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MATHEMATICS

Unified Asymptotic Analysis of Interpolation Errors for Optimal Meshes

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The progress in adaptive mesh generation achieved over recent years makes it possible to attempt to generate optimal meshes or, at least, their approximations. For problems with anisotropic solutions, optimal meshes are also anisotropic. Therefore, estimates of interpolation errors for anisotropic meshes are required for approximation analysis of optimal meshes. Such error estimates in L_∞ for optimal triangulations were derived in [1] and [2] for the two- and three-dimensional cases, respectively. However, the proofs in [1, 2] are not analogous and contain some inaccuracies concerning the lower bound on the interpolation error for an optimal mesh. More specifically, both proofs are somewhat incomplete in the case of indefinite Hessians on *arbitrary* simplices. Since optimal meshes can have arbitrary elements, these proofs have to be refined. Moreover, the estimates for the interpolation error in L_∞ can easily be extended to L_p. We present a complete corrected proof of the error estimate that unifies the two- and three-dimensional cases and, then, extend the result to estimates in L_p.

OPTIMAL MESHES AND THE ERROR IN L_∞

Let $\Omega \in \mathbf{R}^d$ ($d = 2, 3$) be a polyhedral domain and Ω^h be its conformal simplicial partition (triangular for $d = 2$ and tetrahedral for $d = 3$) into $\mathcal{N}(\Omega^h)$ mesh elements. Let $C^k(D)$ be the space of functions defined in $D \subset \overline{\Omega}$ with continuous partial derivatives of up to the order k. The space of functions that are continuous in Ω and linear on each simplex is denoted by $P_1(\Omega^h)$. Let $\mathcal{P}_{\Omega^h}: C^0(\overline{\Omega}) \to P_1(\Omega^h)$ be the linear interpolation operator on the mesh Ω^h.

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Definition 1. Given $u \in C^0(\overline{\Omega})$, a mesh $\Omega^h_{\text{opt}}(N_T, u)$ consisting of no more than N_T elements is called optimal if it solves the optimization problem

$$\Omega^h_{\text{opt}}(N_T, u) = \arg \min_{\Omega_h:\, \mathcal{N}(\Omega_h) \le N_T} \left\| u - \mathcal{P}_{\Omega^h} u \right\|_{L_\infty(\Omega)}. \quad (1)$$

Note that, in practice, $\mathcal{P}_{\Omega^h}$ can be any projector on $P_1(\Omega^h)$, for example, a finite-element projection operator.

Theorem 1. *Suppose that* $u \in C^2(\overline{\Omega})$, *its Hessian H is nonsingular in* Ω, *and any simplex* $\Delta \in \Omega^h_{\text{opt}}$ *satisfies the estimate*

$$\begin{gathered} \| H_{ps} - (H_\Delta)_{ps} \|_{L_\infty(\Delta)} < q_\Delta |\lambda_1(H_\Delta)|, \\ 0 < q_\Delta \le q < 1, \quad p, s = 1, 2, \ldots, d, \end{gathered} \quad (2)$$

where $H_\Delta = H(\operatorname{argmax}_{x \in \Delta} |\det H(x)|)$ *and* $\lambda_1(H_\Delta)$ *is the eigenvalue of* H_Δ *nearest to zero. Then*

$$C(q)\left(\frac{|\Omega|_{|H|}}{\mathcal{N}(\Omega^h_{\text{opt}})}\right)^{\frac{2}{d}} \le \left\| u - \mathcal{P}_{\Omega^h_{\text{opt}}} u \right\|_{L_\infty(\Omega)}. \quad (3)$$

Hereafter, $C(z)$ denotes a positive constant depending on z and independent of the remaining parameters. Note that the upper bound for $\left\| u - \mathcal{P}_{\Omega^h_{\text{opt}}} u \right\|_{L_\infty(\Omega)}$ is similar to the lower bound and was proved in [1] (in the two-dimensional case) and in [2] (in the three-dimensional case).

Before proving the theorem, we formulate and prove the following result.

