Approximations of evolutionary inequality with Lipschitz-continuous functional and minimally regular input data

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Abstract

The convergence and accuracy estimates for an abstract evolutionary inequality with a linear bounded operator and a convex and Lipschitz-continuous functional are investigated.
Four types of approximations are considered:

1. regularization method,
2. Galerkin semi-discrete scheme,
3. Rothe scheme
4. fully discrete scheme.

Approximate problems are studied under rather weak assumptions about the smoothness of the input data.
As an example of applying general theoretical results, we study the finite element approximation of a second order parabolic variational inequality.

ORIGINAL DIFFERENTIAL PROBLEM

Function spaces

- V, H are separable real Hilbert spaces, $V \subset H = H^* \subset V^*$, embeddings are continuous and dense.
- $(\cdot,\cdot)$ means both the inner product in H and the duality pairing between V^* and V;
- $W = L_2(0,T;V) \cap H^1(0,T;V^*)$, $E = L_\infty(0,T;H) \cap L_2(0,T,V)$
  By the definition $W \subset E$ continuously. E is called the energy space, and the norm of the graph in it is called the energy norm.

Problem $(\mathcal{P})$**.**
Given $u_0 \in H$ and $f \in L_2(0,T;V^*)$, find $u \in W$ such that $u(0) = u_0$ and

$$(u'(t) + A(t)u(t) - f(t), v - u(t)) + \phi(v) - \phi(u(t)) \geq 0 \qquad (1)$$

for all $v \in V$ and a.e. $t \in (0,T)$.

Note that we are considering so-called strong solution $(u \in W)$ of the variational inequality.

Assumptions for input data

- Functional $\phi : H \to \mathbb{R}$ is convex and Lipschitz-continuous:

$$|\phi(v) - \phi(u)| \leq L\|v - u\|_H \quad \forall\, v, u \in H. \tag{2}$$

- Linear operators operators $A(t) : V \to V^*$, $t \in [0, T]$, satisfy

$$\begin{cases} \|A(t)v\|_{V^*} \leq M\,\|v\|_V \;\forall\, v \in V; \\ (A(t)v, v) \geq \mu\,\|v\|_V^2 - \lambda\,\|v\|_H^2 \;\; \forall\, v \in V,\ \mu > 0; \\ \text{(Gårding inequality)} \\ \text{function } t \to (A(t)v, w) \text{ is measurable on } (0, T)\ \forall v, w \in V. \end{cases} \tag{3}$$

Theorem 1.

Suppose that assumptions for input data (3), (2) are true. Then

1. there exists a unique solution u of the problem $(\mathcal{P})$;
2. a priori estimates hold:

$$\|u\|_E^2 + \int_0^T \phi(u(t))\,dt \le C\left(\|u_0\|_H^2 + \|f\|^2_{L_2(0,T;V^*)}\right);$$

$$\|u'\|_{L_2(0,T;V^*)} \le C\left(\|u_0\|_H + \|f\|_{L_2(0,T;V^*)} + L\right);$$

3. stability takes place:

$$\|u_1 - u_2\|_E \le C\left(\|u_{10} - u_{20}\|_H + \|f_1 - f_2\|_{L_2(0,T;V^*)}\right),$$

where u_k are the solutions of problem $(\mathcal{P})$ with data u_{k0} and f_k, $k = 0, 1$, respectively.

Remark

Theorem states the existence of **a strong solution** to problem $(\mathcal{P})$ under assumptions (3), (2). A similar result even for the equation is not trivial and proved by A. Bensoussan and J.-L. Lions. For variational inequalities, in our opinion, it is new. In the proof we used the regularization method, which is considered below.

Regularized problem

We use Moreau–Yosida regularization of the functional ϕ:

$$\phi_\varepsilon(u) = \min_{v \in H} \left(\frac{1}{2\varepsilon} \|v - u\|_H^2 + \phi(v) \right), \quad \varepsilon > 0$$

and formulate the regularized problem:

Problem $(\mathcal{P}_\varepsilon)$**.**
Given $u_0 \in H$ and $f \in L_2(0, T; V^*)$, find $u_\varepsilon \in W$ such that $u_\varepsilon(0) = u_0$ and

$$\big(u_\varepsilon'(t) + A(t)u_\varepsilon(t) - f(t), v - u_\varepsilon(t)\big) + \phi_\varepsilon(v) - \phi_\varepsilon(u_\varepsilon(t)) \geq 0 \qquad (4)$$

for all $v \in V$ and a.e. $t \in (0, T)$.

