hat{\psi}$ or $\phi = u_n$ (with $\phi, Q : \Omega \to \mathbb{R}$).

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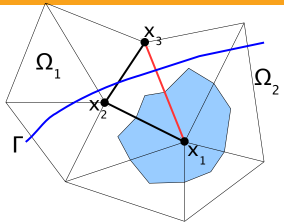
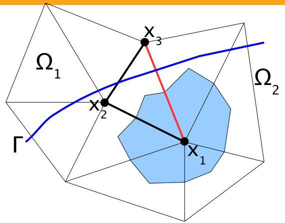
Currently applied method: An explicit FV scheme based on the high-resolution flux-based level set method by P. Frolkovič and K. Mikula with a special treatment of Q. (A 2nd order scheme, explicit in time. A 1st order scheme available.)

Boundary conditions at the interface:

- Extrapolation from the inner to the outer corners in the cut elements
- Otherwise use of the regular spatial FV discretization as for inner elements



Ghost-fluid method interface velocity



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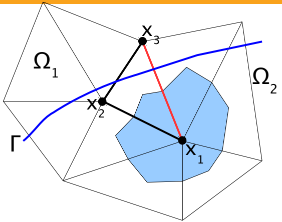
Computation of $\mathbf{q}$ at the interface:

- Accuracy of the velocity evaluated in the cut elements depends on the extrapolation significantly.
- ∇p: Special treatment in the multi-linear elements, $[\nabla p]_E = \mathrm{const.}$

Ghost-fluid method interface velocity



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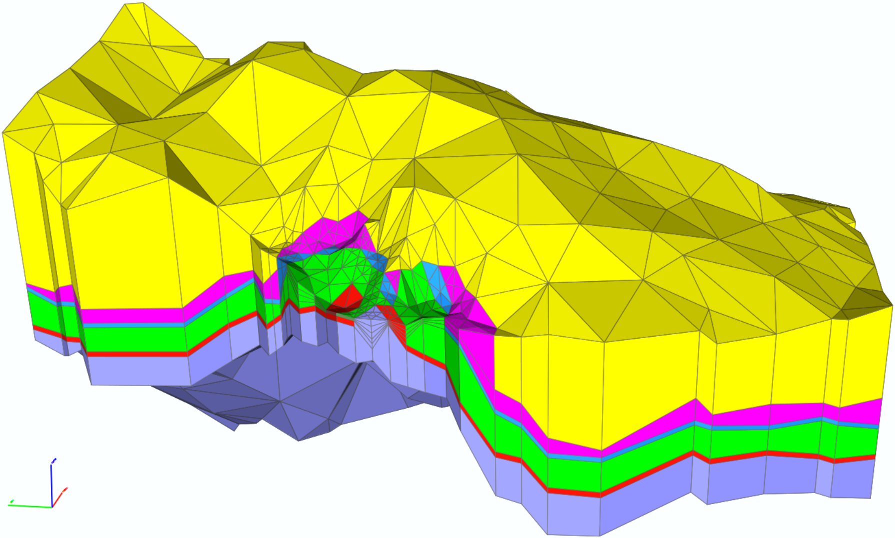
Current issue:

- Mass conservation holds only asymptotically.

3D Example 1: Problem setting



(Scaled $\times 100$ in z-direction.)

