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Structuring preconditioners for unstructured meshes*

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Abstract – An application of a fictitious space technique to the construction of preconditioners for stiffness matrices generated by finite element method on completely unstructured meshes is considered in the paper. Numerical tests for model elliptic operators are presented.

In recent years a number of efficient multilevel techniques [1,3,4,11] have been proposed for either solving or preconditioning systems of grid equations approximating second-order elliptic boundary value problems. Many of these methods are widely used in research and industrial problems. However, multilevel and multigrid methods deal with meshes possessing certain structure or hierarchy. It is the hierarchy which provides efficiency of the method. The multilevel methods, in general, reduce the problem to be solved on a complicated or rather fine mesh to a set of problems which are easy to solve.

In many scientific and engineering applications there is a necessity to solve problems in domains with complicated geometry where it is very hard or even impossible to construct a hierarchical grid. Thus, the finite element discretizations should be performed on completely unstructured grids composed either of triangles in 2D or of tetrahedra in 3D. This is also dictated by available adaptive mesh generators used for discretization of computational domain, which allows a significant reduction of the number of grid nodes (i.e. number of unknowns) while maintaining the quality of approximation.

Recently, several efficient multigrid-type algorithms for finite element elliptic systems associated with the unstructured grid have been presented in [2,5]. They are based on the specific construction of the grid hierarchy for a given unstructured mesh and the application of a multigrid scheme on the produced sequence of grids.

For preconditioning a discrete model operator associated with unstructured triangulation we apply several types of multilevel structured preconditioners taking advantage of the fictitious space technique [18]. Given an unstructured mesh, we generate a structured hierarchical grid which 'approximates' the original mesh in some sense. With special interpolation operators we reduce the problem of constructing a preconditioner in a space associated with the unstructured mesh to the problem of constructing the preconditioner in a fictitious space corresponding to the structured grid. Within a fictitious space we apply multilevel preconditioners of three types: domain decomposition, multigrid, and BPX. A multilevel local refinement preconditioner [3,21] has been chosen as one of domain decomposition algorithms. An algebraic multigrid/substructuring preconditioner [8,11] and BPX preconditioner [4]

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present a multigrid technique and an additive multilevel method, respectively. Since each of the above preconditioners is defined for its own type of structured mesh and fictitious space, we compare the efficiency of the resulting preconditioners on unstructured meshes in terms of the experimental condition number of the preconditioned stiffness matrix and the ratio of CPU time for solving a system with a preconditioner to CPU time for stiffness matrix-vector multiplication.

The paper is organized as follows. In Section 1 we pose the problem to be considered and introduce a fictitious space technique. In Sections 2, 3, and 4 we give descriptions of multilevel methods and related fictitious spaces for the case of algebraic multigrid, BPX, and multilevel local refinement preconditioners, respectively. In Section 5 when considering several types of completely unstructured regular triangulations of a unit square, we make a comparison of the results of numerical experiments for the above preconditioners.

1. FICTITIOUS SPACE TECHNIQUE

Let Π be a unit square and Π^h be its regular unstructured triangulation [6]. Define a finite element space V^h of functions which are continuous in $\bar{\Pi}$, linear at each triangle T_i from Π^h and vanish at $\partial\Pi$.

Let N_0 be a number of interior nodes in Π^h and let $N_0 \times N_0$ symmetric positive definite stiffness matrix A_0 be defined by

$$(A_0 u, v) = \int_{\Pi} \nabla u^h \nabla v^h \, dx \qquad \forall u^h, v^h \in V^h \tag{1.1}$$

where $u,v \in R^{N_0}$ are restrictions of finite element functions $u^h, v^h \in V^h$.

Our goal is to construct a symmetric positive definite operator B_0 called a preconditioner for A_0 such that the condition number of the matrix $B_0^{-1}A_0$ is bounded by a constant independent of Π^h and the system with the matrix B_0 is easy to solve.

Lemma 1.1 [18]. Let H_0 and H be Hilbert spaces with the scalar product $(u_0,v_0)_{H_0}$ and $(u,v)_H$, respectively, and let A_0 and A be selfadjoint positive definite and continuous operators in the spaces H_0 and H:

$$A_0 \colon H_0 \to H_0, \qquad A \colon H \to H.$$

Let R be a linear operator such that

$$R \colon H \to H_0$$

$$(A_0 Rv, Rv)_{H_0} \leqslant c_R (Av, v)_H \qquad \forall v \in H. \tag{1.2}$$

Furthermore, let there exist an operator T such that

$$T \colon H_0 \to H, \qquad RTu_0 = u_0$$

$$c_T (ATu_0, Tu_0)_H \leqslant (A_0 u_0, u_0)_{H_0} \qquad \forall u_0 \in H_0. \tag{1.3}$$

Here c_R and c_T are positive constants.

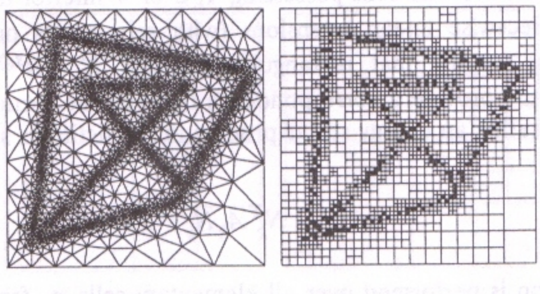


Figure 1. Unstructured mesh Π^h and initial structured mesh Q^h.

Then

$$c_T(A_0^{-1}u_0,u_0)_{H_0} \leqslant (RA^{-1}R^*u_0,u_0)_{H_0} \leqslant c_R(A_0^{-1}u_0,u_0)_{H_0} \quad \forall u_0 \in H_0. \quad (1.4)$$

Here R^* is the operator adjoint to R with respect to the scalar products $(u_0,v_0)_{H_0}$ and $(u,v)_H$ such that

$$R^*: H_0 \rightarrow H$$

$$(R^*u_0,v)_H = (u_0,Rv)_{H_0}.$$

In what follows, along with structured preconditioners we shall consider several types of an original space H_0 and a fictitious space H related with the operator A_0 and a multilevel preconditioner B for the operator A, respectively. We shall also introduce interpolation operators $R: H \rightarrow H_0$ and their right inverses $T: H_0 \rightarrow H$ satisfying conditions (1.2), (1.3).

