Homogenization of linear elasticity with slip displacement conditions

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Outline

Physical motivation

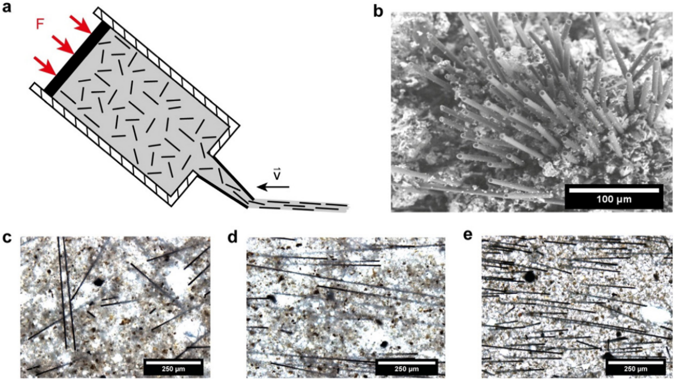


Fig.: (a) By applying a force (F) on the syringe fiber alignment in the extruded cement paste can be achieved; by moving the nozzle in a direction (v) solid samples containing oriented fibers can be fabricated. (b) ESEM micrograph of a fracture edge of a test specimen of nozzle-injected cement paste. Thin sections of (c) randomly distributed carbon fibers and nozzle-aligned carbon fibers at 1 (d) and 3 (e) percent by volume. Source: [Hambach, Möller, Neumann, Volkmer '16]

Physical motivation

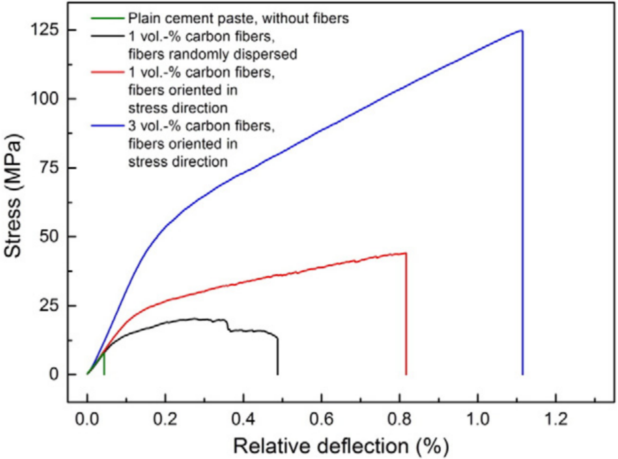
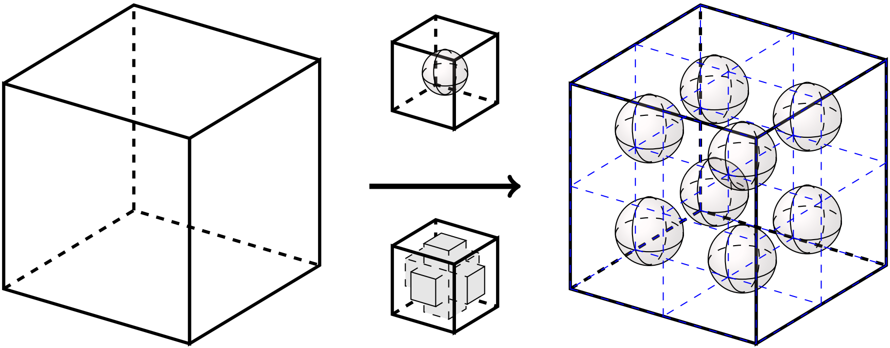


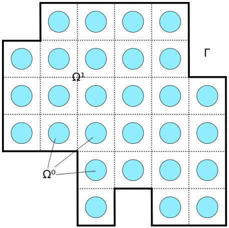
Fig.: Stress-deflection plots of 3-point bending tests for plain cement paste, mold casted and nozzle-injected carbon fiber-reinforced cement paste for 1 and 3 vol.-% carbon fibers. Source: [Hambach, Möller, Neumann, Volkmer '16]