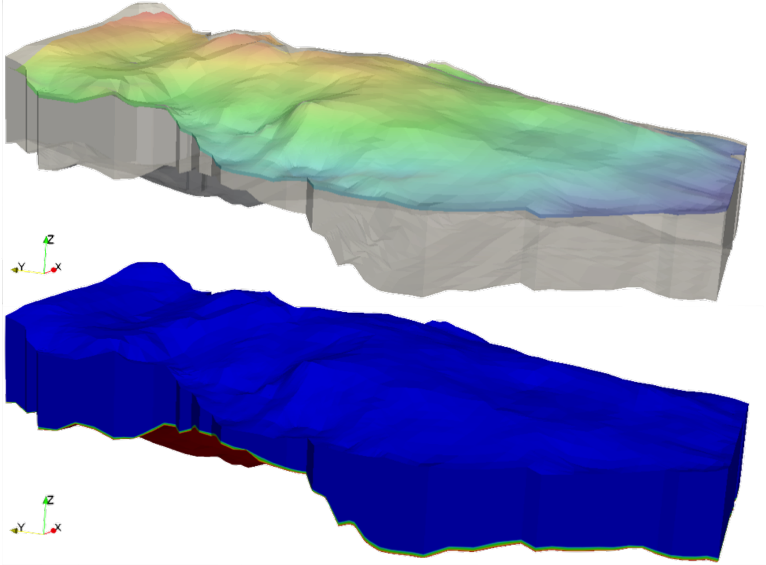
3D Example 1: Problem setting

- Hor. size $\approx$ 100 km, depth $\approx$ 0.5 km. 6 geological layers.
- BC: No-flux for entire fluid phase on the walls and bottom; no-flux for ω on the walls; $\omega = 1$ on the bottom.
- Initial position of the phreatic surface specified by a raster of the depths.
- Time-dependent recharge.
- Unstruct. grid of prisms. Anisotropic refinement required. Projection to the actual outer and inner boundaries.
- The coarsest grid: approx. 2000 grid nodes (2856 prisms).
- Presented result: approx. 10^5 nodes (182784 prisms).
- Time step $\tau = 0.05$ year.

Data and results: GRS/Braunschweig (A. Schneider and Dr. H. Zhao).

Grid: G-CSC/Frankfurt University (Dr. S. Reiter) using ProMesh.

3D Example 1: Results



Init. water table and ω. (Scaled $\times 50$ in z.)

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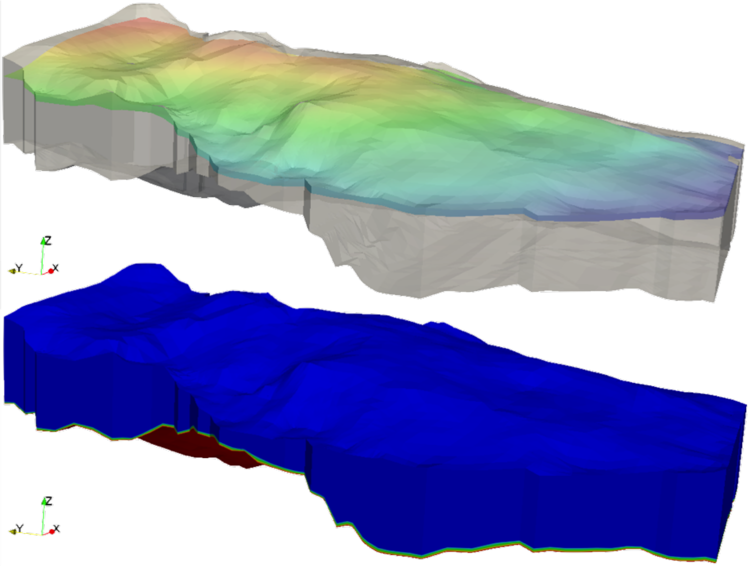
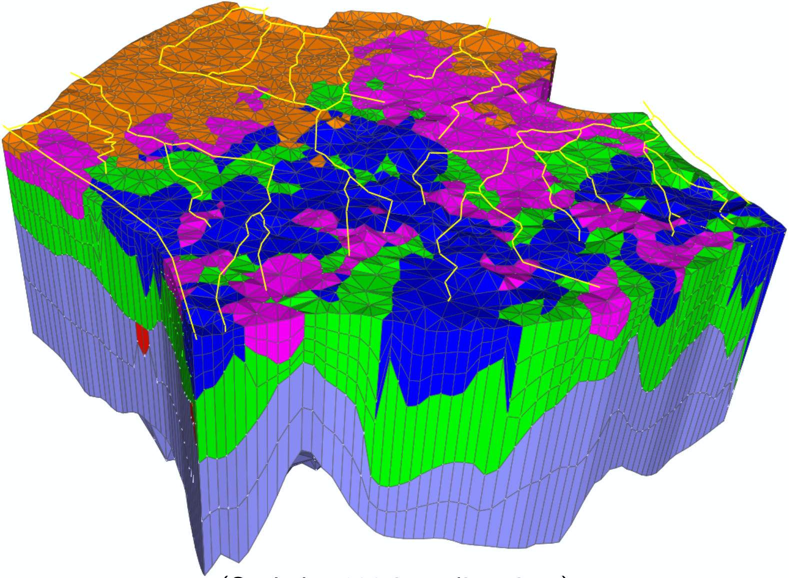


Water table and ω at $t = 150$ years. (Scaled $\times 50$ in z.)

3D Example 2: Problem setting



(Scaled $\times 100$ in z-direction.)