Now, given the unstructured mesh Π^h, we dwell on the construction of a structured mesh Q^h connected with Π^h, which is an initial point for defining the fictitious space. Starting from the single cell coinciding with Π we split this cell into four equal subcells, provided that the cell has more than one vertex from Π^h, and obtain a new set of cells. Then we apply the above splitting procedure to each unsplitted cell having more than one vertex from Π^h and repeat the process as long as this is possible. Thus, we have constructed a mesh Q^h possessing quad-tree structure [14], each elementary (i.e. unsplitted) cell of Q^h containing no more than one vertex from Π^h (see Fig. 1). It is clear that the number of elementary cells is bounded from above by $4N_0$.

2. ALGEBRAIC MULTIGRID PRECONDITIONER

Denote the number of interior nodes in Q^h by N and define an original space $H_0 = R^{N_0}$ and a fictitious space $H = R^N$ with Eucledian scalar products, assuming that there is a one-to-one correspondence between the entries of vectors from R^{N_0} (R^N) and the interior nodes from Π^h (Q^h). Split each elementary cell q_k of Q^h by its 'north–east' diagonal into two triangles and introduce a local stiffness matrix A_k in the cell in a standard FEM way:

$$(A_k u_k,v_k) = \int_{q_k} \nabla u_k^h \nabla v_k^h \, \mathrm{d}x \qquad \forall u_k,v_k \in R^{s_k}, \quad s_k = 1,2,4 \qquad (2.1)$$

where $s_k = 1,2,4$ stands for the cells possessing 1, 2 or 4 interior nodes, respectively, and u_k^h, v_k^h are piecewise linear extensions (with zero traces on $\partial\Pi$) of vectors $u_k, v_k \in R^{s_k}$. Let us assume that rectangular matrices $N_k \in R^{s_k \times N}$ have nonzero entries equal to 1 and provide a correspondence between the vertices of q_k and R^N. The matrix A generating an energy scalar product in the fictitious space H is defined by

$$A = \sum_{q_k} N_k^{\mathrm{T}} A_k N_k$$

where the summation is performed over all elementary cells q_k from Q^h. According to the construction of the structured mesh Q^h we have a one-to-one correspondence between the nodes from Π^h and the subset of all elementary cells in Q^h. Thus, we can set a mapping r of all vertices x_l from Π^h onto the so-called reference nodes y_l in Q^h, which are the 'south–west' corners of reference cells. Then the interpolation operator $R: H \to H_0$ is easy to define:

$$(Ru)[x_l] = u[r(x_l)] \tag{2.2}$$

for all interior nodes x_l from Π^h, where $\mathrm{a}[x_l]$ denotes the value of a vector a entry associated with the node x_l.

There are several ways of defining an operator T. We shall use the simplest one:

$$(Tu)[y_l] = \begin{cases} u[r^{-1}(y_l)] & \text{if } y_l \text{ is a reference node} \\ u[r^{-1}(y_k)] & \text{otherwise, } y_k \text{ is the closest reference node to } y_l. \end{cases} \tag{2.3}$$

It can be shown that there exist constants $c_R > 0$, $c_T > 0$ depending on minimal angle of regular triangulation Π^h such that conditions (1.2), (1.3) are satisfied. The technique for proving it can be found, for instance, in [15–18]. Now, based on the fictitious space lemma statement one can use an operator $RA^{-1}R^{\mathrm{T}}$ as a preconditioner for A_0^{-1}. However, the matrix A is not easily invertible and therefore we have to substitute it by its preconditioner.

Preparatory to considering a multigrid preconditioner for A, we first discuss a two-grid one. Suppose we have two structured meshes Q_{l-1}^h and Q_l^h, the latter being obtained from Q_{l-1}^h by partitioning several elementary cells of minimal size into smaller subcells (see Fig. 2).

Split all the nodes from Q_l^h into 3 groups. The first group is formed by the nodes from Q_{l-1}^h, the second one contains 'new' nodes which belong to edges from Q_{l-1}^h, while the third one contains the rest.

In accordance with the splitting, the stiffness matrix $A^{(l)}$ associated with Q_l^h can be represented in the block form:

$$A^{(l)} = \begin{bmatrix} A_{11}^{(l)} & A_{12}^{(l)} & 0 \\ A_{21}^{(l)} & A_{22}^{(l)} & A_{23}^{(l)} \\ 0 & A_{32}^{(l)} & A_{33}^{(l)} \end{bmatrix}.$$

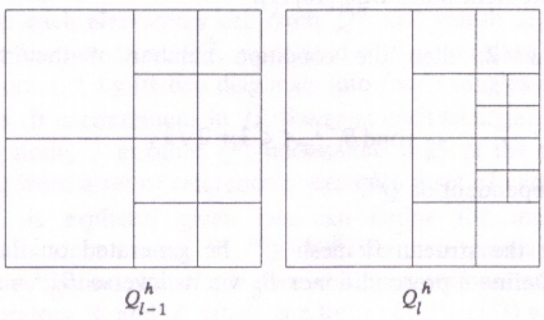


Figure 2. The 'coarse' and the 'fine' grids Q_{l-1}^h, Q_l^h.