Statement of the problem



We consider the problem

$$\begin{cases} -\nabla \cdot \sigma^{\varepsilon} = f^{\varepsilon} \text{ in } \Omega_0^{\varepsilon} \cup \Omega_1^{\varepsilon}, \\ u^{\varepsilon} = 0 \text{ on } \Gamma_1, \\ \sigma^{\varepsilon} \cdot \nu = g \text{ on } \Gamma_2 \end{cases}$$



with jump condition[1] on the interface Σ^{ε}

$$\begin{cases} \varepsilon \, [u_n^{\varepsilon}]_{\Sigma^{\varepsilon}} = \dfrac{1}{K_N} \sigma_n^{\Sigma^{\varepsilon}}, \\ \varepsilon \, [u_{\tau^i}^{\varepsilon}]_{\Sigma^{\varepsilon}} = \dfrac{1}{K_T} \sigma_{\tau^i}^{\Sigma^{\varepsilon}}, \ \ i = 1, 2, \\ [\sigma_n^{\varepsilon}]_{\Sigma^{\varepsilon}} = 0, \\ [\sigma_{\tau_i}^{\varepsilon}]_{\Sigma^{\varepsilon}} = 0, \ \ i = 1, 2, \end{cases}$$

where $K_N, K_T > 0$ are called the normal and tangential stiffness and $\sigma^{\Sigma^{\varepsilon}}$ the stress tensor of the interface.

[1] Lombard, Piraux: Numerical modeling of elastic waves across imperfect contacts.

Notation and assumptions

- scaling factor ε such that $\varepsilon^{-1}\Omega$ can be represented as finite union of axis-parallel cuboids with corner coordinates in $\mathbb{Z}^3$
- $\sigma^\varepsilon = (\sigma^\varepsilon_{ij})_{1\leq i,j\leq 3}$ stress tensor with

$$\sigma^\varepsilon_{ij} = \sum_{k,l=1}^{3} a^\varepsilon_{ijkl} e_{kl}(u^\varepsilon) = \sum_{k,l=1}^{3} a^\varepsilon_{ijkl} \frac{1}{2} (\partial_k u^\varepsilon_l + \partial_l u^\varepsilon_k),$$

- $\varphi_i = \varphi|_{\Omega^i_\varepsilon}$
- $[\varphi]_{\Sigma^\varepsilon} := \left(\varphi^1 - \varphi^0\right)|_{\Sigma^\varepsilon}$ jump on the interface.

Weak formulation

The weak fomulation in the disconnected case is:

Find u^ε such that for all test functions φ

$$\int_{\Omega_0^\varepsilon} A^\varepsilon e(u_0^\varepsilon) e(\varphi_0) \mathrm{d}x + \int_{\Omega_1^\varepsilon} A^\varepsilon e(u_1^\varepsilon) e(\varphi_1) \mathrm{d}x$$

$$+ \varepsilon \int_{\Sigma^\varepsilon} \left(K_{\mathrm{N}} \left[u_n^\varepsilon\right]_{\Sigma^\varepsilon} n + K_{\mathrm{T}} \sum_{i=1}^{2} \left[u_{\tau^i}^\varepsilon\right]_{\Sigma^\varepsilon} \tau^i \right) \cdot (\varphi_1 - \varphi_0) \, \mathrm{d}S(x)$$

$$= \int_{\Omega_0^\varepsilon} f^\varepsilon \cdot \varphi_0 \, \mathrm{d}x + \int_{\Omega_1^\varepsilon} f^\varepsilon \cdot \varphi_1 \, \mathrm{d}x + \int_{\Gamma_2} g \cdot \varphi_1 \mathrm{d}S(x)$$