3D Example 2: Problem setting

- Hor. size $\approx$ 40 km, depth $\approx$ 0.2 km. 6 geological layers.
- BC: No-flux for entire fluid phase and salt on the walls and bottom up to one wall where $\omega = 1$ and hydrostatic p is imposed.
- Initial position of the phreatic surface specified by a raster of the depths.
- Position- and time-dependent recharge.
- 1d objects: channels ("rivers") modelled as the recharge depending on the depth of the phreatic surface.
- Unstruct. grid of prisms. Anisotropic refinement required.
- The coarsest grid: approx. 11500 grid nodes (18252 prisms).
- Presented result: approx. 82000 nodes (146016 prisms).
- Parallel computation on 20 cores.

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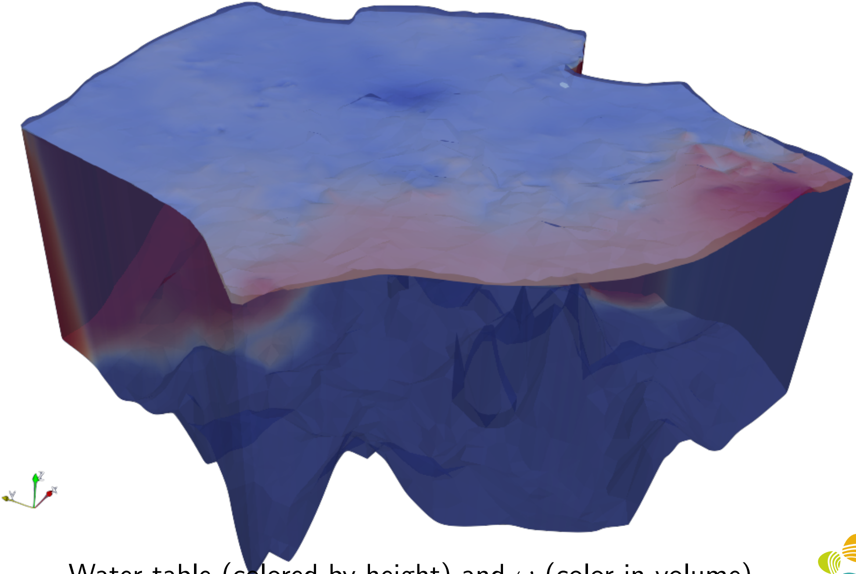

Water table (colored by height) and ω (color in volume) at $t = 80$ years. (Scaled $\times 100$ in z.)

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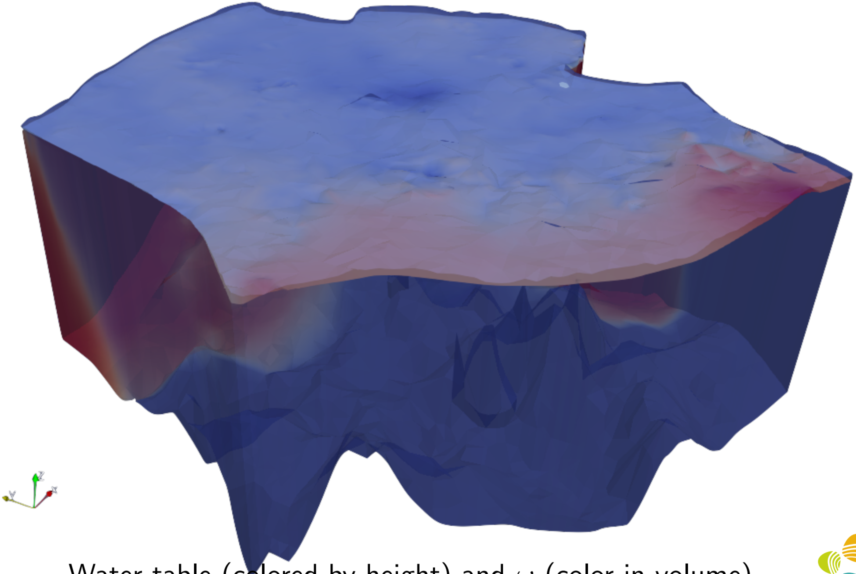

Water table (colored by height) and ω (color in volume) at $t = 180$ years. (Scaled $\times 100$ in z.)

Conclusions

We have presented a coupling of the density-driven flow with the level-set technique for the simulation of the phreatic boundary in the real-world geometries.

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- P. Frolkovič (Slovak University of Technology, Bratislava)

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Conclusions

We have presented a coupling of the density-driven flow with the level-set technique for the simulation of the phreatic boundary in the real-world geometries.

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Thank you for your attention!