Introduce a matrix $B^{(l)}$ by the formula:

$$B^{(l)} = F_l^{\mathrm{T}} \begin{bmatrix} \frac{1}{2}A^{(l-1)} & 0 & 0 \\ 0 & B_{22}^{(l)} & 0 \\ 0 & 0 & A_{33}^{(l)} \end{bmatrix} F_l, \quad F_l = \begin{bmatrix} I_{11}^{(l)} & 0 & 0 \\ [B_{22}^{(l)}]^{-1}A_{21}^{(l)} & I_{22}^{(l)} & 0 \\ 0 & [A_{33}^{(l)}]^{-1}A_{32}^{(l)} & I_{33}^{(l)} \end{bmatrix}$$

where $I_{jj}^{(l)}$, $j = 1,2,3$, are identity matrices of the corresponding orders, $B_{22}^{(l)}$ is certain diagonal matrix (for more details, see [9–11]), $A^{(l-1)}$ is a matrix resulting from the assembling of local stiffness matrices A_k, related to elementary cells q_k, over all q_k from Q_{l-1}^h. The matrix $B^{(l)}$ is referred to as a two-grid preconditioner [10,11], since to solve a system with the matrix $B^{(l)}$ one has to solve a system with $A^{(l-1)}$ corresponding to a 'coarse' grid and several systems with the diagonal matrices $B_{22}^{(l)}$, $A_{33}^{(l)}$.

Let h_M be a size of the smallest cell in the structured mesh Q^h corresponding to Π^h, and $M = \log_2 h_M^{-1}$. Let us take $s \geqslant 1$ to be some integer. Define sequentially the matrices for $l = 2,3,\dots,M$:

$$\frac{1}{2}[\hat{B}_{11}^{(l)}]^{-1} = \left(I^{(l-1)} - \prod_{j=1}^{s}\left(I^{(l-1)} - \tau_j^{(l)}[\hat{B}^{(l-1)}]^{-1}A^{(l-1)}\right)\right)[\hat{A}^{(l-1)}]^{-1}$$

$$\hat{B}^{(l-1)} = F_l^{\mathrm{T}} \begin{bmatrix} \hat{B}_{11}^{(l-1)} & 0 & 0 \\ 0 & B_{22}^{(l)} & 0 \\ 0 & 0 & A_{33}^{(l)} \end{bmatrix} F_l, \quad \hat{B}^{(1)} = B^{(1)}$$

where $\tau_j^{(l)}$ are the known iterative parameters [10,11]. The matrix $B_{\mathrm{AMG}} = \hat{B}^{(M)}$ is called the algebraic multigrid preconditioner for the matrix $A \equiv A^{(M)}$. For $s \geqslant 2$ the algorithm for solving the system with the matrix B_{AMG} can be viewed schematically as s-fold W-cycle.

The following statement holds true [10,11].

Theorem 2.1. If $s = 2$, then the condition number of the matrix $B_{\mathrm{AMG}}^{-1}A$ is estimated by

$$\operatorname{cond} B_{\mathrm{AMG}}^{-1}A \leqslant 3 + 2\sqrt{3}\,.$$

This estimate is independent of Q^h.

Corollary 2.1. Let the structured mesh Q^h be generated on the basis of regular triangulation Π^h. Define a preconditioner B_0 via its inverse $B_0^{-1} = RB_{\mathrm{AMG}}^{-1}R^{\mathrm{T}}$. Then

$$\operatorname{cond} B_0^{-1}A_0 \leqslant \frac{c_R}{c_T}(3 + 2\sqrt{3})$$

where the constants c_R, c_T depend on the minimal angle of Π^h.

Denote by n_l, $l = 1,\dots,M$, the number of interior nodes in all (not only elementary) cells of size 2^{-l} in Q^h. The arithmetical complexities, w_{AMG} and w_{B_0}, of solving systems with the matrices B_{AMG} and B_0, respectively, can be estimated by the formula

$$w_{B_0} \simeq w_{\mathrm{AMG}} \leqslant C(n_M + s(n_{M-1} + s(n_{M-2} + \dots)))\,.$$

Hereinafter we denote by C a generic positive constant.

Consider three types of regular triangulations Π^h:

(1) Π^h is quasi-uniform, then $n_l/n_{l-1} \simeq 4$ for almost all l, and

$$w_{B_0} \simeq w_{\mathrm{AMG}} \leqslant CN_0\,;$$

(2) Π^h is refined towards an arbitrary curve, the splitting of the curve being quasi-uniform. Then $n_l/n_{l-1} \geqslant 2$ for almost all l, and $w_{B_0} \simeq w_{\mathrm{AMG}} \leqslant CN_0 \log_2 N_0$;

(3) Π^h is refined towards a set of isolated points. In this case the computational cost of the algebraic multigrid preconditioner may be rather large, and there is no point in using it when $n_l \simeq n_{l-1}$ for many l. However, certain improvement of the preconditioner seems to be possible. This may be a subject of further investigations.

3. ADDITIVE MULTILEVEL PRECONDITIONER

Since the BPX preconditioner [4,19] has been introduced by using finite element spaces, we will choose both fictitious and original spaces as finite element spaces. For the original space we take $H_0 = V^h$ equipped with L_2 scalar product. Now we dwell on the construction of the fictitious space. The vertices of the structured mesh Q^h can be grouped as nodes contributing degrees of freedom into a space of continuous functions which are bilinear at each elementary cell from Q^h, and the rest is called 'slave' nodes. A slave node appears when an interior edge point for some elementary cell becomes a vertex of another elementary cell from Q^h. Following the construction in the previous section we associate each reference cell from the structured mesh Q^h with its 'south–west' corner. However, if the reference node is a slave one, we divide the reference cell into four equal subcells and make the middle of the cell a reference node. The resulting structured mesh $\tilde{Q}^h$ is a basis for the construction of the fictitious space H which is a space W^h consisting of the functions which are continuous in $\bar{\Pi}$, piecewise linear at each elementary cell from $\tilde{Q}^h$ and vanish at $\partial\Pi$. To specify the shape of the basis function from W^h, associated with a non-slave node, we split each elementary cell from $\tilde{Q}^h$ by its two diagonals into four triangles and define the basis function as follows. It is continuous in $\bar{\Pi}$, linear at each triangle, and is equal to 1 at the corresponding node, 0 at other $\tilde{Q}^h$ nodes, and 0.25 at the support cell centres. Since the mapping from a set of reference nodes onto a set of vertices of unstructured triangulation Π^h is explicitly given, we can define the interpolation operator $R\colon W^h \to V^h$ and its right inverse $T\colon V^h \to W^h$ via their matrix formulation as it is done in the previous section. Let $N\times N$-matrix A be a stiffness matrix associated with W^h, then the operators R and T satisfy conditions (1.2), (1.3) with the constants c_R and c_T depending on minimal angle of triangulation Π^h.

