*Math. Model. Nat. Phenom.*
**Vol. 5, No. 7, 2010, pp. 91-96**
DOI: 10.1051/mmnp/20105715

# Edge-based a Posteriori Error Estimators for Generating Quasi-optimal Simplicial Meshes

**A. Agouzal[1], K. Lipnikov[2] and Yu. Vassilevski[3] ***

[1] University Lyon1, Institute Camille Jordan, UMR 5208, 69100 Villeurbanne, France
[2] Los Alamos National Laboratory, Los Alamos, NM 87545, U.S.A.
[3] Institute of Numerical Mathematics, Gubkina str. 8, Moscow 119333, Russia

**Abstract.** We present a new method for generating a $d$-dimensional simplicial mesh that minimizes the $L^p$-norm, $p > 0$, of the interpolation error or its gradient. The method uses edge-based error estimates to build a tensor metric. We describe and analyze the basic steps of our method.

**Key words:** finite elements, anisotropic meshes, a posteriori error estimates
**AMS subject classification:** 65D05, 65D15, 65N15, 65N50

## 1. Introduction

Generation of a mesh adapted to a given function requires a specially designed metric. We develop further a new methodology, proposed originally in [1, 2], for generating a tensor metric $\mathfrak{M}$ from error estimates *prescribed to mesh edges*. The volume and the perimeter of a $d$-simplex measured in this metric control the norm of error or its gradient. The error equidistribution principle suggests to balance $\mathfrak{M}$-volumes and $\mathfrak{M}$-perimeters to produce a $\mathfrak{M}$-quasi-uniform mesh [3, 4].

The edge-based error estimates come usually from postprocessing or a posteriori error analysis of a discrete solution. Critical advantage of edge-based error estimates over cell-based error estimates is that they provide local directional information about the error. In this paper, we consider the problem of minimizing the $P_1$-interpolation error or its gradient, where cell-based and edge-based errors can be easily defined. Most methods that convert the cell-based errors into a metric lose directional information. We describe a new method that uses the edge-based errors to

*Corresponding author. E-mail: yuri.vassilevski@gmail.com

build a metric that preserves directional information. We illustrate the optimal error reduction with numerical experiments. We also summarize the existing error estimates.

## 2. Edge-based interpolation error estimates

Let $\Omega \subset \Re^d$ be a bounded polyhedral domain and $\Omega_h$ be a conformal simplicial mesh with $N_h$ $d$-simplexes. The volume of a $d$-simplex $\Delta$ and the total length of its edges in a metric $\mathfrak{M}$ are denoted by $|\Delta|_{\mathfrak{M}}$ and $|\partial\Delta|_{\mathfrak{M}}$, respectively. Let $\mathcal{I}_1 u$ be the continuous piecewise linear interpolant of $u$, and $\mathcal{I}_{1,\Delta} u$ be its restriction to $\Delta$. Our goal is to generate meshes that minimize the $L^p$-norm, $p \in (0, \infty]$, of the interpolation error $e = u - \mathcal{I}_1 u$ or its gradient $\nabla e$. Let us consider a $d$-simplex $\Delta$ with vertices $\mathbf{v}_i$, $i = 1, \ldots, d+1$, edge vectors $\mathbf{e}_k = \mathbf{v}_i - \mathbf{v}_j$, $1 \le i < j \le d+1$, and mid-edge points $\mathbf{c}_k$, $k = 1, \ldots, n_d$, where $n_d = d(d+1)/2$. Let $\lambda_i$, $i = 1, \ldots, d+1$, be the linear functions on $\Delta$ such that $\lambda_i(\mathbf{v}_j) = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker symbol. For every edge $\mathbf{e}_k$, we define the quadratic bubble function $b_k = \lambda_i \lambda_j$. Let $u$ be a continuous function. On each simplex $\Delta$, we consider its quadratic approximation $u_2 = \mathcal{I}_{2,\Delta} u$, where $\mathcal{I}_{2,\Delta} u$ is the Lagrange interpolant of $u$. The interpolation error for the linear approximation of $u_2$ is

$$e_2 = u_2 - \mathcal{I}_{1,\Delta} u_2 = 4 \sum_{k=1}^{n_d} (u_2(\mathbf{c}_k) - \mathcal{I}_{1,\Delta} u_2(\mathbf{c}_k))\, b_k \equiv \sum_{k=1}^{n_d} \gamma_k\, b_k.$$