Lemma 1. *Let* Δ *be a simplex with edges* e_i, *and let* $u_2 \in P_2(\Delta)$ *be a quadratic function with a nonsingular Hessian* H_2 *that has a spectral decomposition* $H_2 = W_2^T \Lambda_2 W_2$. *Then*

$$C|\hat{\Delta}|^{\frac{2}{d}} \le \frac{1}{8} \max_{e_i} |(H_2 e_i, e_i)| \le \| u_2 - \mathcal{P}_\Delta u_2 \|_{L_\infty(\Delta)}, \quad (4)$$

where $\hat{\Delta}$ is the image of Δ *under the mapping* $\hat{x} = R(x)$ *with* $R = \sqrt{|\Lambda_2|}\, W_2$, *which reduces* H_2 *to the canonical form*

$$\hat{H}_2 = \begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix},\ d = 2,\ \hat{H}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix},\ d = 3.$$

Proof. According to the multipoint Taylor formula [3, 4] for quadratic functions, we have

$$u_2(x) - \mathcal{P}_\Delta u_2(x) = -\frac{1}{2}\sum_{i=1}^{d+1}(H_2(x-a_i), (x-a_i))p_i(x), \tag{5}$$

where $p_i(x)$ is a linear function on Δ that reaches 1 at the vertex a_i and vanishes at the remaining vertices of Δ.

Consider the two-dimensional case first. Using (5), we obtain

$$\max_{x \in \Delta}|u_2(x) - \mathcal{P}_\Delta u_2(x)| \ge \max_{e_i}\max_{x \in e_i}|u_2(x) - \mathcal{P}_\Delta u_2(x)| = \max_{e_i}\frac{1}{8}|(H_2 e_i, e_i)|$$

because the error on an edge is maximal at the midpoint of the edge. If $\det H_2 > 0$, we have

$$|\hat{\Delta}| \le \frac{\sqrt{3}}{4}(\operatorname{diam}\hat{\Delta})^2 = \frac{\sqrt{3}}{4}\max_{e_i}|(H_2 e_i, e_i)| \le 2\sqrt{3}\|u_2 - \mathcal{P}_\Delta u_2(x)\|_{L_\infty(\Delta)}.$$

For $\det H_2 < 0$, the two-dimensional result in [5] states that, for any quadratic function u_2 and a prescribed value ε of the interpolation error, the maximum possible area of a triangle is $\frac{\sqrt{5}\varepsilon}{4}$ and the maximum error on the triangle is reached at one of the midpoints of e_i and is equal to $\max_{e_i}\frac{1}{8}|(H_2 e_i, e_i)|$. Therefore,

$$\frac{4}{\sqrt{5}}|\hat{\Delta}| \le \max_{e_i}\frac{1}{8}|(H_2 e_i, e_i)| = \|u_2 - \mathcal{P}_\Delta u_2\|_{L_\infty(\Delta)},$$

and the proof of estimate (4) in the two-dimensional case is completed. Since $|\hat{\Delta}| = |\Delta|_{|H_2|}$, where $|H_2|$ is defined as $|H_2| = W_2^T|\Lambda_2|W_2$, estimate (4) can be rewritten as

$$C|\Delta|_{|H_2|}^{\frac{2}{d}} \le \frac{1}{8}\max_{e_i}|(H_2 e_i, e_i)| \le \|u_2 - \mathcal{P}_\Delta u_2(x)\|_{L_\infty(\Delta)}.$$

In the three-dimensional case, the faces of Δ are denoted by f_i ($i = 1, 2, 3, 4$). Then, using the two-dimensional result, we have

$$\max_{x \in \Delta}|u_2(x) - \mathcal{P}_\Delta u_2(x)| \ge \max_{i=1,2,3,4}\ \max_{x \in f_i}|u_2(x) - \mathcal{P}_\Delta u_2(x)|$$
$$\ge \max_{i=1,2,3,4}\ \max_{j=1,2,3}\frac{1}{8}|(H_2 e_{ij}, e_{ij})| \ge C\max_{i=1,2,3,4}|f_i|_{|H_2|},$$

where e_{ij} are the edges of f_i. Moreover,

$$\max_{i=1,2,3,4}|f_i|_{|H_2|} \ge |f_{i^*}|_{|H_2|}\frac{|h_{i^*}|_{|H_2|}}{|h_{i^*}|_{|H_2|}} \ge C\frac{|\Delta|_{|H_2|}}{|h_{i^*}|_{|H_2|}},$$

where f_{i^*} is the face for which the displacement of the opposite vertex in a plane parallel to f_{i^*} generates a $|H_2|$-equilateral tetrahedron Δ_{i^*} with an face $f_{i^{**}}$ belonging to f_{i^*}. We have $|\Delta_{i^*}|_{|H_2|} \le |\Delta|_{|H_2|}$ because $|f_{i^{**}}|_{|H_2|} \le |f_{i^*}|_{|H_2|}$ and Δ has the same height h_{i^*} as Δ_{i^*}. Therefore,

$$|h_{i^*}|_{|H_2|} \le C|\Delta_{i^*}|_{|H_2|}^{\frac{1}{3}} \le C|\Delta|_{|H_2|}^{\frac{1}{3}}$$

and

$$\max_{x \in \Delta}|u_2(x) - \mathcal{P}_\Delta u_2(x)| \ge \max_{e_i}\frac{1}{8}|(H_2 e_i, e_i)| \ge C|\Delta|_{|H_2|}^{\frac{2}{3}} = C|\hat{\Delta}|^{\frac{2}{3}}.$$

The lemma is proved.

