Numerical solution of steady-state nonlinear groundwater flow equation

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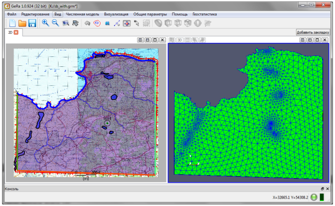
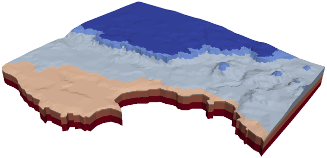
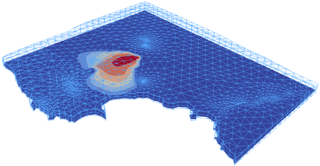
Problem importance

GeRa (**Ge**omigration of **Ra**dionuclides) – software for hydrogeological modeling

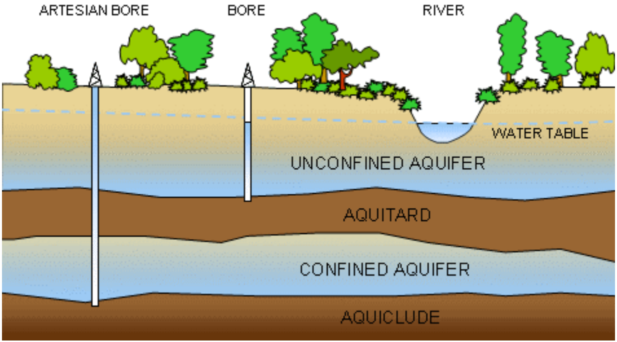






Modeling in GeRa includes various models of groundwater flow, contaminant transport, DDF, chemical reactions, heat transfer, ...

Variably saturated flow



- important for near-surface objects
- medium pores are partially filled with water
- water content and medium permeability depend on hydraulic water head

Mathematical problem

Variably saturated flow is governed by Richards equation:

$$\frac{\partial \theta}{\partial t} + S \cdot s_{stor} \cdot \frac{\partial h}{\partial t} - \nabla \cdot (K_r(\theta)\mathbb{K}\nabla h)) = Q,$$

- θ – volumetric water content, $[-]$
- h – water head, $[L]$
- S – water saturation, $[-]$
- s_{stor} – storage coefficient, $[L^{-1}]$
- $\mathbb{K}$ – hydraulic conductivity tensor, $[LT^{-1}]$, $K_r(\theta)$ – relative permeability, $[-]$
- Q – specific sinks and sources, $[T^{-1}]$

Consitutive relationships

Relations between water head, water content and relative permeability are required

$$\theta = \theta(h)$$

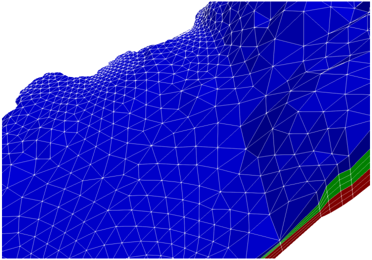
$$K_r = K_r(h)$$

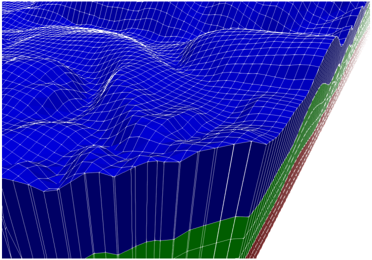
- nonlinear functions (van Genuchten, Mualem) – can model capillary effects
- simpler piecewise linear functions – suitable for real problems

Numerical solution: spatial discretization

We use finite volume (FV) schemes on unstructured meshes:

- linear TPFA
- linear MPFA (O-scheme)
- nonlinear monotone TPFA





Numerical solution: time-stepping and nonlinear solvers

We use fully implicit scheme and need to solve a nonlinear system at each time step.