The $L^2$-norm of this error is given by

$$\|e_2\|^2_{L^2(\Delta)} = |\Delta| (\mathbb{B}\, \boldsymbol{\gamma}, \boldsymbol{\gamma}),$$

where $\boldsymbol{\gamma}$ is the vector with $n_d$ components $\gamma_k$ and $\mathbb{B}$ is the $n_d \times n_d$ symmetric positive definite matrix with positive entries $\mathbb{B}_{k,l} = |\Delta|^{-1} \int_\Delta b_k b_l \,\mathrm{d}V$. This error is only a number; therefore, it does not provides any directional information. To recover this information, we split this error into $n_d$ edge-based error estimates $\alpha_k \ge 0$ such that

$$\|e_2\|_{L^2(\Delta)} = |\Delta|^{1/2} \sum_{k=1}^{n_d} \alpha_k \qquad \text{and} \qquad \sum_{k=1}^{n_d} \alpha_k = (\mathbb{B}\boldsymbol{\gamma}, \boldsymbol{\gamma})^{1/2}. \tag{2.1}$$

In the next section, we motivate the following choice of $\alpha_k$:

$$\alpha_k = |\gamma_k|\, (\mathbb{B}\boldsymbol{\gamma},\, \boldsymbol{\gamma})^{1/2} \Big( \sum_{k=1}^{n_d} |\gamma_k| \Big)^{-1}. \tag{2.2}$$

Similarly, the $L^2$-norm of gradient of $e_2$ is given by

$$\|\nabla e_2\|^2_{L^2(\Delta)} = \| \sum_{k=1}^{n_d} \gamma_k \nabla b_k \|^2_{L^2(\Delta)} = |\Delta| (\widetilde{\mathbb{B}}\, \boldsymbol{\gamma}, \boldsymbol{\gamma}),$$

where $\widetilde{\mathbb{B}}$ is the symmetric positive definite matrix with entries $\widetilde{\mathbb{B}}_{k,l} = |\Delta|^{-1} \int_\Delta \nabla b_k \cdot \nabla b_l \, \mathrm{d}V$. Again, we split the cell-based error (a number) into $n_d$ edge-based error estimates $\tilde{\alpha}_k \geq 0$ such that

$$\|\nabla e_2\|^2_{L^2(\Delta)} = |\Delta| \sum_{k=1}^{n_d} \tilde{\alpha}_k \qquad \text{and} \qquad \sum_{k=1}^{n_d} \tilde{\alpha}_k = (\widetilde{\mathbb{B}}\boldsymbol{\gamma}, \boldsymbol{\gamma}). \tag{2.3}$$

In the next section, we motivate the following choice of $\tilde{\alpha}_k$:

$$\tilde{\alpha}_k = |\gamma_k| \, (\widetilde{\mathbb{B}}\boldsymbol{\gamma}, \, \boldsymbol{\gamma}) \Big( \sum_{k=1}^{n_d} |\gamma_k| \Big)^{-1}. \tag{2.4}$$

# 3. Metric derivation from edge-based error estimates

The next lemma shows existence of a tensor metric generated by errors associated with mesh edges.

**Lemma 1.** *Let $\alpha_k$, $k = 1, \ldots, n_d$, be the values prescribed to edges of a $d$-simplex $\Delta$ such that $\alpha_k \geq 0$ and $\sum_{k=1}^{n_d} \alpha_k > 0$. Then, there exists a constant tensor metric $\mathfrak{M}_\Delta$ such that*

$$\left( \frac{d!}{(d+1)(d+2)} \right)^{1/d} |\Delta|^{2/d}_{\mathfrak{M}_\Delta} \leq \sum_{k=1}^{n_d} \alpha_k \leq |\partial \Delta|^2_{\mathfrak{M}_\Delta}. \tag{3.1}$$