Proof of Theorem 1. Let Δ be an arbitrary simplex of an optimal triangulation Ω_{opt}^h and $\mathbf{l} = \overrightarrow{a_{k_1}a_{k_2}}$ be a directed edge of Δ. According to the one-dimensional multipoint Taylor formula,

$$\max_{x \in \Delta}|u_2(x) - \mathcal{P}_\Delta u_2(x)| \ge \max_{x \in [a_{k_1}, a_{k_2}]}|u(x) - \mathcal{P}_\Delta u(x)|$$
$$= \max_{x \in [a_{k_1}, a_{k_2}]}\frac{1}{2}\left|\sum_{j=1}^{2}(H(\tilde{x}_j)(x-a_{k_j}), (x-a_{k_j}))p_j(x)\right|,$$

where $\tilde{x}_j \in [a_{k_1}, a_{k_2}]$. By virtue of (2), we have

$$|((H(\tilde{x}_j) - H_\Delta)(x-a_{k_j}), (x-a_{k_j}))| \le Cq|(H_\Delta(x-a_{k_j}), (x-a_{k_j}))|. \tag{6}$$

Therefore, for sufficiently small q such that $Cq < 1$, the values of $(H(\tilde{x}_j)(x - a_{k_j}), (x - a_{k_j})$ and $(H_\Delta(x - a_{k_j}), (x - a_{k_j}))$ and have the same sign. Moreover, the values of $(H_\Delta(x - a_{k_j}), (x - a_{k_j}))$, $j = 1, 2$, have the same sign because the vectors $x - a_{k_1}$ and $x - a_{k_2}$ are parallel. Therefore,

$$\max_{x \in [a_{k_1}, a_{k_2}]} \frac{1}{2}\left|\sum_{j=1}^{2}(H(\tilde{x}_j)(x - a_{k_j}), (x - a_{k_j}))p_j(x)\right|$$

$$\geq (1 - Cq)\max_{x \in [a_{k_1}, a_{k_2}]} \frac{1}{2}\left|\sum_{j=1}^{2}(H_\Delta(x - a_{k_j}), (x - a_{k_j}))p_j(x)\right|$$

$$\geq C(1 - Cq)|(H_\Delta \mathbf{l}, \mathbf{l})|. \quad (7)$$

By Lemma 1, for the quadratic functions

$$u_2(x) = \mathcal{P}_\Delta u(x) - \frac{1}{2}\sum_{i=1}^{d+1}(H_\Delta(x - a_i), (x - a_i))p_i(x), \quad (8)$$

there is an edge $\mathbf{l}$ such that

$$|(H_\Delta \mathbf{l}, \mathbf{l})| \geq \tilde{B}|\Delta|_{|H_\Delta|}^{\frac{2}{d}}.$$

Then, in view of (2), there exists a constant $B(q, \tilde{B})$ such that

$$|(H_\Delta \mathbf{l}, \mathbf{l})| \geq \tilde{B}|\Delta|_{|H_\Delta|}^{\frac{2}{d}} \geq B(q, \tilde{B})|\Delta|_{|H|}^{\frac{2}{d}}.$$

Here, $|H|$ is the spectral modulus of H. Therefore, a simplex with the maximum volume $|\Delta|_{|H|}$ satisfies the estimate

$$\max_{x \in \Delta}|u(x) - \mathcal{P}_\Delta u(x)| \geq B(q, \tilde{B})|\Delta|_{|H|}^{\frac{2}{d}}$$

and

$$\|u - \mathcal{P}_{\Omega^h_{\text{opt}}} u\|_{L_\infty(\Omega)} \geq B(q, \tilde{B}) \max_{\Delta \subset \Omega^h_{\text{opt}}} |\Delta|_{|H|}^{\frac{2}{d}}$$

$$\geq B(q, \tilde{B})\left(\frac{|\Omega|_{|H|}}{\mathcal{N}(\Omega^h_{\text{opt}})}\right)^{\frac{2}{d}},$$

which proves (3).