Weak formulation

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$$+ \varepsilon \int_{\Sigma^\varepsilon} \left(K_\mathrm{N} \, [u_n^\varepsilon]_{\Sigma^\varepsilon} \, n + K_\mathrm{T} \sum_{i=1}^{2} [u_{\tau^i}^\varepsilon]_{\Sigma^\varepsilon} \, \tau^i \right) \cdot (\varphi_1 - \varphi_0) \, \mathrm{d}S(x)$$

$$= \int_{\Omega_0^\varepsilon} f^\varepsilon \cdot \varphi_0 \, \mathrm{d}x + \int_{\Omega_1^\varepsilon} f^\varepsilon \cdot \varphi_1 \, \mathrm{d}x + \int_{\Gamma_2 \cap \partial\Omega_0^\varepsilon} g \cdot \varphi_0 \mathrm{d}S(x)$$

$$+ \int_{\Gamma_2 \cap \partial\Omega_1^\varepsilon} g \cdot \varphi_1 \mathrm{d}S(x)$$

Existence result for the disconnected case

Since infinitesimal rotations induce no forces, we can assume that there are no rotations. We define the solution space

$$\mathcal{W}_{\mathrm{d}}(\Omega^\varepsilon) = \{u \in \left[L^2(\Omega^\varepsilon)\right]^3 : u_0 \in \left[H^1(\Omega_0^\varepsilon)\right]^3, u_1 \in \left[H^1(\Omega_1^\varepsilon, \Gamma_1)\right]^3, \\ \nabla \times u_0 = 0 \text{ in } \Omega_0^\varepsilon\},$$

equipped with the norm

$$\|u\|^2_{\mathcal{W}_{\mathrm{d}}(\Omega^\varepsilon)} := \|e(u_0)\|^2_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times 3}} + \|e(u_1)\|^2_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}} + \varepsilon \|\, [u]_{\Sigma^\varepsilon}\, \|^2_{[L^2(\Sigma^\varepsilon)]^3},$$

which is a Hilbert space.

The operator $\nabla \times \cdot$ is the usual curl operator, i.e.

$$\nabla \times u = \begin{pmatrix} \partial_{x_2} u_3 - \partial_{x_3} u_2 \\ \partial_{x_3} u_1 - \partial_{x_1} u_3 \\ \partial_{x_1} u_2 - \partial_{x_2} u_1 \end{pmatrix}.$$

Theorem

Let $f^\varepsilon \in \left[L^2(\Omega^\varepsilon)\right]^3$, $g \in \left[L^2(\Gamma_2)\right]^3$. Then, there exists a unique weak solution $u \in \mathcal{W}_{\mathrm{d}}(\Omega^\varepsilon)$ for all admissible $0 < \varepsilon \leq 1$. Furthermore there exists an ε-independent constant C with $\|u^\varepsilon\|_{\mathcal{W}_{\mathrm{d}}(\Omega^\varepsilon)} \leq C$.

Existence result for the connected case

Instead of

$$\mathcal{W}_{\mathrm{d}}(\Omega^\varepsilon) = \{u \in \left[L^2(\Omega^\varepsilon)\right]^3 : u_0 \in \left[H^1(\Omega_0^\varepsilon)\right]^3, u_1 \in \left[H^1(\Omega_1^\varepsilon, \Gamma_1)\right]^3, \\ \nabla \times u_0 = 0 \text{ in } \Omega_0^\varepsilon\},$$

endowed with the norm

$$\|u\|^2_{\mathcal{W}_{\mathrm{c}}(\Omega^\varepsilon)} = \|e(u_0)\|^2_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times 3}} + \|e(u_1)\|^2_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}} + \varepsilon \|[u]_{\Sigma^\varepsilon}\|^2_{[L^2(\Sigma^\varepsilon)]^3},$$

which is a Hilbert space.