Preparatory to specifying the BPX preconditioner for the matrix A, we first introduce a nodal basis function Φ_i^l of level l. The mesh $\tilde{Q}^h$ is a composition of M uniform grids $\tilde{Q}_l^h$, $l = 1,\ldots,M$, which are unions of all the cells of size 2^{-l} from $\tilde{Q}^h$. Denote by $\bar{\Omega}_l$ the geometric locus of points belonging to closures of cells from $\tilde{Q}_l^h$ and by $\Omega_l = \bar{\Omega}_l \backslash \partial\bar{\Omega}_l$, $l = 1,\ldots,M$. Let W_l^h, $l = 1,\ldots,M$, be a finite element space consisting of the functions (see above) which are continuous in $\bar{\Omega}_l$, piecewise linear at each cell from $\tilde{Q}_l^h$ and vanish at $\partial\Omega_l$. For any node x_i^l from $\tilde{Q}_l^h$, $x_i^l \notin \partial\Omega_l$, we introduce a nodal basis function $\Phi_i^l \in W_l^h$, $\Phi_i^l(x_j^l) = \delta_{ij}$. The BPX multilevel preconditioner [4,19] is defined via its inverse

$$(B_{\text{BPX}}^{-1} u)^h = \sum_{l=1}^{M} \sum_{\Phi_i^l \in W_l^h} (\Phi_i^l, u^h)\Phi_i^l \tag{3.1}$$

where $(\mathbf{a})^h$ is the extension of the vector $\mathbf{a} \in R^N$ to W^h.

Theorem 3.1 ([19], see also references therein). The condition number of the matrix $B_{\text{BPX}}^{-1}A$ is bounded from above by the constant C_{BPX} independent of $\tilde{Q}^h$:

$$\operatorname{cond} B_{\text{BPX}}^{-1} A \leqslant C_{\text{BPX}}.$$

The numerical solution of a system with the BPX preconditioner can be realized as a V-cycle [4], therefore the arithmetical cost of the preconditioning $N\times N$-matrix A is $O(N)$ ops.

Corollary 3.1. For the fictitious space preconditioner B_0 defined via its inverse $B_0^{-1} = R\hat{B}_{\text{BPX}}^{-1}R^{\text{T}}$, the following estimate holds true:

$$\operatorname{cond} B_0^{-1} A_0 \leqslant \frac{c_R}{c_T} C_{\text{BPX}}$$

where C_{BPX} is a positive constant independent of Π^h, c_R, c_T are positive constants depending on the minimal angle of Π^h, while the arithmetical complexity of solving a system with the preconditioner B_0 is proportional to N_0.

4. MULTILEVEL LOCAL REFINEMENT PRECONDITIONER

We have chosen the multilevel local refinement technique [3,21] as one of domain decomposition algorithms. The distinguishing feature of these methods is that we solve discrete boundary value problems in subdomains of an initial domain. A peculiarity of the multilevel local refinement preconditioner is that it should be defined on the structured meshes with moderate changes in local sizes of cells forming a mesh. More precisely, the sizes of any two adjacent elementary cells may differ by no more than two times. These meshes are referred to as balanced. Given an initial structured mesh Q^h, using certain balancing procedure we obtain the balanced mesh $\hat{Q}^h$ with the arithmetical cost proportional to the number of elementary cells in Q^h. We define the multilevel local refinement preconditioner on the basis of a finite element space and hence we apply to $\hat{Q}^h$ the procedure of exhausting slave nodes from the set of reference ones, as it was described in the previous section. On repeating the balancing procedure we obtain a balanced structured mesh $\tilde{Q}^h$ with reference nodes which are not slaves. We split each elementary cell of $\tilde{Q}^h$ by its 'north–east' diagonal and produce a triangulation $\tilde{\omega}^h$ which is a basis for construction of a fictitious space. The fictitious space H is a finite space W^h equipped with L_2 scalar product and consisting of the functions which are continuous in $\bar{\Pi}$, linear at each triangle from $\tilde{\omega}^h$, and vanish at $\partial\Pi$. The original space H_0 is chosen to be V^h (see the previous section).

We define a $N \times N$-matrix A as a stiffness matrix related to W^h. Note that $N < CN_0$. The interpolation operator R and its right inverse T are defined via their matrix counterparts in Section 2. By construction $\tilde{Q}^h$ is finer, than $\tilde{Q}^h$ in the above section and therefore the operators R and T satisfy conditions (1.2) and (1.3) with the same positive constants c_R and c_T.

In order to specify a multilevel local refinement preconditioner for A, we introduce the following notation. Let Q_l be a set of all the cells of size 2^{-l}, $l = 1,\dots,M$, from $\tilde{Q}^h$, V_{Q_l} be a set of their vertices (except slave nodes), $\bar{\Omega}_l$ be a geometric locus of points belonging to the closures of elements from Q_l (i.e. maximal 'subdomain' of Π, where the elementary cell size does not exceed 2^{-l}). Denote by m the maximal l, for which $\Omega_l = \Pi$. We split the set

$$\bigcup_{l=m}^{M} V_{Q_l}$$

into the sum of subsets:

$$V_m \subset \Theta_m = \bar{\Omega}_m \backslash \bar{\Omega}_{m+1} \backslash \partial\Pi$$

$$V_{\Gamma_m} \subset \Gamma_m = \bar{\Theta}_m \cap \bar{\Omega}_{m+1} \backslash \partial\Pi$$

$$V_{m+1} \subset \Theta_{m+1} = \bar{\Omega}_{m+1} \backslash \Gamma_m \backslash \bar{\Omega}_{m+2} \backslash \partial\Pi$$

$$\cdots\cdots\cdots\cdots$$

$$V_{\Gamma_{M-1}} \subset \Gamma_{M-1} = \bar{\Theta}_{M-1} \cap \bar{\Omega}_M \backslash \partial\Pi$$

$$V_M \subset \Theta_M = \bar{\Omega}_M \backslash \Gamma_{M-1} \backslash \partial\Pi .$$