OPTIMAL MESHES AND THE ERROR IN L_p

Definition 2. Given $u \in C^0(\overline{\Omega})$ and $p \in\ ]0, +\infty]$, a mesh $\Omega^h_{\text{opt}}(N_T, u)$ consisting of no more than N_T elements is called optimal with respect to the L_p norm if it solves the optimization problem

$$\Omega^h_{\text{opt}}(N_T, u) = \arg \min_{\Omega_h:\, \mathcal{N}(\Omega_h) \leq N_T} \|u - \mathcal{P}_{\Omega^h} u\|_{L_p(\Omega)}. \quad (9)$$

Theorem 2. *Let the assumptions of Theorem* 1 *be satisfied. For an indefinite Hessian H, it is additionally assumed that, for any simplex* $\Delta \subset \Omega^h_{\text{opt}}$,

$$c_0(H_\Delta\zeta, \zeta) \leq (H(x)\zeta, \zeta) \leq c_1(H_\Delta\zeta, \zeta) \ \forall \zeta \in \mathbf{R}^d, \quad \text{a.e. } x \in \Delta, \quad c_0 > 0, \quad c_1 > 0. \quad (10)$$

Then, for optimal meshes with respect to the L_p *norm,*

$$C(q, c_0, c_1)\frac{|\Omega|_{|\overline{H}|}^{\frac{d+2p}{pd}}}{\mathcal{N}^{\frac{2}{d}}(\Omega^h_{\text{opt}})} \leq \|u - \mathcal{P}_{\Omega^h_{\text{opt}}} u\|_{L_p(\Omega)}, \quad (11)$$

where $|\overline{H}| := (\det|H|)^{\frac{-1}{2p+d}}|H|$.

Proof. It is based on the following estimate for any quadratic function u_2 with a Hessian H_2:

$$\|u_2 - \mathcal{P}_\Delta u_2\|_{L_p(\Delta)} \geq C|\Delta|_{|\overline{H}_2|}^{\frac{2p+d}{pd}}, \quad (12)$$

where $|\overline{H}_2| = (\det|H_2|)^{\frac{-1}{2p+d}}|H_2|$. Indeed, for $p \in [1, +\infty[$, we have

$$\|u_2 - \mathcal{P}_\Delta u_2\|_{L_p(\Delta)} \geq \frac{\|u_2 - \mathcal{P}_\Delta u_2\|_{L_1(\Delta)}}{|\Delta|^{1-\frac{1}{p}}}$$

$$\geq C|\Delta|\frac{\|u_2 - \mathcal{P}_\Delta u_2\|_{L_\infty(\Delta)}}{|\Delta|^{1-\frac{1}{p}}} = C|\Delta|^{\frac{1}{p}}\|u_2 - \mathcal{P}_\Delta u_2\|_{L_\infty(\Delta)}.$$

Applying Lemma 1, we derive

$$\|u_2 - \mathcal{P}_\Delta u_2\|_{L_p(\Delta)} \geq C|\Delta|^{\frac{1}{p}}|\hat{\Delta}|^{\frac{2}{d}}$$

$$= C\left(|\Delta|_{|H_2|}|\det H_2|^{-\frac{1}{2}}\right)^{\frac{1}{p}}|\Delta|_{|H_2|}^{\frac{2}{d}} = \frac{C|\Delta|_{|H_2|}^{\frac{2p+d}{pd}}}{|\det H_2|^{\frac{1}{2p}}} = C|\Delta|_{|\overline{H}_2|}^{\frac{2p+d}{pd}}.$$

For $p \in\ ]0, 1[$, we have $\text{meas}_d\{x \in \Delta | u_2(x) - \mathcal{P}_\Delta u_2(x) = 0\} = 0$ since $\det(H_2) \neq 0$ and

$$|u_2 - \mathcal{P}_\Delta u_2|^p(x) = |u_2 - \mathcal{P}_\Delta u_2|^{p-1}(x)|u_2 - \mathcal{P}_\Delta u_2|(x)$$

$$\geq \|u_2 - \mathcal{P}_\Delta u_2\|_{L_\infty(\Delta)}^{p-1}|u_2 - \mathcal{P}_\Delta u_2|(x), \text{ a.e. } x \in \Delta,$$

which implies

$$\|u_2 - \mathcal{P}_\Delta u_2\|_{L_\infty(\Delta)} = \|u_2 - \mathcal{P}_\Delta u_2\|_{L_p(\Delta)}^{\frac{p-1}{p}}\|u_2 - \mathcal{P}_\Delta u_2\|_{L_1(\Delta)}^{\frac{1}{p}}$$

$$\geq C|\Delta|^{\frac{1}{p}}\|u_2 - \mathcal{P}_\Delta u_2\|_{L_\infty(\Delta)} \geq C|\Delta|_{|\overline{H}_2|}^{\frac{2p+d}{pd}}.$$