Theorem

Let $f^\varepsilon \in \left[L^2(\Omega^\varepsilon)\right]^3$, $g \in \left[L^2(\Gamma_2)\right]^3$. Then there exists a unique solution $u^\varepsilon \in \mathcal{W}_{\mathrm{c}}(\Omega^\varepsilon)$ of for all admissible $0 < \varepsilon \leq 1$. Furthermore there exists an ε-independent constant C with $\|u^\varepsilon\|_{\mathcal{W}_{\mathrm{c}}(\Omega^\varepsilon)} \leq C$.

Existence result for the connected case

We define

$$\mathcal{W}_{\mathrm{c}}(\Omega^\varepsilon) = \{u \in \left[L^2(\Omega^\varepsilon)\right]^3 : u_0 \in \left[H^1(\Omega_0^\varepsilon, \Gamma_1 \cap \partial\Omega_0^\varepsilon)\right]^3, \quad u_1 \in \left[H^1(\Omega_1^\varepsilon, \Gamma_1 \cap \partial\Omega_1^\varepsilon)\right]^3\}.$$

endowed with the norm

$$\|u\|^2_{\mathcal{W}_{\mathrm{c}}(\Omega^\varepsilon)} = \|e(u_0)\|^2_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times 3}} + \|e(u_1)\|^2_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}} + \varepsilon \|[u]_{\Sigma^\varepsilon}\|^2_{[L^2(\Sigma^\varepsilon)]^3},$$

which is a Hilbert space.

Theorem

Let $f^\varepsilon \in \left[L^2(\Omega^\varepsilon)\right]^3$, $g \in \left[L^2(\Gamma_2)\right]^3$. Then there exists a unique solution $u^\varepsilon \in \mathcal{W}_{\mathrm{c}}(\Omega^\varepsilon)$ of for all admissible $0 < \varepsilon \leq 1$. Furthermore there exists an ε-independent constant C with $\|u^\varepsilon\|_{\mathcal{W}_{\mathrm{c}}(\Omega^\varepsilon)} \leq C$.

Two-scale convergence

Definition (Nguetseng '89, Allaire '92)

Let $\{u^\varepsilon\}$ be a sequence of functions in $L^2(\Omega)$ with $\Omega \subset \mathbb{R}^d, d \geq 1$, open and $Y = (0,1)^d$ the unit cube. The sequence is said to two-scale converge to $u \in L^2(\Omega \times Y)$ if

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y u(x,y)\varphi(x,y)\, \mathrm{d}y\mathrm{d}x$$

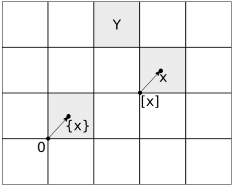
for every $\varphi \in L^2(\Omega; C^\infty_{per}(Y))$ and we write $u^\varepsilon \xrightarrow{2} u$.

Theorem (Allaire '92)

Let $\{u^\varepsilon\}$ be a bounded sequence in $H^1(\Omega)$ such that $u^\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega)$. Then $\{u^\varepsilon\}$ two-scale converges to u and there exists up to a subsequence a function $\hat{u} \in L^2(\Omega, H^1_{per}(Y)/\mathbb{R})$ that satisfy

$$u^\varepsilon \xrightarrow{2} u(x) \quad \text{and} \quad \nabla u^\varepsilon \xrightarrow{2} \nabla u(x) + \nabla_y \hat{u}(x,y).$$

Periodic unfolding method



Let $x \in \mathbb{R}^3$, then $x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right] + \left\{\frac{x}{\varepsilon}\right\}\right)$ with

$$[x] = \sum_{j=1}^{3} \xi_j e_j, \text{ so that } \{x\} := x - [x] \in Y.$$

Definition (Donato,Nguyen,Tardieu '11)

Let $i = 0, 1$. For any Lebesgue measurable function ϕ on Ω_i^ε the periodic unfolding operator $\mathcal{T}_i^\varepsilon$ is defined by the formula

$$\mathcal{T}_i^\varepsilon(\phi)(x,y) := \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) & \text{for a.e. } (x,y) \in \hat{\Omega}^\varepsilon \times Y_i, \\ 0 & \text{for a.e. } (x,y) \in \Pi^\varepsilon \times Y_i, \end{cases}$$

where $\hat{\Omega}^\varepsilon = \bigcup_{\xi \in \Lambda^\varepsilon} \varepsilon(\xi + \overline{Y})$ and $\Pi^\varepsilon = \Omega \backslash \hat{\Omega}^\varepsilon$.