This splitting generates the following block representation of the matrix A:

$$A = \begin{bmatrix} A_m & A_{m,\Gamma_m} & & & & \\ A_{\Gamma_m,m} & A_{\Gamma_m} & A_{\Gamma_m,m+1} & & & \\ & A_{m+1,\Gamma_m} & A_{m+1} & A_{m+1,\Gamma_{m+1}} & & \\ & & A_{\Gamma_{m+1},m+1} & A_{\Gamma_{m+1}} & \ddots & \\ & & & \ddots & \ddots & A_{\Gamma_{M-1},M} \\ & & & & A_{M,\Gamma_{M-1}} & A_M \end{bmatrix}. \quad (4.1)$$

For each subdomain Ω_j, $j = m+1,\dots,M$, we consider uniform meshes $Q^h_{j,j}$ and $Q^h_{j,j-1}$ with the cell size $h_{j,j} = h_j = 2^{-j}$ and $h_{j,j-1} = h_{j-1} = 2^{1-j}$, respectively.

The multilevel local refinement preconditioner [21] can be defined by the matrix

$$B_{\mathrm{DD}} = A + \begin{bmatrix} 0 & & & & & \\ & \tilde{S}_{\Gamma_m} - \hat{S}_{\Gamma_m} & & & & \\ & & 0 & & & \\ & & & \tilde{S}_{\Gamma_{m+1}} - \hat{S}_{\Gamma_{m+1}} & & \\ & & & & \ddots & \\ & & & & & \tilde{S}_{\Gamma_{M-1}} - \hat{S}_{\Gamma_{M-1}} & \\ & & & & & & 0 \end{bmatrix}. \quad (4.2)$$

where

$$\hat{S}_{\Gamma_j} = \hat{A}_{\Gamma_j} - \hat{A}_{\Gamma_j,j+1}\hat{A}^{-1}_{j+1}\hat{A}_{j+1,\Gamma_j}$$

$$\tilde{S}_{\Gamma_j} = \tilde{A}_{\Gamma_j} - \tilde{A}_{\Gamma_j,j+1}\tilde{A}^{-1}_{j+1}\tilde{A}_{j+1,\Gamma_j}$$

and the matrices

$$\begin{bmatrix} \hat{A}_{\Gamma_j} & \hat{A}_{\Gamma_j,j+1} \\ \hat{A}_{j+1,\Gamma_j} & \hat{A}_{j+1} \end{bmatrix}, \quad \begin{bmatrix} \tilde{A}_{\Gamma_j} & \tilde{A}_{\Gamma_j,j+1} \\ \tilde{A}_{j+1,\Gamma_j} & \tilde{A}_{j+1} \end{bmatrix}$$

are the stiffness matrices on $Q^h_{j+1,j+1}$ and $Q^h_{j+1,j}$, respectively, $j = m,\dots,M-1$.

Assumption 4.1. The spectrum $\mathrm{Sp}\, B^{-1}_{\mathrm{DD}} A$ belongs to the interval

$$\mathrm{Sp}\, B^{-1}_{\mathrm{DD}} A \subset [c_{\mathrm{DD}}, 1]$$

where the constant $c_{\mathrm{DD}} > 0$ is independent of $\tilde{Q}^h$. The numerical experiments show that $c^{-1}_{\mathrm{DD}} < 1.5$.

It can be shown that the procedure for solving the system with the matrix B_{DD} has the form of V-cycle, at each node of which one solves the systems with the matrices $\hat{A}_l$, $l = M,\ldots,m+1,m,m+1,\ldots,M$, where $\hat{A}_m$ is the stiffness matrix associated with Q_m. However, to obtain the exact solutions of systems with $\hat{A}_l$, $l = M,\ldots,m+1$, may be very expensive due to shape irregularity of the subdomain Ω_l. That is why we introduce the preconditioner $B_{\mathrm{DD},\varepsilon}$ which differs algorithmically from B_{DD} in that the preconditioner provides approximate solutions of systems with matrices $\hat{A}_l$, $l = M,\ldots,m+1$, with certain precision $\varepsilon > 0$.

Let $\mathrm{Sp}\hat{A}_l \subset [\lambda_l, \Lambda_l]$, $\lambda_l > 0$, $l = M,\ldots,m+1$. Introduce the operator [1,20]

$$\hat{A}_l^{\varepsilon} = \hat{A}_l\left(I_l - \prod_{i=1}^{\nu_l}(I_l - \tau_{l,i}\hat{A}_l)\right)^{-1} \tag{4.3}$$

where $\tau_{l,i}$ is the reciprocal of the roots of the polynomial

$$P_{\nu_l}(x) = \left(1 + T_{\nu_l}\left(\frac{\Lambda_l + \lambda_l - 2x}{\Lambda_l - \lambda_l}\right)\right)\Big/\left(1 + T_{\nu_l}\left(\frac{\Lambda_l + \lambda_l}{\Lambda_l - \lambda_l}\right)\right)$$

specially arranged [13] to avoid the instability of the iterative process.