The proof is sketched below and the detailed proof can be found in [2]. Let us consider the quadratic function $v_2 = -\frac{1}{2} \sum_{k=1}^{n_d} \alpha_k b_k$ and denote its Hessian by $\mathbb{H}(v_2)$. If $det(\mathbb{H}(v_2)) \neq 0$, we set $\mathfrak{M}_\Delta = |\mathbb{H}(v_2)|$ where $|\mathbb{H}(v_2)|$ is the spectral module of $\mathbb{H}(v_2)$. Otherwise, we increase slightly the largest $\alpha_k$ so that the modified function $v_2$ has a non-singular Hessian. In practice, increase by 1% was sufficient for all numerical experiments.

The derivation of metric $\mathfrak{M}_\Delta$ suggests a simple motivation for the choices (2.2) and (2.4) when $\mathbb{H}(v_2)$ is definite. Since the bubble function $b_k$ is non-zero only on one edge, we get

$$(\mathfrak{M}_\Delta \mathbf{e}_k, \mathbf{e}_k) = \frac{1}{2} \alpha_k (|\mathbb{H}(b_k)| \mathbf{e}_k, \mathbf{e}_k) = 4 \frac{\alpha_k}{|\gamma_k|} \|e_2\|_{L^\infty(\mathbf{e}_k)}.$$

When $\Delta$ is the $\mathfrak{M}_\Delta$-equilateral simplex, we have

$$\frac{\alpha_1}{|\gamma_1|} \|e_2\|_{L^\infty(\mathbf{e}_1)} = \cdots = \frac{\alpha_{n_d}}{|\gamma_{n_d}|} \|e_2\|_{L^\infty(\mathbf{e}_{n_d})}.$$

Thus, the choice (2.2) means that in a mesh consisting on $\mathfrak{M}_\Delta$-equilateral simplexes, we equidistribute $L^2$-norm of error over cells and $L^\infty$-norm of error over edges.
Let $\widetilde{\mathfrak{M}}_\Delta$ be the metric corresponding to $\tilde{\alpha}_k$. Repeating the above arguments, we obtain (2.4).
Combining (2.1), (2.3) and (3.1), we get the geometric representation of $L^2$-norm of error and its gradient:

$$c_d |\Delta|^{1/2} |\Delta|^{2/d}_{\mathfrak{M}_\Delta} \leq \|e_2\|_{L^2(\Delta)} \leq |\Delta|^{1/2} |\partial \Delta|^2_{\mathfrak{M}_\Delta}, \tag{3.2}$$

$$c_d |\Delta|^{1/2} |\Delta|^{1/d}_{\widetilde{\mathfrak{M}}_\Delta} \leq \|\nabla e_2\|_{L^2(\Delta)} \leq |\Delta|^{1/2} |\partial \Delta|_{\widetilde{\mathfrak{M}}_\Delta}. \tag{3.3}$$

In other words, both norms of the error are controlled from above by $\mathfrak{M}_\Delta$-perimeter and from below by $\mathfrak{M}_\Delta$-volume of simplex $\Delta$. Now we show how to modify the metric so that the controlling quantities will be measured in the same metric.

# 4. Generalizations to $L^p$-norms and $C^2$-functions

Let $p \in (0; \infty]$. To control various $L^p$-norms of $e_2$ and $\nabla e_2$, we use the scaling result from [2]:

$$\mathfrak{M}_{\Delta,p} = (\det(\mathfrak{M}_\Delta))^{-1/(d+2p)}\,\mathfrak{M}_\Delta \quad \text{and} \quad \widetilde{\mathfrak{M}}_{\Delta,p} = (\det(\widetilde{\mathfrak{M}}_\Delta))^{-1/(d+p)}\widetilde{\mathfrak{M}}_\Delta.$$