Homogenization result for the connected case

Let $\{u^\varepsilon\}$ be a sequence of weak solutions with $u^\varepsilon \in \mathcal{W}_{\mathrm{c}}(\Omega^\varepsilon)$. Then there exist functions $u_0, u_1 \in \left[H^1(\Omega, \Gamma_1)\right]^3$ and $\hat{u}_0, \hat{u}_1 \in \left[L^2(\Omega, H^1_{per}(Y)/\mathbb{R})\right]^3$ such that

$$\begin{cases} \tilde{u}_1^\varepsilon \xrightarrow{2} u_1, \\ e(\tilde{u}_1^\varepsilon) \xrightarrow{2} e(u_1) + e_y(\hat{u}_1), \\ \tilde{u}_0^\varepsilon \xrightarrow{2} u_0, \\ e(\tilde{u}_0^\varepsilon) \xrightarrow{2} e(u_0) + e_y(\hat{u}_0), \end{cases}$$

whereby $\tilde{\cdot}$ is the extension to Ω defined in [Höpker '16].

Theorem

Let $\{u^\varepsilon\}$ *be the sequence of weak solutions with* $u^\varepsilon \in \mathcal{W}_c(\Omega^\varepsilon)$ *and* $\{f^\varepsilon\} \subset \left[L^2(\Omega^\varepsilon)\right]^3$ *such that* $f^\varepsilon \xrightarrow{2} f$ *for some* $f \in \left[L^2(\Omega \times Y)\right]^3$*. Then the restriction* $u = (u_0, \hat{u}_0|_{\Omega\times Y_0}, u_1, \hat{u}_1|_{\Omega\times Y_1})$ *is the unique solution of the problem*

$$\int_\Omega \int_{Y_0} A\left(e(u_0) + e_y(\hat{u}_0)\right)\left(e(v_0) + e_y(\hat{v}_0)\right) \mathrm{d}y\mathrm{d}x$$
$$+ \int_\Omega \int_{Y_1} A\left(e(u_1) + e_y(\hat{u}_1)\right)\left(e(v_1) + e_y(\hat{v}_1)\right) \mathrm{d}y\mathrm{d}x$$
$$+ \int_\Omega \int_{\Sigma_Y} \left(K_N(u_1 \cdot n - u_0 \cdot n)n + K_T \sum_{i=1}^{2} (u_1 \cdot \tau^i - u_0 \cdot \tau^i)\tau^i \right) \cdot (v_1 - v_0)\, \mathrm{d}S(y)\mathrm{d}x$$
$$= \int_\Omega \int_{Y_0} f\, \mathrm{d}y \cdot v_0\, \mathrm{d}x + \int_\Omega \int_{Y_1} f\, \mathrm{d}y \cdot v_1\, \mathrm{d}x + \int_{\Gamma_2} g \cdot h_0 v_0\, \mathrm{d}S(x) + \int_{\Gamma_2} g \cdot h_1 v_1\, \mathrm{d}S(x)$$

for all $v_0, v_1 \in \left[H^1(\Omega, \Gamma_1)\right]^3$, $\hat{v}_0 \in \left[L^2(\Omega, H^1_{per}(Y_0)/\mathbb{R})\right]^3$, $\hat{v}_1 \in \left[L^2(\Omega, H^1_{per}(Y_1)/\mathbb{R})\right]^3$.