The number of iterations at l-level depends on ε, Λ_l, λ_l:

$$\nu_l = \left[\frac{\log \varepsilon/2}{\log \rho_l}\right], \qquad \rho_l = \frac{\sqrt{\Lambda_l} - \sqrt{\lambda_l}}{\sqrt{\Lambda_l} + \sqrt{\lambda_l}}.$$

Theorem 4.1. Let $\rho(A^{-1})$, $\rho(\hat{A}_l)$ be spectral radii of the matrices A^{-1} and $\hat{A}_l$, $l = m,\ldots,M$, respectively. If there exists $\delta > 0$ such that

$$\sigma\delta < 1, \quad \sigma = \rho(A^{-1})\sum_{l=m}^{M}\rho(\hat{A}_l), \quad \hat{A}_l \leqslant \hat{A}_l^{\varepsilon} \leqslant (1+\delta)\hat{A}_l, \qquad l = m,\ldots,M \tag{4.4}$$

then

$$\mathrm{Sp}(B_{\mathrm{DD},\varepsilon}^{-1}A) \subset \left[\frac{c_{\mathrm{DD}}}{1 + c_{\mathrm{DD}}\sigma\delta}, \frac{1}{1-\sigma\delta}\right]$$

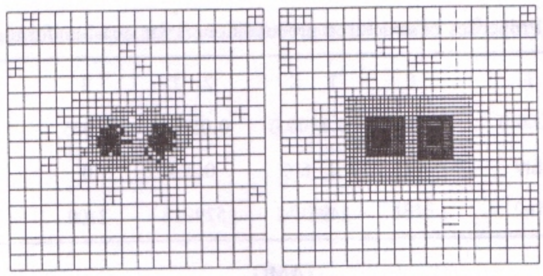
where the constant c_{DD} is taken from Assumption 4.1.

Corollary 4.1. Let the hypotheses of Theorem 4.1 hold, then for the fictitious space preconditioner B_0 specified by the relation $B_0^{-1} = RB_{\mathrm{DD},\varepsilon}^{-1}R^{\mathrm{T}}$, the following estimate is valid:

$$\mathrm{cond}\, B_0^{-1}A_0 \leqslant \frac{c_R}{c_T}\,\frac{1 + c_{\mathrm{DD}}\sigma\delta}{(1-\sigma\delta)c_{\mathrm{DD}}}$$

where c_R, c_T are the positive constants depending on the minimal angle of Π^h.

Since further refinement of the structured mesh $\tilde{Q}^h$ does not increase the ratio c_R/c_T, we can change the shape of the domain $\Omega_l \to \Omega_l'$, $l = M,\ldots,m+1$, for the sake of the efficiency of $\hat{A}_l$ solver. We propose an empiric algorithm which minimizes the arithmetical cost of V-cycle. Starting from the most expensive discrete boundary value problem solvers at the finest mesh level, the algorithm modifies $\tilde{Q}^h$ (if it makes



(a) (b)

Figure 3. (a) An initial structured balanced mesh and (b) a final modified mesh.

sense) by extending the connected parts of Ω_l to minimal covering rectangles (see Fig. 3) and employs a fast direct solver there (see, for example, [12]). This procedure is performed over all the levels $l = M,\ldots,m+1$, which are used in the preconditioner, and is rather expensive since it evaluates the arithmetical cost for an inexact solver of any discrete boundary value problem, which appears in the algorithm. However, the above procedure is called only at the initialization step and therefore it does not affect the computational efficiency of the preconditioner $B_{\mathrm{DD},\varepsilon}$. Since the proposed multi-level preconditioner exploits inexact solvers on subdomains, its efficiency depends on the precision δ (4.4) of inexact solvers. In our numerical experiments we use $\delta \simeq 0.01 - 0.05$ that yields both the small condition number of $B_{\mathrm{DD},\varepsilon}^{-1}A$ and reasonable efficiency.

In general, it is next to impossible to make a rational estimate for the arithmetical complexity $w_{\mathrm{DD},\varepsilon}$ of the above preconditioner. However, for certain classes of unstructured meshes Π^h we can evaluate the efficiency of the method. Let n_l be orders of the matrices $\hat{A}_l$, $l = M,\ldots,m$. Consider three types of regular triangulations Π^h:

(1) Π^h is a quasi-uniform triangulation with a mesh parameter h. Then the computational cost of solving a system with the preconditioner in the fictitious space is less (due to the optimization procedure) than the cost on the uniform grid with the meshstep equal to the smallest cell size in $\tilde{Q}^h$, which is proportional to h. Since the fast direct solver requires $O(n\log_2 n)$ operations, where n is the number of nodes in the uniform grid, $n = O(h^{-2})$, we can easily obtain

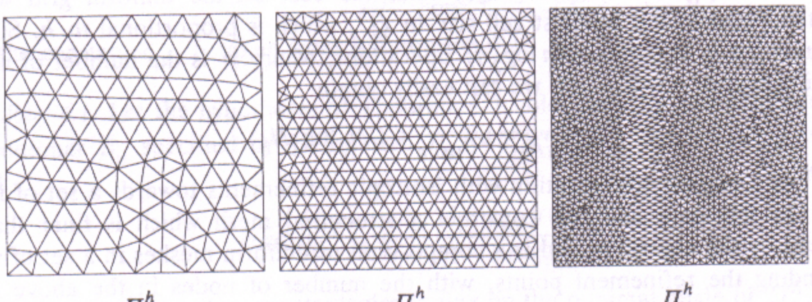
$$w_{B_0} \simeq w_{B_{\mathrm{DD},\varepsilon}} \leqslant CN_0\log_2 N_0;$$

(2) Π^h is a regular triangulation with isotropic refinements towards a set of isolated points. In this case one can construct a structured mesh which is finer than that produced by the algorithm and is a composition of uniform meshes in a set of squares surrounding the refinement points, with the number of nodes in the above meshes being proportional to n_l at each level of refinement $l = m,\ldots,M$. For this type of structured mesh, fast direct solvers can be applied, which yields the estimate for less expensive preconditioner produced by the optimization algorithm

$$w_{B_0} \simeq w_{B_{\mathrm{DD},\varepsilon}} \leqslant C\sum_{l=m}^{M} n_l\log_2 n_l \leqslant CN_0\log_2 N_0;$$

Table 1. Properties of structured preconditioners for quasi-uniform meshes.