- Solution at the previous time step is used as the initial estimate
- Time step size may vary to improve convergence

Picard method

The nonlinear system can be rewritten as

$$A(h)h = f$$

and solved with this method:

$$A(h^k)h^{k+1} = f.$$

- Convergence rate of Picard method is determined by the time step size, we may need too small steps

Newton's method

The nonlinear system can be rewritten as

$$A(h)h - f \equiv F(h) = 0.$$

At each iteration a linear system with Jacobian matrix has to be solved:

$$J(h^k)\delta h = -F(h^k).$$

- We can often take significantly larger time steps compared to Picard
- Linear systems are harder to solve due to their non-symmetry and supposedly larger condition numbers of J
- For nonlinear monotone TPFA J may be denser than A

Relaxation

Water head values h^* at iteration k+1 are changed to a convex linear combination with the values from the previous iteration:

$$h^{k+1} = \Omega h^* + (1 - \Omega) h^k, \quad 0 < \Omega \leq 1.$$

For Newton method it is equivalent to changing the update δh:

$$h^{k+1} = h^k + \Omega \delta h.$$

Solution of steady-state problems

Numerical solution of steady-state equation

$$-\nabla \cdot (K_r(h)\mathbb{K}\nabla h) = Q$$

directly with Newton method is practically impossible due to difficulty of choosing good initial estimate, so we seek another ways.

Time-stepping method

We can try to solve original time-dependent equation until solution stops to change in time.

- **Drawbacks:** we don't know modeling time apriori and we can get stuck at small time steps
- **Example:** $T \approx 30000$ days with $\Delta t \approx 0.001$ days

Continuation method

Original equation:

$$-\nabla \cdot (K_r(h)\mathbb{K}\nabla h) = Q$$

A *continuation parameter* q is introduced to control "complexity" of the equation:

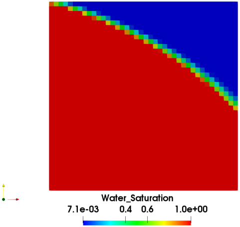
$$-\nabla \cdot (K_r^q(h))\mathbb{K}\nabla h) = Q, \quad 0 \leq q \leq 1$$

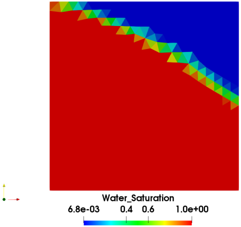
or

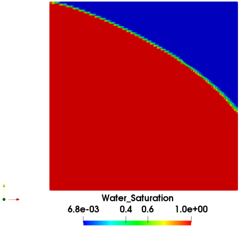
$$-\nabla \cdot ((1 + q(K_r(h) - 1))\mathbb{K}\nabla h) = Q, \quad 0 \leq q \leq 1$$

- We start with a linear problem
- After solving equation with some q, we move to a greater q and use previous solution as initial estimate

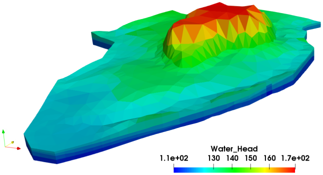
2D water flow through a dam

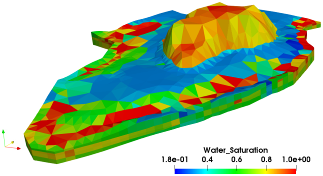






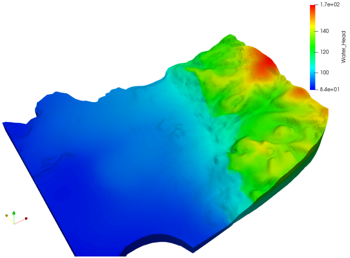
Landfill problem

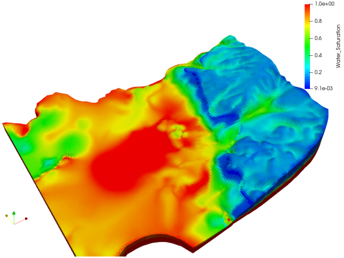




- Heterogenous domain with anisotropic hydraulic conductivity tensor
- Rivers

Real-world problem





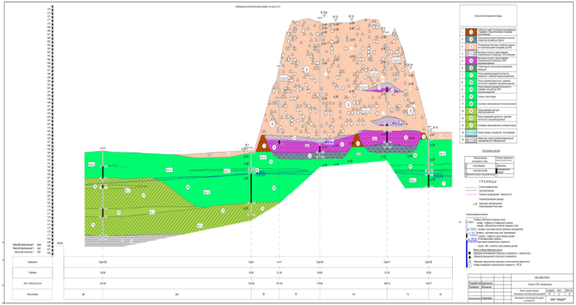
- Heterogenous domain with anisotropic hydraulic conductivity tensor
- Lakes, rivers
- Relaxation helps reduce number of steps in the continuation method
- Continuation method allows to use MPFA!