Remark

The regularization method is among other things one of the approximate methods for solving problem $(\mathcal{P})$, which reduces it to the solving an equation. In fact, the functional ϕ_ε is convex and Fréshet differentiable on H, let $\nabla\phi_\varepsilon$ be its differential. So, variational inequality (4) is equivalent to the equation

$$u_\varepsilon'(t) + A(t)u_\varepsilon(t) + \nabla\phi_\varepsilon(u_\varepsilon(t)) = f(t) \quad \text{a.e. } t \in (0, T).$$

Theorem 2.

Suppose that assumptions for input data (2), (3) are true. Then

1. there exists a unique solution u_ε of the problem $(\mathcal{P}_\varepsilon)$;
2. $\{u_\varepsilon\}$ converges to u:

$$u_\varepsilon \to u \ \text{ in } \ E, \quad u'_\varepsilon \rightharpoonup u' \ \text{ weakly in } \ L_2(0,T;V^*) \ \text{ as } \ \varepsilon \to 0\,;$$

3. error estimate holds:

$$\|u - u_\varepsilon\|_E \le C\,L\,\varepsilon^{1/2}\,;$$

Spaces of approximate solutions

- $\{V_h\}_{h>0}$ is a family of finite-dimensional subspaces of V;
- H_h is the same as V_h linear space equipped with topology of H.

The embeddings $V_h \subset H_h = H_h^* \subset V_h^*$ are continuous and dense, and the constants in the embedding inequalities don't depend on h. We identify V_h^* with a subspace of V^* and keep the notation $(\cdot, \cdot)$ for the duality pairing between V_h^* and V_h.

Projection operator $P_h : V^* \to V_h$:

$$(P_h v - v, v_h) = 0 \quad \forall\, v_h \in V_h.$$

The restriction of P_h to the space H coincides with the orthogonal projector $H \to V_h$, so,

$$\|P_h v\|_H \leq \|v\|_H \;\forall\; v \in H.$$

Semidiscrete Galerkin scheme for the problem $(\mathcal{P})$:

Problem $(\mathcal{P}_h)$**.**
Given $u_0 \in H$ and $f \in L_2(0,T;V^*)$, find $u_h \in L_2(0,T;V_h)$ such that $u_h' \in L_2(0,T;V^*)$, $u_h(0) = P_h u_0$ and

$$(u_h'(t) + A(t)u_h(t) - f(t), v - u_h(t)) + \phi(v) - \phi(u_h(t)) \geq 0 \qquad (5)$$

for all $v \in V_h$ and a.e. $t \in (0,T)$.

Theorem 3.

Suppose that assumptions for input data (2), (3) are true. Then

1. there exists a unique solution u_h of the problem $(\mathcal{P}_h)$;
2. it satisfies a priory estimate

$$\|u_h\|_E^2 + \int_0^T \phi(u_h(t))\,dt \leq C\left(\|u_0\|_H^2 + \|f\|_{L_2(0,T;V^*)}^2 + L^2\right);$$

3. stability estimate holds:

$$\|u_{h,1} - u_{h,2}\|_E \leq C\left(\|u_{10} - u_{20}\|_H + \|f_1 - f_2\|_{L_2(0,T;V^*)}\right),$$

where $u_{h,k}$ are the solutions of the problem $(\mathcal{P}_h)$ with input data $u_{0,k}$ and f_k, $k = 0, 1$.

Convergence and accuracy

Approximation assumptions:

$$\lim_{h\to 0} \inf_{v_h \in V_h} \|u - v_h\|_V = 0 \text{ for any } u \in V. \tag{6}$$

$$\|P_h v\|_V \le C_V \, \|v\|_V \quad \forall\, v \in V. \tag{7}$$

By definition and due to (7) the following inequalities hold for all $v_h \in V_h$:

$$\|u - P_h u\|_H \le \|u - v_h\|_H,$$
$$\|u - P_h u\|_V \le (1 + C_V)\, \|u - v_h\|_V.$$

Remark.

