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On the L^q-saturation property for functions from $W^{2,p}(\Omega)$

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ABSTRACT

This short note is devoted to the proof of the L^q-saturation property for functions from the Sobolev space $W^{2,p}(\Omega)$, $p > d/2$, on sequences of conformal, possibly anisotropic, simplicial meshes for spaces of arbitrary dimension d. The proof completes the theory of optimal meshes minimizing the L^q-error ($0 < q \le p$) of P_1-interpolation and the theory of quasi-optimal meshes which are achievable approximations of the optimal meshes (Agouzal et al., 2009, 2010 [2,3], Agouzal et al., 2010 [4]).

1. Prerequisites

Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain and $\mathcal{T}_N$ be a conformal simplicial mesh with N d-simplexes. The volume of a d-simplex Δ and its diameter are denoted by $|\Delta|$ and $\mathrm{diam}(\Delta)$, respectively. The vertices of Δ are denoted by $\mathbf{v}_i$, $i = 1, \dots, d+1$. We define $h_N = \sup_{\Delta \in \mathcal{T}_N} \mathrm{diam}(\Delta)$ and consider a sequence of meshes $\mathcal{T}_N$ such that $N \to \infty$.

Let $\mathcal{I}_N^1 u$ ($\mathcal{I}_N^2 u$) be the continuous piecewise linear (quadratic) interpolant of a continuous function u on a mesh $\mathcal{T}_N$, and $\mathcal{I}_\Delta^1 u$ ($\mathcal{I}_\Delta^2 u$) be its restriction to Δ. In particular,

$$(\mathcal{I}_\Delta^1 u)(\mathbf{x}) = \sum_{i=1}^{d+1} u(\mathbf{v}_i)\lambda_i(\mathbf{x}), \quad \mathbf{x} \in \Delta,$$

where $\lambda_i(\mathbf{x})$, $i = 1, \dots, d+1$ are linear functions on Δ such that $\lambda_i(\mathbf{v}_j) = \delta_{ij}$, δ_{ij} is the Kronecker symbol.

In this note we shall deal with functions u from the Sobolev space $W^{2,p}(\Omega)$ for $p > d/2$. Due to the embedding theorem, they are continuous functions and therefore the conventional Lagrange interpolation can be defined for them. Each entry $H_{ij}(\mathbf{x})$ of the Hessian matrix $H(\mathbf{x})$ of u is a function from $L^p(\Omega)$. The space $W^{2,p}(\Omega)$, $p > d/2$, is feasible for applications. For instance, the solution u of the Poisson equation in a 2D domain with piecewise smooth boundary ($d = 2$) has the singular part $\phi(r, \theta)r^\alpha$, $\alpha \in [\frac{1}{2}; 1[$, in a local polar coordinate system [1]. Therefore, $u \in W^{1+\alpha,2}(\Omega)$ and $u \in W^{2,p}(\Omega)$ for any $1 \le p < 2/(2-\alpha)$, i.e. $u \in W^{2,\frac{4}{3}-\epsilon}(\Omega)$ for a small $\epsilon > 0$.

The L^q-saturation property, in its simple form, asserts that a smooth function can be approximated asymptotically better with its piecewise quadratic interpolant than with its piecewise linear interpolant. More precisely, there exists $\alpha \in]0, 1[$ such that, for any function $u \in W^{2,p}(\Omega)$ and sequences of meshes $\mathcal{T}_N$, $N \to \infty$, one has for $0 < q \le p$:

$$\frac{\|u - \mathcal{I}_N^2 u\|_{L^q(\Omega)}}{\|u - \mathcal{I}_N^1 u\|_{L^q(\Omega)}} \le \alpha. \tag{1}$$

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The L^q-saturation property is used in the theory of optimal meshes minimizing the L^q-error $(0 < q \le p)$ of P_1-interpolation and the theory of quasi-optimal meshes which are achievable approximations of the optimal meshes [2–4].