Formulation of limit problem with effective tensor

Find $u_0, u_1 \in \left[H^1(\Omega, \Gamma_1)\right]^3$ with

$$
\begin{aligned}
&\int_\Omega A_0^{\mathrm{hom}} e(u_0) e(v_0)\,\mathrm{d}x + \int_\Omega A_1^{\mathrm{hom}} e(u_1) e(v_1)\,\mathrm{d}x \\
&+ \int_\Omega \int_{\Sigma_Y} \left(K_N(u_1 \cdot n - u_0 \cdot n) n + K_T \sum_{i=1}^{2} (u_1 \cdot \tau^i - u_0 \cdot \tau^i)\tau^i \right) \cdot (v_1 - v_0)\,\mathrm{d}S(y)\mathrm{d}x \\
&= \int_\Omega \int_{Y_0} f\,\mathrm{d}y \cdot v_0\,\mathrm{d}x + \int_\Omega \int_{Y_1} f\,\mathrm{d}y \cdot v_1\,\mathrm{d}x + \int_{\Gamma_2} g \cdot h_0 v_0\,\mathrm{d}S(x) + \int_{\Gamma_2} g \cdot h_1 v_1\,\mathrm{d}S(x),
\end{aligned}
$$

Formulation of limit problem with effective tensor

Find $u_0, u_1 \in \left[H^1(\Omega, \Gamma_1)\right]^3$ with

$$\int_\Omega A_0^{\mathrm{hom}} e(u_0) e(v_0)\,\mathrm{d}x + \int_\Omega A_1^{\mathrm{hom}} e(u_1) e(v_1)\,\mathrm{d}x \ldots = \ldots,$$

whereby

$$(A_\alpha^{\mathrm{hom}})_{ijkh} = \int_{Y_\alpha} a_{ijkh}(y) - \sum_{l,m=1}^{3} a_{ijlm} \left(e_y(\chi_\alpha^{kh})\right)_{lm} \mathrm{d}y$$

and $\chi_\alpha^{lm} \in \left[H^1(Y_\alpha)\right]^3$, $l, m \in \{1,2,3\}$ is the unique solution of

$$\begin{cases} \left(-\sum_{j=1}^3 \frac{\partial}{\partial y_j}\left[\left(Ae_y(\chi_\alpha^{lm})\right)_{ij} - a_{ijlm}\right]\right)_{1\le i\le 3} = 0 & \text{in } Y_\alpha, \\ \left(-\sum_{j=1}^3 \left[\left(Ae_y(\chi_\alpha^{lm})\right)_{ij} - a_{ijlm}\right]\right)_{1\le i\le 3} \cdot n = 0 & \text{on } \Sigma_Y, \\ \chi_\alpha^{lm} \text{ is Y-periodic with } \mathcal{M}_{Y_\alpha}(\chi_\alpha^{lm}) = 0. \end{cases}$$

for $\alpha \in \{0, 1\}$.

Homogenization result for the disconnected case

Define the Hilbert space

$$\left[L^2(\Omega, H^1_{per,0}(Y_1))\right]^3 := \{u \in \left[L^2(\Omega, H^1_{per}(Y_1))\right]^3 : \mathcal{M}_{Y_1}(u) = 0\}.$$

Theorem

Let $\{u_1^\varepsilon\}$ be a sequence in $\left[H^1(\Omega_1^\varepsilon, \Gamma_1)\right]^3$ with

$$\|u_1^\varepsilon\|_{\left[L^2(\Omega_1^\varepsilon)\right]^3} + \|e(u_1^\varepsilon)\|_{\left[L^2(\Omega_1^\varepsilon)\right]^{3\times 3}} \leq C$$

for a constant C independent of ε. Then there exists a subsequence (again denoted by ε), $u_1 \in \left[H^1(\Omega, \Gamma_1)\right]^3$ and $\hat{u}_1 \in \left[L^2(\Omega, H^1_{per,0}(Y_1))\right]^3$ such that

$$\mathcal{T}_1^\varepsilon(u_1^\varepsilon) \rightharpoonup u_1 \text{ weakly in } \left[L^2(\Omega, H^1(Y_1))\right]^3,$$

$$\mathcal{T}_1^\varepsilon(e(u_1^\varepsilon)) \rightharpoonup e(u_1) + e_y(\hat{u}_1) \text{ weakly in } \left[L^2(\Omega \times Y_1)\right]^{3\times 3}.$$