The use of projector from V^* onto V_h, which restriction to the space V is supposed to be uniformly in h bounded operator (assumption (7)) is a key point in proving the error estimate. The above assumption is satisfied for a wide class of approximations of Sobolev spaces by finite element method (see [J. H. Bramble, J. E. Pasciak and O. Steinbach (2002)][a], [R. E. Bank and H. Yserentant (2014)]) [b].

The projection operator $P_h : V^* \to V_h$ plays an important role in the estimating the approximation errors of time derivative. Its usefulness in studying the Galerkin scheme for parabolic equations was demonstrated in [K. Chrysafinos and L.S. Hou (2002)] [c].

Theorem 4.

Let assumptions for input data (2), (3) and approximation assumptions (6), (7) hold. Then

1. a priory estimate takes place:

$$\|u_h\|_W^2 + \int_0^T \phi(u_h(t))\,dt \le C\,(\|u_0\|_H^2 + \|f\|_{L_2(0,T;V^*)}^2 + L^2);$$

2. the sequence of solutions $\{u_h\}_h$ of the problem $(\mathcal{P}_h)$ converges to the exact solution of the problem $(\mathcal{P})$:

$$u_h \to u \quad \text{in} \quad E, \qquad u_h' \rightharpoonup u' \quad \text{weakly in} \quad L_2(0,T;V^*) \quad \text{as} \quad h \to 0\,;;$$

3. the accuracy estimate holds:

$$\|u - u_h\|_E \le C\,\varepsilon_h(u), \tag{8}$$

where

$$\varepsilon_h(u) = \inf_{v_h \in L_\infty(0,T;V_h)} \|u - v_h\|_E + \inf_{v_h \in L_2(0,T;V_h)} \|u - v_h\|_{L_2(0,T;H)}^{1/2}.$$

Remark

Estimate (8) is obtained under the minimal smoothness assumptions for input data. No requirements for the regularity of the solutions u and/or u_h with respect to t are assumed.

Rothe scheme (semidiscrete backward Euler scheme)

1. Discretization in time variable
We fix a time step $\tau = T/N$ and a subdivision of $[-\tau, T]$ given by the intervals $I_n = [t_{n-1}, t_n)$, $n = 0, 1, \ldots, N$, where $t_j = j\tau$ for $j = -1, 0, \ldots, N$.

Time step restriction is fulfilled hereafter:

$2\lambda\tau < 1$, where λ is a constant from Gårding inequality (3).

The approximations of the operator and the right-hand side

$$(A^n v, w) = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} (A(t)v, w)dt, \quad f^n = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(t)dt, \quad n = 1, \ldots, N,$$

$$u^{-1} = u_0, \quad f^0 = f^1, \quad A^0 = A^1.$$

Rothe scheme approximated problem $(\mathcal{P})$:

Problem $(\mathcal{P}_\tau)$
Given $u_0 \in H$ and $f \in L_2(0,T;V^*)$,
find $u^n \in V$ for all $n = 0, \dots, N$ such that

$$\Big(\frac{u^n - u^{n-1}}{\tau} + A^n u^n - f^n, v - u^n\Big) + \phi(v) - \phi(u^n) \geq 0 \quad \forall\, v \in V. \tag{9}$$

Remark.

Initial condition u^0 is not assumed to be equal to u_0, but it is a solution to the inequality

$$\big((u^0 - u_0)/\tau + A^1 u^0 - f^1, v - u^0\big) + \phi(v) - \phi(u^0) \geq 0 \quad \forall\, v \in V.$$

We used this choice of the initial condition following [G. Savaré, 1996][a], since it is more convenient for studying the accuracy of the scheme. For a scheme with the usual choice of $u^0 = u_0$, all results remain valid.

[a] G. Savaré, *Weak solutions and maximal regularity for abstract evolution inequalities*, Adv. Math. Sci. Appl. **6**, 377-418

Existence and convergence results

Theorem 5.

Under the assumptions for input data (2), (3) the problem $(\mathcal{P}_\tau)$ has a unique solution.

For a given $u^n \in V$, $n = 0, 1, \ldots, N$, we denote by $\hat{u}_\tau$ the continuous piecewise-linear function that equals u^n at the points t_n. If $\{u^n\}_{n=0}^N$ is a solution of inequality (9), then we call the $\hat{u}_\tau$ as the solution of the problem $(\mathcal{P}_\tau)$ as well.

Theorem 6.