A more popular assumption is the $W^{1,q}$-saturation property [5–8] where the L^q-norm in (1) is replaced by the $W^{1,q}$-seminorm. It is widely used in the classical proof of equivalence of some a posteriori error estimators with the energy error although this property was shown to be superfluous in the case of isotropic meshes [6]. The analysis of the $W^{1,q}$-saturation property on sequences of conformal, possibly anisotropic, simplicial meshes is the subject of future research. The analysis will complete the theory of optimal meshes minimizing the $W^{1,q}$-error of P_1-interpolation and the theory of corresponding quasi-optimal meshes [2–4,9].

In the sequel, we develop the proof of (1) and present the conditions for mesh sequences under which (1) holds.

2. Functions of two arguments and their properties

For a function of two arguments $v(\mathbf{x}, \mathbf{y})$ we introduce a broken norm in Ω

$$[v]^p_{p,\Omega} = \sum_{\Delta\in\mathcal{T}_N} [v]^p_{p,\Delta}, \quad [v]^p_{p,\Delta} = |\Delta|^{-1} \int_\Delta \int_\Delta |v(\mathbf{x}, \mathbf{y})|^p \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}. \tag{2}$$

For functions with one argument, the broken p-norm coincides with the L^p-norm. If v is continuous in Ω with respect to the first argument, we can define the P_1 Lagrange interpolation $\mathcal{I}^1_N v$ via its restriction $\mathcal{I}^1_\Delta v$ on Δ

$$\mathcal{I}^1_\Delta v(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{d+1} v(\mathbf{v}_i, \mathbf{y})\lambda_i(\mathbf{x}). \tag{3}$$

We shall consider two functions of two arguments,

$$\pi(\mathbf{x}, \mathbf{y}) = u(\mathbf{y}) + \nabla u \cdot (\mathbf{x} - \mathbf{y}) + \frac{1}{2}(H(\mathbf{y})(\mathbf{x} - \mathbf{y}), \mathbf{x} - \mathbf{y}), \tag{4}$$

$$\omega(\mathbf{x}, \mathbf{y}) = \|H(\mathbf{x}) - H(\mathbf{y})\|, \tag{5}$$

where the difference of the Hessians is evaluated in the spectral norm.

Lemma 1. *For any conformal triangulation $\mathcal{T}_N$ and any $u \in W^{2,q}(\Omega)$, $q > 0$, it holds:*

$$C(d, q)^{\frac{1}{q}} \| \det H\|^{\frac{1}{d}}_{L^{\frac{q}{2q+d}}(\Omega)} \le N^{\frac{2}{d}} [\pi - \mathcal{I}^1_N \pi]_{q,\Omega} \tag{6}$$

where $C(d, q)$ is a positive constant depending on d and q only.

Proof. Let

$$t = \frac{q}{2q+d}, \qquad r = 1 + \frac{2q}{d}, \qquad s = 1 + \frac{d}{2q}, \qquad \beta = \frac{2q}{2q+d}$$

for which it holds:

$$tr = \frac{q}{d}, \qquad \beta r = \frac{2q}{d}, \qquad \beta s = 1.$$

Using the Hölder inequality, $|\det H|^t = (|\det H|^t |\Delta|^\beta)|\Delta|^{-\beta}$ and the fact that

$$\sum_{\Delta\in\mathcal{T}_N} \int_\Delta |\Delta|^{-1} \mathrm{d}\mathbf{x} = N$$

we obtain

$$\int_\Omega |\det H(\mathbf{y})|^t \mathrm{d}\mathbf{y} \le \left(\sum_{\Delta\in\mathcal{T}_N} \int_\Delta |\det H(\mathbf{y})|^{\frac{q}{d}} |\Delta|^{\frac{2q}{d}} \mathrm{d}\mathbf{y} \right)^{\frac{1}{r}} N^{\frac{1}{s}}. \tag{7}$$

Let us show that there exists such a positive constant $C(d, q)$ that

$$C(d, q) \int_\Delta |\det H(\mathbf{y})|^{\frac{q}{d}} |\Delta|^{\frac{2q}{d}} \mathrm{d}\mathbf{y} \le [\pi - \mathcal{I}^1_\Delta \pi]^q_{q,\Delta}. \tag{8}$$