Homogenization result for the disconnected case

Theorem

Let $\{u_0^\varepsilon\}$ *be a sequence in* $\left[\tilde{H}^1(\Omega_0^\varepsilon)\right]^3 := \{u \in \left[H^1(\Omega_0^\varepsilon)\right]^3 : \nabla \times u = 0 \text{ a.e. in } \Omega_0^\varepsilon\}$ *with*

$$\|u_0^\varepsilon\|_{\left[L^2(\Omega_0^\varepsilon)\right]^3} + \|e(u_0^\varepsilon)\|_{\left[L^2(\Omega_0^\varepsilon)\right]^{3\times 3}} \leq C$$

for a constant C *independent of* ε*. Then, there exists a subsequence (again denoted by* ε*) and* $u_0 \in \left[L^2(\Omega)\right]^3$ *such that*

$$\mathcal{T}_0^\varepsilon(u_0^\varepsilon) \rightharpoonup u_0 \text{ weakly in } \left[L^2(\Omega, H^1(Y_0))\right]^3,$$

and

$$\varepsilon \mathcal{T}_0^\varepsilon(\nabla u_0^\varepsilon) \to 0 \text{ strongly in } \left[L^2(\Omega \times Y_0)\right]^{3\times 3}.$$

Homogenization result for the disconnected case

Theorem

Let $\{u_0^\varepsilon\}$ *be a bounded sequence in* $\left[\tilde{H}^1(\Omega_0^\varepsilon)\right]^3$ *with*

$$\mathcal{T}_0^\varepsilon(u_0^\varepsilon) \rightharpoonup u_0 \text{ weakly in } \left[L^2(\Omega, H^1(Y_0))\right]^3$$

for some $u_0 \in \left[L^2(\Omega)\right]^3$*. Then, the weak limit satisfies* $\nabla \times u_0 = 0$*.*

Homogenization result for the disconnected case

Let $\{u^\varepsilon\}$ be the sequence of weak solutions with $u^\varepsilon \in \mathcal{W}_{\mathrm{d}}(\Omega^\varepsilon)$. Then there exists $u = (u_1, \hat{u}_1, u_0, \hat{u}_0) \in \mathcal{Z}(\Omega, Y_1, Y_0)$, so that

$$\begin{cases} \mathcal{T}_1^\varepsilon(u_1^\varepsilon) \rightharpoonup u_1 \text{ weakly in } \left[L^2(\Omega, \tilde{H}^1(Y_1))\right]^3, \\ \mathcal{T}_1^\varepsilon(e(u_1^\varepsilon)) \rightharpoonup e(u_1) + e_y(\hat{u}_1) \text{ weakly in } \left[L^2(\Omega \times Y_1)\right]^{3\times 3}, \\ \mathcal{T}_0^\varepsilon(u_0^\varepsilon) \rightharpoonup u_0 \text{ weakly in } \left[L^2(\Omega, \tilde{H}^1(Y_0))\right]^3, \\ \mathcal{T}_0^\varepsilon(e(u_0^\varepsilon)) \rightharpoonup e_y(\hat{u}_0) \text{ weakly in } \left[L^2(\Omega \times Y_0)\right]^{3\times 3}. \end{cases}$$