**Lemma 2.** *It holds*

$$c|\Delta|^{2/d+1/p}_{\mathfrak{M}_{\Delta,p}} \le \|e_2\|_{L^p(\Delta)} \le C\,|\Delta|^{1/p}_{\mathfrak{M}_{\Delta,p}}\,|\partial\Delta|^2_{\mathfrak{M}_{\Delta,p}}, \tag{4.1}$$

$$\tilde{c}|\Delta|^{1/d+1/p}_{\widetilde{\mathfrak{M}}_{\Delta,p}} \le \|\nabla e_2\|_{L^p(\Delta)} \le \tilde{C}\,|\Delta|^{1/p}_{\widetilde{\mathfrak{M}}_{\Delta,p}}\,|\partial\Delta|_{\widetilde{\mathfrak{M}}_{\Delta,p}}, \tag{4.2}$$

*where constants $c$, $C$, $\tilde{c}$, and $\tilde{C}$ depend only on $d$.*

Up to this moment, we derived the geometric representation of various norms of $e_2 = \mathcal{I}_{2,\Delta}u - \mathcal{I}_{1,\Delta}u$. It was shown in [2] that the norm of $e_2$ provides a good approximation for the corresponding norm of the true error $e = u - \mathcal{I}_{1,\Delta}u$. For completeness, we summarize these important results. Let $\mathcal{F}$ be the space of symmetric $d \times d$ matrices. For a vector $\mathbf{e}_k$, we define the following norm:

$$|||\mathbf{e}_k|||^2_{|\mathbb{H}|} = \max_{\mathbf{x}\in\Delta}\,(|\mathbb{H}(\mathbf{x})|\mathbf{e}_k, \mathbf{e}_k). \tag{4.3}$$

Then, $|||\partial\Delta|||^2_{|\mathbb{H}|}$ means formally the sum of (4.3) over edges $\mathbf{e}_k$ of $\Delta$.

**Lemma 3.** *Let $u \in C^2(\bar{\Delta})$. Then, there exist positive constant $c_o$ depending only on $d$ such that*

$$\frac{d+1}{2d}\|e_2\|_{L^\infty(\Delta)} \le \|e_\Delta\|_{L^\infty(\Delta)} \le \|e_2\|_{L^\infty(\Delta)} + \frac{1}{4}\inf_{\mathbb{F}\in\mathcal{F}} |||\partial\Delta|||^2_{|\mathbb{H}-\mathbb{F}|}, \tag{4.4}$$

$$\|\nabla e_2\|_{L^\infty(\Delta)} - c_o\text{osc}(\mathbb{H},\Delta) \le \|\nabla e_\Delta\|_{L^\infty(\Delta)} \le \|\nabla e_2\|_{L^\infty(\Delta)} + c_o\text{osc}(\mathbb{H},\Delta), \tag{4.5}$$

*where*

$$\text{osc}(\mathbb{H},\Delta) = \frac{|\partial\Delta|^{d-1}}{|\Delta|}\inf_{\mathbb{F}\in\mathcal{F}} |||\partial\Delta|||^2_{|\mathbb{H}-\mathbb{F}|}.$$

The oscillation terms are conventional in contemporary error analysis. Their value depend on the simplex and particular features of the function. For instance, if $u \in C^2(\bar{\Delta})$, and $\Delta$ is shape regular, one has $\text{osc}(\mathbb{H},\Delta) \le C|\partial\Delta| \inf_{\mathbb{F}\in\mathcal{F}} |\mathbb{H}-\mathbb{F}|_\infty$. Thus, the oscillation terms are smaller than the errors. Similar analysis can be performed for $L^p$-norms, $p > 0$.

# 5. Asymptotic error estimates

Let $\Omega_h$ and $\tilde{\Omega}_h$ be simplicial meshes with $N_h$ cells that balance the volume and perimeter of cells:

$$N_h^{-1}|\Omega|_{\mathfrak{M}_p} \simeq |\Delta|_{\mathfrak{M}_{\Delta,p}} \simeq |\partial\Delta|^d_{\mathfrak{M}_{\Delta,p}} \qquad \forall\Delta \in \Omega_h,$$

$$N_h^{-1}|\Omega|_{\widetilde{\mathfrak{M}}_p} \simeq |\Delta|_{\widetilde{\mathfrak{M}}_{\Delta,p}} \simeq |\partial\Delta|^d_{\widetilde{\mathfrak{M}}_{\Delta,p}} \qquad \forall\Delta \in \tilde{\Omega}_h.$$