Indeed, π is a quadratic function in $\mathbf{x}$ and due to (3) it is sufficient to analyze functions $\pi_M = \frac{1}{2}(M\mathbf{x}, \mathbf{x})$ for any symmetric nonsingular matrix M of order d on any d-simplex. We consider the matrix $\widehat{M} = |\det M|^{-\frac{1}{d}} M$, such that $|\det \widehat{M}| = 1$, and the d-simplex $\widehat{\Delta} = \{\hat{\mathbf{x}} \mid \hat{\mathbf{x}} = |\Delta|^{-\frac{1}{d}}\mathbf{x},\ \mathbf{x} \in \Delta\}$, such that $|\widehat{\Delta}| = 1$. Then

$$\|\pi_M - \mathcal{I}^1_\Delta \pi_M\|^q_{L^q(\Delta)} = |\Delta|^{\frac{2q}{d}+1} |\det M|^{\frac{q}{d}} \|\pi_{\widehat{M}} - \mathcal{I}^1_{\widehat{\Delta}} \pi_{\widehat{M}}\|^q_{L^q(\widehat{\Delta})} \ge C(q, d)|\Delta|^{\frac{2q}{d}+1} |\det M|^{\frac{q}{d}},$$

where

$$C(q,d) := \inf_{|\det \widehat{M}|=1} \inf_{|\widehat{\Delta}|=1} \|\pi_{\widehat{M}} - \mathcal{I}^1_{\widehat{\Delta}}\pi_{\widehat{M}}\|^q_{L^q(\widehat{\Delta})}.$$

Averaging the last estimate over Δ, we get (8). Plugging (8) into (7) and using $r = \frac{q}{td}$, $\frac{r}{s} = \frac{2q}{d}$ we get (6). □

We denote $\epsilon_N = [\omega(\mathbf{x},\mathbf{y})]_{p,\Omega}$ which is the L^p modulus of continuity of $\omega(\mathbf{x},\mathbf{y})$. It is well known that:

$$\lim_{h_N\to 0} \epsilon_N = 0. \tag{9}$$

Lemma 2. *Let $u \in W^{2,p}(\Omega)$, $p > d/2$, $0 < q \le p$. Then*

$$[u - \pi - \mathcal{I}^1_N(u-\pi)]_{q,\Omega} \le \frac{6p}{2p-d} h_N^2 |\Omega|^{\frac{1}{q}-\frac{1}{p}} \epsilon_N. \tag{10}$$

Proof. Consider the function $v = u - \pi$ as a function of $\mathbf{x}$ argument. We have $v \in W^{2,p}(\Delta)$ and the Hessian of v is $H(\mathbf{x}) - H(\mathbf{y})$. According to [10] (p. 413, Theorems 2–1) it holds:

$$\|v - \mathcal{I}^1_\Delta v\|^p_{p,\Delta} \le \left(\frac{6p}{2p-d}\right)^p (\operatorname{diam}\Delta)^{2p} \int_\Delta \|H(\mathbf{x}) - H(\mathbf{y})\|^p d\mathbf{x}, \quad a.e.\ \mathbf{y} \in \Delta. \tag{11}$$

The definition of the broken norm implies that

$$[v - \mathcal{I}^1_\Delta v]_{p,\Delta} \le \frac{6p}{2p-d} h_N^2 [\omega(\mathbf{x},\mathbf{y})]_{p,\Delta}$$

and

$$[v - \mathcal{I}^1_N v]_{q,\Omega} \le |\Omega|^{\frac{1}{q}-\frac{1}{p}} [v - \mathcal{I}^1_N v]_{p,\Omega} \le |\Omega|^{\frac{1}{q}-\frac{1}{p}} \frac{6p}{2p-d} h_N^2 [\omega(\mathbf{x},\mathbf{y})]_{p,\Omega} = \frac{6p}{2p-d} |\Omega|^{\frac{1}{q}-\frac{1}{p}} h_N^2 \epsilon_N. \quad \square$$

The simple consequence of Lemmas 1 and 2 is the corollary.