Solution time comparison

Problem	Time-stepping	Continuation	Speed-up
Dam, tri 900, TPFA	3.9	1.8	2
Dam, tri 900, MPFA	–	77	–
Dam, hex 10000, TPFA	115	36	3.2
Landfill, tri 5700, TPFA	401	58	6.9
RWP, tri 3900, TPFA	17.5	4.9	3.6
RWP, tri 3900, MPFA	–	46.2	–
RWP, tri 28500, TPFA	3126	33	94

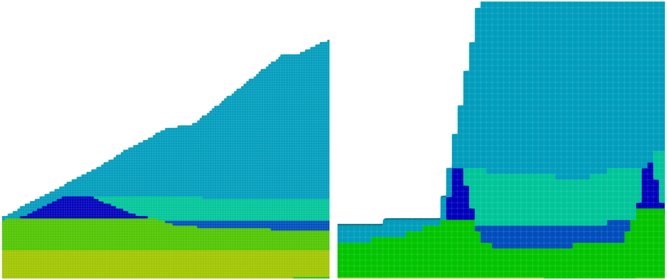
Solution time, s

2D landfill problem



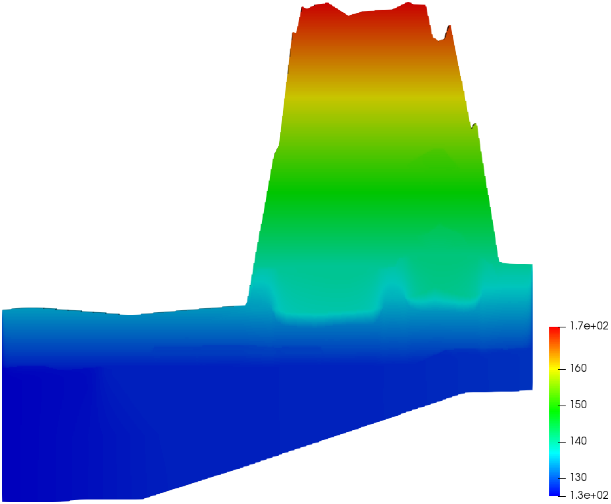
- Heterogenous anisotropic hydraulic conductivity tensor with jumps up to 5 orders of magnitude, different water retention curves for each material
- It is practically impossible to solve the problem using time-stepping even on coarse cubic meshes!
- The continuation method allows us to solve the problem even on fine meshes

2D landfill problem: meshes

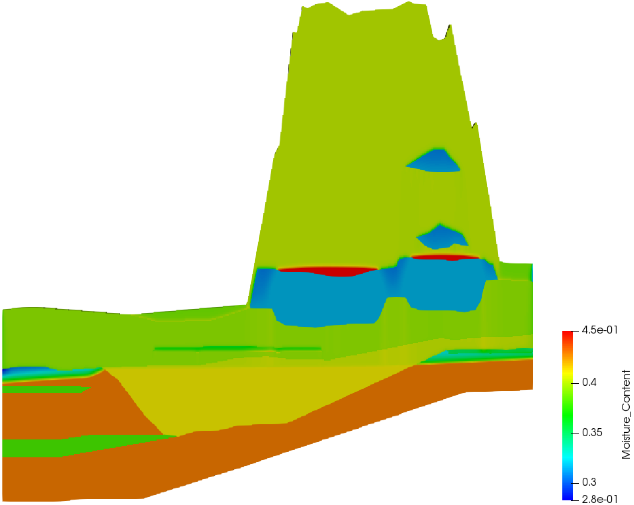


- Mesh resolution in vertical direction is more important
- The domain is shrinked 10 times in horizontal direction
- Hydraulic conductivity is changed accordingly

2D landfill problem: water head



2D landfill problem: water content



Conclusions

- We considered numerical modeling of variably saturated flow in GeRa software
- We compared two methods for solution of the steady-state equation
- The continuation method can significantly reduce modeling time
- The continuation method allows for the use of complicated discretization schemes
- The continuation method gives principal possibility to solve some hard problems

Thank you for your attention!