Π_i^h	1	2	3	4
$h_{\min}$	0.1	0.05	0.025	0.01
Decrement	0	0	0	0
N_{Π^h}	149	529	2100	12784
AMG				
N	295	1001	3887	18226
$\tau(B_0^{-1})/\tau(A_0)$	2.1	2.1	2.2	1.9
$\tau(Q^h)/\tau(A_0)$	1.3	1.4	1.3	1.2
$\operatorname{cond} B_0^{-1}A_0$	9.6	12.8	9.1	10.8
BPX				
N	211	822	3592	17064
$\tau(B_0^{-1})/\tau(A_0)$	1.2	1.2	1.2	1.0
$\tau(Q^h)/\tau(A_0)$	2.7	2.1	1.9	2.1
$\operatorname{cond} B_0^{-1}A_0$	6.5	7.1	8.3	8.6
DD, $\delta = 0.05$				
N	231	978	4000	17086
$\tau(B_0^{-1})/\tau(A_0)$	1.4	1.7	2.1	1.8
$\tau(Q^h)/\tau(A_0)$	31	24	35	58
$\operatorname{cond} B_0^{-1}A_0$	3.6	4.3	4.1	5.4

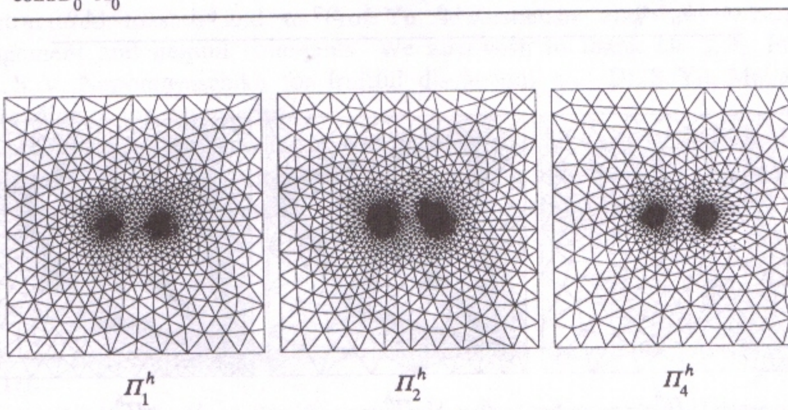


Π_1^h Π_2^h Π_3^h

Figure 4. Quasi-uniform triangulations.

Table 2. Properties of structured preconditioners for meshes refined to two points.

Π_i^h	1	2	3	4	5
$h_{\min}$	0.005	0.001	0.0005	0.0005	0.0005
Decrement	0.2	0.2	0.2	0.3	0.6
N_{Π^h}	819	1493	1816	947	386
AMG					
N	1394	2547	3097	1638	745
$\tau(B_0^{-1})/\tau(A_0)$	5.7	14.3	24.2	36.8	32.4
$\tau(Q^h)/\tau(A_0)$	1.1	1.1	1.2	1.3	1.3
$\operatorname{cond} B_0^{-1}A_0$	12.3	10.1	11.4	9.6	12.4
BPX					
N	1425	2595	3228	1737	757
$\tau(B_0^{-1})/\tau(A_0)$	1.4	1.4	1.5	1.7	1.9
$\tau(Q^h)/\tau(A_0)$	2.1	2.2	2.0	2.1	2.4
$\operatorname{cond} B_0^{-1}A_0$	7.6	7.8	9.3	8.6	6.3
DD, $\delta = 0.05$					
N	2071	3829	4828	2055	898
$\tau(B_0^{-1})/\tau(A_0)$	6.3	7.2	7.4	9.4	6.3
$\tau(Q^h)/\tau(A_0)$	27	26	27	22	25
$\operatorname{cond} B_0^{-1}A_0$	4.7	4.8	4.9	4.7	4.5

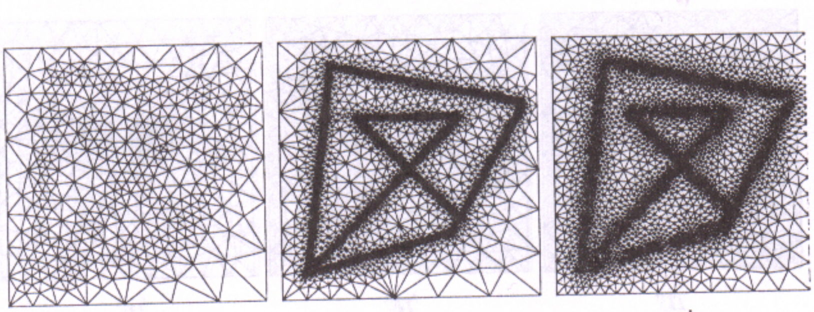


Π_1^h Π_2^h Π_4^h

Figure 5. Regular triangulations refined to points.

Table 3. Properties of structured preconditioners for regular meshes refined to a curve.

Π_i^h	1	2	3	4
$h_{\min}$	0.03	0.01	0.003	0.01
Decrement	0.8	0.8	0.8	0.3
N_{Π^h}	446	1707	6443	3396
AMG				
N	775	3291	11729	6068
$\tau(B_0^{-1})/\tau(A_0)$	2.1	1.8	3.5	3.1
$\tau(Q^h)/\tau(A_0)$	1.2	1.1	1.2	1.2
$\mathrm{cond}\, B_0^{-1}A_0$	9.3	11.9	11.4	9.1
BPX				
N	752	3325	11689	6043
$\tau(B_0^{-1})/\tau(A_0)$	1.1	1.4	1.4	1.3
$\tau(Q^h)/\tau(A_0)$	1.8	2.2	2.6	2.0
$\mathrm{cond}\, B_0^{-1}A_0$	6.5	9.3	9.5	8.9
DD, $\delta = 0.05$				
N	751	3444	11907	7018
$\tau(B_0^{-1})/\tau(A_0)$	8.6	12	15	7.9
$\tau(Q^h)/\tau(A_0)$	35	36	38	43
$\mathrm{cond}\, B_0^{-1}A_0$	4.0	5.7	6.1	4.9



Π_1^h Π_3^h Π_4^h

Figure 6. Regular triangulations refined to a curve.