Let the assumptions for input data (2), (3) hold. Then

1. Rothe scheme $(\mathcal{P}_\tau)$ has a unique solution.
2. if u and $\hat{u}_\tau$ are the solutions of the problems $(\mathcal{P})$ and $(\mathcal{P}_\tau)$, then

$$\hat{u}_\tau \to u \quad \text{in} \quad E, \quad \hat{u}'_\tau \rightharpoonup u' \quad \text{weakly in} \quad L_2(0,T;V^*) \quad \text{as} \quad \tau \to 0\,.$$

This theorem establishes only the convergence of approximate solutions. The assumptions for input data (2), (3) are not enough to obtain error estimates.

To obtain the error estimates, we impose **stricter assumptions on the data,** than (2), (3):

$$\begin{cases} \|A(t)v\|_{V^*} \leq M\,\|v\|_V \;\forall\, v \in V; \\ (A(t)v, v) \geq \mu\,\|v\|_V^2 - \lambda\,\|v\|_H^2 \;\; \forall\, v \in V,\, \mu > 0; \\ \|(A(t_1) - A(t_2))v\|_{V^*} \leq M_\alpha |t_1 - t_2|^\alpha\,\|v\|_V, \qquad 0 < \alpha \leq 1, \\ \qquad\qquad \forall\, v \in V, \;\; \forall t_1, t_2 \in (0,T); \end{cases} \tag{10}$$

(Hölder-continuity of $A(t)$ with respect to t is an additional assumption);

$$\begin{cases} f \in B^\beta_{2\infty}(0,T;V^*) \quad \text{with} \quad 0 < \beta \leq 1; \\ \exists\, T_0 \in (0,T) :\; f|_{(0,T_0)} \in L_\infty(0,T_0;V^*). \end{cases} \tag{11}$$

Used interpolation spaces
For the arbitrary Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and the parameters $s \in [0,1]$, $p \in [1,\infty]$ we denote by $(\mathcal{H}_1,\mathcal{H}_2)_{s,p}$ the interpolation space constructed through real method (see books of J. Bergh, J. Löfstrõm [1976] and P.L. Butzer, H. Berens [1967] for more details). In particular, for a Hilbert space $\mathcal{H}$ we set

$$B^s_{2p}(0,T;\mathcal{H}) = (L_2(0,T;\mathcal{H}), H^1(0,T;\mathcal{H}))_{s,p}\,.$$

Theorem 7.

Let u and $\hat{u}_\tau$ be the solutions of the problems $(\mathcal{P})$ and $(\mathcal{P}_\tau)$, respectively and assumptions (10), (11) be fulfilled, and $u_0 \in V$. Then

$$\|u - \hat{u}_\tau\|_E \leq C\,\varepsilon_\tau, \tag{12}$$

where

$$\varepsilon_\tau = \tau^{1/2}\left(\|u_0\|_V + \|f\|_{L_\infty(0,T_0;V^*)} + L\right) + \tau^\beta\,\|f\|_{B^\beta_{2\infty}(0,T;V^*)} +$$

$$+\tau^\alpha\,M_\alpha\left(\|u_0\|_H + \|f\|_{L_2(0,T;V^*)}\right).$$

Remark.

Approximate scheme $(\mathcal{P}_\tau)$ for problem $(\mathcal{P})$ was studied by [Savar'e (1996)] in the case of general proper, convex and lower semicontinuous functional ϕ. The existence of a unique solution from the space $H^1(0,T;V) \cap W^1_\infty(0,T;H)$ was proved under corresponding regularity assumptions imposed on the data. The estimate

$$\|u - \hat{u}_\tau\|_E \leq C\,\tau$$

was proved in the supposition of these regularity assumptions for the exact solution.

Our results for Rothe scheme are proved under weaker assumptions for input data and complement these results.

Fully discrete implicit scheme

The **fully discrete** implicit approximation of the problem $(\mathcal{P})$ is the combination of semidiscrete schemes $(\mathcal{P}_h)$ and $(\mathcal{P}_\tau)$:

Problem $(\mathcal{P}_{h\tau})$ Given $u_0 \in H$ and $f \in L_2(0,T;V^*)$, find $u_{h\tau}^n \in V_h$ such that $u_{h\tau}^{-1} = P_h u_0 \in V_h$, and for $n = 0, 1, \ldots, N$ the following variational inequalities hold:

$$((u_{h\tau}^n - u_{h\tau}^{n-1})/\tau + A^n u_{h\tau}^n - f^n, v - u_{h\tau}^n) + \phi(v) - \phi(u_{h\tau}^n) \geq 0 \quad \forall\, v \in V_h.$$

Above we use the previous notations for the operator A^n and function f^n.