Homogenization result for the disconnected case

Theorem

Let $\{u^\varepsilon\}$ be the sequence of weak solutions with $u^\varepsilon \in \mathcal{W}_{\mathrm{d}}(\Omega^\varepsilon)$. Then $u = (u_1, \hat{u}_1, u_0)$ is the solution of the problem

$$\int_\Omega \int_{Y_1} A(y)(e(u_1) + e_y(\hat{u}_1))(e(v_1) + e_y(\hat{v}_1)) \,\mathrm{d}y\mathrm{d}x$$

$$+ \int_\Omega \int_{\Sigma_Y} \left(K_{\mathrm{N}} \left[u_1 \cdot n - u_0 \cdot n\right] n + K_{\mathrm{T}} \sum_{i=1}^{2} \left[u_1 \cdot \tau^i - u_0 \cdot \tau^i\right] \tau^i \right) \cdot (v_1 - v_0) \,\mathrm{d}S(y) \,\mathrm{d}x$$

$$= \int_\Omega \int_{Y_1} f \,\mathrm{d}y \cdot v_1 \,\mathrm{d}x + \int_\Omega \int_{Y_0} f \,\mathrm{d}y \cdot v_0 \,\mathrm{d}x + \int_{\Gamma_2} g \cdot v_1 \,\mathrm{d}S(x).$$

for all $v = (v_1, \hat{v}_1, v_0) \in \mathcal{Z}(\Omega, Y_1, \Gamma_1)$.

Homogenization result for the disconnected case

We can reformulate the homogenized problem:

Find $u_1 \in \left[H^1(\Omega, \Gamma_1)\right]^3$, $u_0 \in \left[L^2_{curl}(\Omega)\right]^3$ such that

$$\int_\Omega A_1^{\mathrm{hom}} e(u_1) e(v_1)\,\mathrm{d}x$$
$$+ \int_\Omega \int_{\Sigma_Y} \left(K_{\mathrm{N}}(u_1 \cdot n - u_0 \cdot n)n + K_{\mathrm{T}} \sum_{i=1}^{2} (u_1 \cdot \tau^i - u_0 \cdot \tau^i)\tau^i \right) \cdot (v_1 - v_0)\,\mathrm{d}S(y)\mathrm{d}x$$
$$= \int_\Omega \int_{Y_0} f\,\mathrm{d}y \cdot v_0\,\mathrm{d}x + \int_\Omega \int_{Y_1} f\,\mathrm{d}y \cdot v_1\,\mathrm{d}x + \int_{\Gamma_2} g \cdot v_1\,\mathrm{d}S(x)$$

for all $v_1 \in \left[H^1(\Omega, \Gamma_1)\right]^3$, $v_0 \in \left[L^2_{curl}(\Omega)\right]^3$.

Comparison

Although the test functions are curl-free, by using the unique decomposition of functions $v \in L^2(\Omega)$, namely $v = \nabla p + \nabla \times w$ for some functions p, w, we can show that u_0 can not include rigid body motions. The strong formulation of the differential equations is

$$-\nabla \cdot \left(A_1^{\mathrm{hom}} e(u_1)\right) + \int_{\Sigma_Y} K_{\mathrm{N}}(u_1 - u_0) \cdot nn + K_{\mathrm{T}} \sum_{i=1}^{2} (u_1 - u_0) \cdot \tau^i \tau^i \,\mathrm{d}S(y) = \int_{Y_1} f \,\mathrm{d}y,$$

$$-\nabla \cdot \left(A_0^{\mathrm{hom}} e(u_0)\right) - \int_{\Sigma_Y} K_{\mathrm{N}}(u_1 - u_0) \cdot nn + K_{\mathrm{T}} \sum_{i=1}^{2} (u_1 - u_0) \cdot \tau^i \tau^i \,\mathrm{d}S(y) = \int_{Y_0} f \,\mathrm{d}y$$

Summary

- Periodic homogenization was used to upscale the problem of linear elasticity in a two-component solid with linear slip displacement coupling conditions.
- One component was connected while the other one was either connected or disconnected.
- The effective models for the connected and the disconnected case are qualitatively different.
- In passing, general compactness results in the context of periodic unfolding for sequences in curl-free spaces were derived.