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5. NUMERICAL

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REFERENCES

1. O. Axelsson a
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2. R. Bank and J
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(3) Π^h is a regular triangulation with strong refinements towards a set of arbitrary curves. Then it can be shown that starting from some coarse level $k \geqslant m$, $k \ll M$, all the matrices $\hat{A}_l$, $l = k,\ldots,M$, to be approximately inverted, have the bounded condition numbers. The total cost of solvers for $\hat{A}_l$, $l = m,\ldots,k-1$, can be evaluated by the computational cost of the fast direct solver on a uniform mesh of size $h_{k-1} = 2^{-k+1}$. Since for the above meshes the number of nodes lying outside the vicinity of the curve is small compared to the total number of vertices ($kh_{k-1}^{-2} < CN$), one can derive

$$w_{B_0} \simeq w_{B_{\mathrm{DD},\varepsilon}} \leqslant C\left(\sum_{l=k}^{M} n_l \log_2 \delta^{-1} + kh_{k-1}^{-2}\right) \leqslant CN_0 \log_2 \delta^{-1}.$$

5. NUMERICAL EXPERIMENTS

In the tables presented below we show the condition numbers of preconditioned stiffness matrices associated with unstructured triangulations, which have been analysed in the previous sections. These triangulations are quasi-uniform meshes and regular triangulations refined towards either a set of isolated points or a set of arbitrary curves. We vary the number of nodes N_{Π^h} in a mesh by changing the minimal mesh size $h_{\min}$ and the power of refinement which is defined as relative decrement of mesh size in the direction to a refinement set (i.e. points or curves). We also present the dimension N of fictitious space as well as the efficiency of preconditioners in terms of the ratio of time $\tau(B_0)$ for solving a system with the preconditioner B_0 to time $\tau(A_0)$ for stiffness matrix A_0-vector multiplication by the assembling procedure. Besides, we give the duration $\tau(Q^h)$ of the initialization procedure compared to stiffness matrix-vector multiplication time.

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REFERENCES

1. O. Axelsson and P. Vassilevski, Algebraic multilevel preconditioning methods. *Numer.Math.* (1989) **56**, 157–177.
2. R. Bank and J. Xu, The hierarachical basis multigrid method and incomplete LU decomposition. In: *Domain Decomposition Methods in Scientific and Engineering Computing, Contemporary Mathematics* (1994) **180**, 163–173.
3. J. H. Bramble, R. E. Ewing, J. E. Pasciak, and A. H. Schatz, A preconditioning technique for the efficient solution of problems with local grid refinement. *Comp. Meth. Appl. Mech. Eng.* (1988) **67**,149–159.

4. J. H. Bramble, J. E. Pasciak, and J. Xu, Parallel multilevel preconditioners. *Math. Comp.* (1990) **55**, 1–22.
5. T. Chan and B. Smith, Domain decomposition and multigrid algorithms for elliptic problems on unstructured meshes. In: *Domain Decomposition Methods in Scientific and Engineering Computing, Contemporary Mathematics* (1994) **180**, 175–189.
6. Ph. Ciarlet, *The Finite Element Method for Elliptic Problems*. North Holland, Amsterdam, 1978.
7. R. P. Fedorenko, Relaxation method of solving finite difference elliptic equations. *Zh. Vychisl. Mat. Mat. Fiz.* (1961) **1**, No. 5, 922–926 (in Russian).
8. Yu. R. Hakopian and Yu. A. Kuznetsov, Algebraic multigrid/substructuring preconditioners. *Sov. J. Numer. Anal. Math. Modelling* (1991) **6**, 453–483.
9. Yu. I. Iliash, Algebraic multigrid method for finite element appoximation of elliptic equations on composite grids. In: *Chislennye Metody i Prilozheniya*, Inst. Numer. Math., Russian Acad. Sci., Moscow, 1995, pp. 81–91 (in Russian).
10. Yu. A. Kuznetsov, Algebraic multigrid domain decomposition method, *Sov. J. Numer. Anal. Math. Modelling* (1989) **4**, 351–380.
11. Yu. A. Kuznetsov, Multigrid domain decomposition methods for elliptic problems. In: *Comput. Meth. for Applied Science and Engineering, VII*, 1987, pp. 605–616.
12. Yu. A. Kuznetsov and S. A. Finogenov, The package for solving elliptic problems with separable variables. In: *Chislennye Metody i Programmnoe Obespechenie*, Dept. Numer. Math., USSR Acad. Sci., Moscow, 1990, pp. 95–105 (in Russian).
13. V. I. Lebedev and S. A. Finogenov, On the application of arranged Chebyshev iterative parameters in iterative processes. *Zh. Vychisl. Mat. Mat. Phis.* (1976) **16**, No. 4, 845–905 (in Russian).
14. R. Löhner, Some useful data structures for the generation of unstructured grids. *Comm. Applied Numer. Methods* (1988) **4**, 123–135.
15. A. M. Matsokin, Norm preserving extension of grid functions. In: *Variacionnye Metody v Zadachah Chislennogo Analiza*, Novosibirsk, 1986, pp. 111–132 (in Russian).
16. A. M. Matsokin, Fictitious component and subspace alternating methods. *Doctoral thesis*, Novosibirsk, 1988 (in Russian).
17. S. V. Nepomnyaschikh, Domain decomposition and Schwarz method in a subspace for obtaining an approximate solution of elliptic boundary value problem. *PhD thesis*, Novosibirsk, 1986 (in Russian).
18. S. V. Nepomnyaschikh, Mesh theorems of traces, normalizations of functions traces and their inversion. *Sov. J. Numer. Anal. Math. Modeling* (1991) **6**, No. 3, 1–25.
19. P. Oswald, Multilevel finite element approximation: theory and applications. *B. G. Teubner*, Stuttgart, 1994.
20. Yu. V. Vassilevski, Domain decomposition method with inexact solvers in subdomains. In: *Chislennye Metody i Programmnoe Obespechenie*, Dept. Numer. Math., USSR Acad. Sci., Moscow, 1990, pp. 23–29 (in Russian).
21. O. Widlund, Optimal iterative refinement methods. In: *Proceedings of the Second Int. Symposium on DDM*, SIAM, Philadelphia, 1989, pp. 114–126.

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