On such meshes, the following error estimates are held

$$\|e\|_{L^p(\Omega)} = \Big(\sum_{\Delta\in\Omega_h} \|e\|^p_{L^p(\Delta)}\Big)^{\frac{1}{p}} \lesssim \Big(\sum_{\Delta\in\Omega_h} |\Delta|^{1+\frac{2p}{d}}_{\mathfrak{M}_{\Delta,p}}\Big)^{\frac{1}{p}} \lesssim |\Omega|^{\frac{1}{p}+\frac{2}{d}}_{\mathfrak{M}_p} N_h^{-\frac{2}{d}},$$

$$\|\nabla e\|_{L^p(\Omega)} = \Big(\sum_{\Delta\in\tilde{\Omega}_h} \|\nabla e\|^p_{L^p(\Delta)}\Big)^{\frac{1}{p}} \lesssim \Big(\sum_{\Delta\in\tilde{\Omega}_h} |\Delta|^{1+\frac{p}{d}}_{\widetilde{\mathfrak{M}}_{\Delta,p}}\Big)^{\frac{1}{p}} \lesssim |\Omega|^{\frac{1}{p}+\frac{1}{d}}_{\widetilde{\mathfrak{M}}_p} N_h^{-\frac{1}{d}},$$

where $a \lesssim b$ means existence of constants $c$ and $C$ independent of mesh such that $c\,a \le b \le C\,a$. In other words, the $\mathfrak{M}_p$ (resp., $\widetilde{\mathfrak{M}}_p$)-quasi-uniform meshes provide asymptotically optimal rate for reduction of the $L^p$-norm of the error (resp., the gradient of the error).

# 6. Metric-based mesh adaptation

We use Algorithm 1 to build an adaptive mesh minimizing the $L^p$-norm of error or its gradient. It provides faster convergence and results in smoother meshes when the metric is continuous. To define a continuous metric we suggest a method of shifts. For every node $\mathbf{a}_i$ in $\Omega_h$, we define the superelement $\sigma_i$ as the union of all $d$-simplexes sharing $\mathbf{a}_i$. Then, $\mathfrak{M}(\mathbf{a}_i)$ is defined as one of the metrics in $\sigma_i$ with the largest determinant. This method always chooses the worst metric in the superelement. To generate a $\mathfrak{M}$-quasi-uniform mesh, we use local mesh modifications described in [3] and implemented in package `Ani2D` (sourceforge.net/projects/ani2d).

**Algorithm 1** Adaptive mesh generation

```
Generate an initial mesh Ω_h and compute the metric 𝔐.
loop
    Generate a 𝔐-quasi-uniform mesh Ω_h with the prescribed number of simplexes.
    Recompute the metric 𝔐.
    If Ω_h is 𝔐-quasi-uniform, then exit the loop                                  .
end loop
```

For the numerical illustration, we consider the analytical function

$$u = (x^2 y + y^3)/16^3 + \tanh(2(\sin(6y) - 3x)(\sin(6x) - 3y)) \tag{6.1}$$

in the Texas-shape domain inscribed in $[-\frac{3}{2}; \frac{3}{2}]$. The spider-like isolines of $u$ (see Fig. 1) show that this function has both isotropic and anisotropic regions. Table 1 verifies the theoretical estimates from Section 5. The $L^\infty$-norm of the interpolation error on meshes $\Omega_h$ built with Algorithm 1 and metric $\mathfrak{M}_p$ is proportional to $N_h^{-1}$. The $L^\infty$-norm of its gradient on meshes $\tilde{\Omega}_h$ built with Algorithm 1 and metric $\widetilde{\mathfrak{M}}_p$ is proportional to $N_h^{-1/2}$. The last column in Table 1 shows that the $L^\infty$-norm of the interpolation error on mesh $\tilde{\Omega}_h$, which is not the optimal mesh for this error, exhibits still the optimal convergence rate; albeit, the error is larger than that on mesh $\Omega_h$. We also note that the mesh $\tilde{\Omega}_h$ (Fig.1, right) is denser than the mesh $\Omega_h$ (Fig.1, left) in regions where the solution is sharp.