Theorem 8.

Suppose the assumptions for input data (2), (3) hold. Then

1. problem $(\mathcal{P}_{h\tau})$ has a unique solution $u_h^n \in V_h$, $n = 0, 1, \ldots N$.
2. the following stability estimate takes place:

$$\|\hat{u}_{h\tau,1} - \hat{u}_{h\tau,2}\|_E \leq C\left(\|u_{10} - u_{20}\|_H + \|f_1 - f_2\|_{L_2(0,T;V^*)}\right) \quad (13)$$

where $u_{h\tau,k}$ are the piecewise linear interpolation (with respect to t) of two solutions of the problem $(\mathcal{P}_{h\tau})$ with data $u_{0,k}$ and f_k (k=1,2).

Convergence and the accuracy estimates

Theorem 9.

Suppose the assumptions for input data (2), (3) and approximation assumptions (6),(7) hold. Let u and $\hat{u}_{h\tau}$ be the solutions of the problems $(\mathcal{P})$ and $(\mathcal{P}_{h\tau})$, respectively. Then

$$\hat{u}_{h\tau} \to u \quad \text{in} \quad E, \quad \hat{u}'_{h\tau} \rightharpoonup u' \quad \text{in} \quad L_2(0,T;V^*) \;\; \text{as} \;\; h \to 0, \;\tau \to 0\,.$$

The result of the following theorem is proved by appropriate combination of the error estimates for the Galerkin problem $(\mathcal{P}_h)$ and the error estimate of Rothe method applied to Galerkin problem $(\mathcal{P}_h)$.

Theorem 10.

Let the assumptions for the data (10), (11) and the approximation assumptions (6), (7) hold. Then

$$\|u - \hat{u}_{h\tau}\|_E \leq C \left(\varepsilon_h(u) + \tau^{\gamma/2}\, \|u_0\|_{(H,V)_{\gamma,\infty}} + \varepsilon_{0\tau}\right) \tag{14}$$

with

$$\varepsilon_h(u) = \inf_{v_h \in L_\infty(0,T;V_h)} \|u - v_h\|_E + \inf_{v_h \in L_2(0,T;V_h)} \|u - v_h\|^{1/2}_{L_2(0,T;H)}.$$

$$\varepsilon_{0\tau} = \tau^{1/2}(\|f\|_{L_\infty(0,T_0;V^*)} + L) + \tau^\beta\, \|f\|_{B^\beta_{2}\,(0,T;V^*)} + \tau^\alpha\, M_\alpha \left(\|u_0\|_H + \|f\|_{L_2(0,T;V^*)}\right)$$

Convergence and the accuracy estimates

For proving (14) we use error estimate for the solution of the Galerkin problem $(\mathcal{P}_h)$ ((8) in Theorem 11):

$$\|u - u_h\|_E \le C\,\varepsilon_h(u)\,. \tag{15}$$

Next, since $(\mathcal{P}_{h\tau})$ is Rothe method applied to Galerkin problem $(\mathcal{P}_h)$, then we can use the result of theorem 16, (changing the spaces H and V by H_h and V_h, and taking $u_0 = P_h u_0$, $u = u_h$ and $\hat{u}_\tau = \hat{u}_{h\tau}$) to obtain the estimate

$$\|u_h - \hat{u}_{h\tau}\|_E \le C\,\varepsilon_{h\tau}, \tag{16}$$

where

$$\varepsilon_{h\tau} = \tau^{1/2}\,(\|P_h u_0\|_V + \|f\|_{L_\infty(0,T_0;V_h^*)} + L) + \tau^{\beta}\,\|f\|_{B^{\beta}_{2\infty}(0,T;V_h^*)} \\ + \tau^{\alpha}\,M_\alpha\left(\|P_h u_0\|_{H_h} + \|f\|_{L_2(0,T;V_h^*)}\right).$$

To estimate $\varepsilon_{h\tau}$ by a value which depends only on τ, we use the inequalities

$$\|P_h u_0\|_{H_h} \le \|u_0\|_H\,, \quad \|P_h u_0\|_V \le C_V\|u_0\|_V$$

and get the inequality

$$\varepsilon_{h\tau} \le C_V\,\tau^{1/2}\,\|u_0\|_V + \varepsilon_{0\tau}. \tag{17}$$

Combination of (15), (16) and (17) results in the estimate (14).