To extend this result to arbitrary functions $u \in C^2(\overline{\Omega})$ that satisfy (2) and (10) (for indefinite H), we use the multipoint Taylor formula

$$u(x) - \mathcal{P}_\Delta u(x) = -\frac{1}{2}\sum_{i=1}^{d+1}(H(\tilde{x}_i)(x-a_i),(x-a_i))p_i(x), \quad x \in \Delta.$$

Defining

$$u_2(x) = \mathcal{P}_\Delta u(x) - \frac{1}{2}\sum_{i=1}^{d+1}(H_\Delta(x-a_i),(x-a_i))p_i(x), \quad x \in \Delta,$$

we have $\mathcal{P}_\Delta u = \mathcal{P}_\Delta u_2$. Then, by virtue of (10),

$$c_2(c_0, c_1)|u_2(x) - \mathcal{P}_\Delta u_2(x)| \le |u(x) - \mathcal{P}_\Delta u(x))|, \quad \text{a.e. } x \in \Delta, \quad c_2 = \min\{c_0, c_1\}$$

for functions u with indefinite H and, by virtue of (2),

$$c_2(q)|u_2(x) - \mathcal{P}_\Delta u_2(x)| \le |u(x) - \mathcal{P}_\Delta u(x))|, \quad x \in \Delta$$

for functions u with definite H. This implies that $\forall p \in]0, +\infty[$

$$\|u - \mathcal{P}_\Delta u\|_{L_p(\Delta)} \ge c_2\|u_2 - \mathcal{P}_\Delta u_2\|_{L_p(\Delta)} \ge c_2 C|\Delta|_{|\overline{H}_\Delta|}^{\frac{2p+d}{pd}} \ge C(q, c_0, c_1)|\Delta|_{|\overline{H}|}^{\frac{2p+d}{pd}}, \tag{13}$$

where

$$|\overline{H}| = (\det|H|)^{\frac{-1}{2p+d}}|H|,$$

$$|\overline{H}_\Delta| = (\det|H_\Delta|)^{\frac{-1}{2p+d}}|H_\Delta|.$$

Using the Hölder inequality with $r = 1 + \frac{2p}{d}$ and $s = 1 + \frac{d}{2p}$ $\left(\frac{1}{r} + \frac{1}{s} = 1\right)$, we obtain

$$|\Omega|_{|\overline{H}|} = \sum_{\Delta \in \Omega^h}|\Delta|_{|\overline{H}|} \le \mathcal{N}^{\frac{1}{s}}(\Omega^h_{\text{opt}})\left(\sum_{\Delta \in \Omega^h}|\Delta|^r_{|\overline{H}|}\right)^{\frac{1}{r}}.$$

By using (13), this can be rewritten as

$$\frac{|\Omega|_{|\overline{H}|}^{\frac{d+2p}{d}}}{\mathcal{N}^{\frac{2p}{d}}(\Omega^h_{\text{opt}})} \le \sum_{\Delta \in \Omega^h}|\Delta|_{|\overline{H}|}^{\frac{2p+d}{d}} \le C^{-p}(q, c_0, c_1)\sum_{\Delta \in \Omega^h}\|u - \mathcal{P}_\Delta u\|^p_{L_p(\Delta)} = C^{-p}(q, c_0, c_1)\|u - \mathcal{P}_{\Omega^h}u\|^p_{L_p(\Omega)}.$$

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REFERENCES

1. Yu. V. Vasilevskii and K. N. Lipnikov, Zh. Vychisl. Mat. Mat. Fiz. **39**, 1532–1551 (1999) [Comput. Math. Math. Phys. **39**, 1468–1486 (1999)].
2. A. Agouzal, K. Lipnikov, and Y. Vassilevski, East-West J. Numer. Math. **7**, 223–244 (1999).
3. P. Ciarlet and C. Wagschal, Numer. Math. **17**, 84–100 (1971).
4. P. Ciarlet, *The Finite Element Method for Elliptic Problems* (North-Holland, Amsterdam, 1977; Mir, Moscow, 1980).
5. E. D'Azevedo, SIAM J. Sci. Stat. Comput. **12**, 755–786 (1991).