| $N_h$ | $\|e\|_{L^\infty(\Omega_h)}$ | $\|\nabla e\|_{L^\infty(\tilde\Omega_h)}$ | $\|e\|_{L^\infty(\tilde\Omega_h)}$ |
|---|---|---|---|
| 2000 | 3.72e-2 | 1.23e+0 | 2.04e-1 |
| 4000 | 1.76e-2 | 8.00e-1 | 9.24e-2 |
| 8000 | 8.12e-3 | 5.36e-1 | 4.06e-2 |
| 16000 | 4.60e-3 | 4.00e-1 | 1.99e-2 |
| 32000 | 2.19e-3 | 2.72e-1 | 1.11e-2 |
| 64000 | 1.22e-3 | 1.92e-1 | 6.47e-3 |
| rate | 0.99 | 0.53 | 1.00 |

Table 1: Convergence of the $L^\infty$-norm of the interpolation error and its gradient.

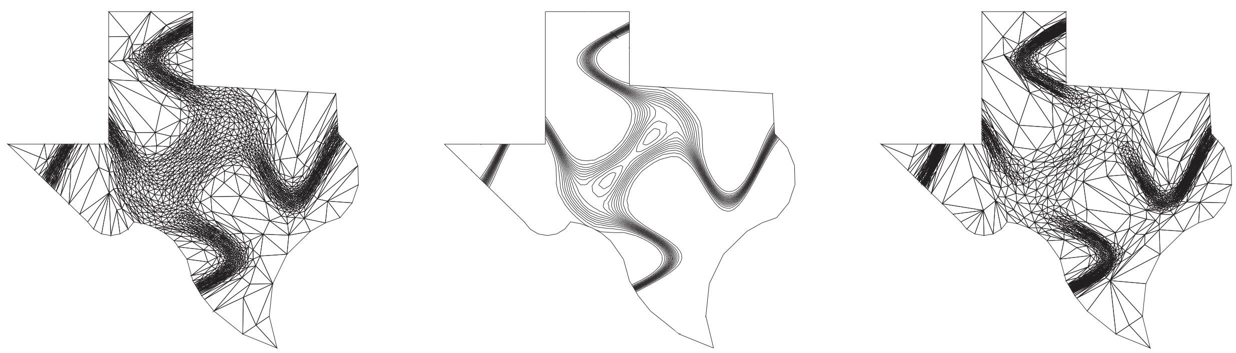

Figure 1: Left: The mesh $\Omega_h$ with roughly 2000 triangles minimizing $\|e\|_{L^\infty(\Omega_h)}$. Middle: The isolines of function (6.1). Right: The mesh $\tilde\Omega_h$ with roughly 2000 triangles minimizing $\|\nabla e\|_{L^\infty(\tilde\Omega_h)}$.

## Acknowledgements

This research was partly supported by the Russian Foundation for Basic Research through grant 08-01-00159-a and by RAS program “Optimal methods for problems of mathematical physics”.

## References

[1] A. Agouzal, K. Lipnikov, Y. Vassilevski. *Generation of quasi-optimal meshes based on a posteriori error estimates.* In: Proceedings of 16th International Meshing Roundtable. M.Brewer and D.Marcum (eds.), Springer, (2007), 139–148.

[2] A. Agouzal, K. Lipnikov, Y. Vassilevski. *Hessian-free metric-based mesh adaptation via geometry of interpolation error.* To appear in Comp. Math. Math. Phys., 50 (2010).

[3] Y. Vassilevski, K. Lipnikov. *Adaptive algorithm for generation of quasi-optimal meshes*. Comp. Math. Math. Phys., 39 (1999), 1532–1551.

[4] Y. Vassilevski, A. Agouzal. *An unified asymptotic analysis of interpolation errors for optimal meshes*. Doklady Mathematics, 72 (2005), 879–882.