Example

Consider an example of problem $(\mathcal{P})$ with the following input data:

- $\Omega \subset R^d$ is a bounded polyhedral domain, $d = 2$ or 3, $Q_T = \Omega \times (0, T)$, $H = L_2(\Omega)$, $V = H_0^1(\Omega)$, $V^* = H^{-1}(\Omega)$;
- Elliptic operator

$$A(t)u = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} a_i(x,t)\frac{\partial u}{\partial x_i} + a_0(x,t)u,$$

  with the coefficients that satisfy the assumptions

$$a_{ij}(x,t), a_i(x,t), a_0 \in W_\infty^1(Q_T), \quad a_{ij}(x,t) = a_{ji}(x,t);$$

$$\sum_{i,j=1}^{d} a_{ij}(x,t)\xi_i\xi_j \geq c_0|\xi|^2 \quad \text{for all } \xi \in R^d, \quad \text{a.e. } (x,t) \in Q_T,\ c_0 > 0.$$

  Operator $A(t) : H_0^1(\Omega) \to H^{-1}(\Omega)$ satisfies the assumptions (10): it is bounded, satisfies Gårding inequality and Lipschitz-continuous with respect to t.
- Functional

$$\phi(v) = \int_\Omega g(x)\,(v - \psi(x))^-\,dx, \quad g \in L_2(\Omega),\ g \geq 0,\ \psi \in L_\infty(\Omega),$$

  where $v^- = \max\{0, -v\}$, is convex and Lipschitz-continuous.
- Right hand side

$$f \in B_{2\infty}^{1/2}(0,T;L_2(\Omega)) \cap L_\infty(0,T;H^{-1}(\Omega))$$

  satisfies assumption (11) with $\beta = 1/2$.
- Initial value $u_0 \in H_0^1(\Omega)$.

Let T_h be a locally quasi-uniform simplicial partition of Ω,

$$V_h = \{v \in H_0^1(\Omega) : \; v|_T \in P_1(T) \text{ for all } T \in T_h\}$$

the space of piecewise linear continuous functions that vanish on the boundary of Ω. For these triangulation and finite element space the assumptions (6) and (7) for the projection operator $P_h : H^{-1}(\Omega) \to V_h$ are satisfied[1], [2]

In more detail the results of the presentation can be found in
Dautov R. Z., Lapin A.V. *Approximations of evolutionary inequality with Lipschitz-continuous functional and minimally regular input data, Lobachevskii J. Math.*, **40** *(4) 425-438 (2019)*.
In particular, for $v \in H^2(\Omega)$ we have the error estimates:

$$\|v - P_h v\|_{L_2(\Omega)} \leq c\,h\,\|v\|_{H^1(\Omega)}, \quad \|v - P_h v\|_{L_2(\Omega)} \leq c\,h^2\,\|v\|_{H^2(\Omega)},$$

$$\|v - P_h v\|_{H^1(\Omega)} \leq c\,h\,\|v\|_{H^2(\Omega)}.$$

We construct fully discrete schemes $(\mathcal{P}_{h\tau})$, based on finite element subspace V_h of V. Its accuracy estimate is given by (14) in the theorem 18 with

$$\varepsilon_h(u) = \inf_{v_h \in V_h} \|u - v_h\|_E + \inf_{v_h \in V_h} \|u - v_h\|_{L_2(Q_T)}^{1/2} \leq$$
$$\leq C\,h\,(\|u_0\|_{H^1(\Omega)} + \|f\|_{L_2(0,T;H^1(\Omega))} + L).$$

The accuracy estimate is

$$\|u - \hat{u}_{h\tau}\|_E \leq C\,(h + \sqrt{\tau}).$$

[1] J. H. Bramble, J. E. Pasciak, and O. Steinbach. *On the stability of the L_2 projection in $H^1(\Omega)$*, Math. Comp., **71**,147–156 (2002)
[2] R. E. Bank and H. Yserentant, *On the H^1-stability of the L_2-projection onto finite element spaces*, Numer. Math., **126** 361–381 (2014)

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