Corollary 3. *Under conditions of Lemma 2 it holds:*

$$C(d,q)^{\frac{1}{q}} \|\det H\|^{\frac{1}{d}}_{L^{\frac{q}{2q+d}}(\Omega)} \le N^{\frac{2}{d}} \|u - \mathcal{I}^1_N u\|_{L^q(\Omega)} + \frac{6p}{2p-d} N^{\frac{2}{d}} h_N^2 |\Omega|^{\frac{1}{q}-\frac{1}{p}} \epsilon_N. \tag{12}$$

Proof. Since $\pi - \mathcal{I}^1_N\pi = \pi - u - \mathcal{I}^1_N(\pi - u) + u - \mathcal{I}^1_N u$, we apply the triangular inequality to get from (6) and (10):

$$\begin{aligned} C(d,q)^{\frac{1}{q}} \|\det H\|^{\frac{1}{d}}_{L^{\frac{q}{2q+d}}(\Omega)} &\le N^{\frac{2}{d}} \left([u - \mathcal{I}^1_N u]_{q,\Omega} + [\pi - u - \mathcal{I}^1_N(\pi - u)]_{q,\Omega}\right) \\ &\le N^{\frac{2}{d}} \left([u - \mathcal{I}^1_N u]_{q,\Omega} + \frac{6p}{2p-d} h_N^2 |\Omega|^{\frac{1}{q}-\frac{1}{p}} \epsilon_N\right) \\ &= N^{\frac{2}{d}} \left(\|u - \mathcal{I}^1_N u\|_{L^q(\Omega)} + \frac{6p}{2p-d} h_N^2 |\Omega|^{\frac{1}{q}-\frac{1}{p}} \epsilon_N\right). \quad \square \end{aligned}$$

3. Upper and lower bounds of interpolation errors

An upper bound for the P_2-interpolation error can be established as follows.

Lemma 4. *Let $u \in W^{2,p}(\Omega)$, $p > d/2$ and a conformal triangulation $\mathcal{T}_N$ of Ω be given. Then for $0 < q \le p$ it holds:*

$$\|u - \mathcal{I}^2_N u\|_{L^q(\Omega)} \le \frac{12p}{2p-d} |\Omega|^{\frac{1}{q}-\frac{1}{p}} h_N^2 \epsilon_N. \tag{13}$$

Proof. The proof is based on the local estimate [10], (p. 413, Theorems 2–1) valid for any d-simplex $\Delta \in \mathcal{T}_N$:

$$\|u - \mathcal{I}^2_\Delta u\|^p_{L^p(\Delta)} \le \left(\frac{12p}{2p-d}\right)^p (\operatorname{diam}\Delta)^{2p} \int_\Delta \|H(\mathbf{x})\|^p d\mathbf{x}. \tag{14}$$

Since $\pi(\mathbf{x},\mathbf{y})$ is the quadratic function of argument $\mathbf{x}$ and its Hessian with respect to $\mathbf{x}$ is $H(\mathbf{y})$, we have

$$\|u - \mathcal{I}^2_\Delta u\|^p_{L^p(\Delta)} = \|u - \pi - \mathcal{I}^2_\Delta(u-\pi)\|^p_{L^p(\Delta)} \le \left(\frac{12p}{2p-d}\right)^p (\operatorname{diam}\Delta)^{2p} \int_\Delta \|H(\mathbf{x}) - H(\mathbf{y})\|^p d\mathbf{x}, \quad a.e.\ \mathbf{y} \in \Delta.$$

Averaging over Δ we obtain

$$\|u - \mathcal{I}^2_\Delta u\|^p_{L^p(\Delta)} \leq \left(\frac{12p}{2p-d}\right)^p (\mathrm{diam}\Delta)^{2p} \frac{1}{|\Delta|} \int_\Delta \int_\Delta \|H(\mathbf{x}) - H(\mathbf{y})\|^p \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y},$$

and summing over all $\Delta \in \mathcal{T}_N$ we get

$$\begin{aligned} \|u - \mathcal{I}^2_\Delta u\|_{L^q(\Omega)} &= [u - \mathcal{I}^2_\Delta u]_{q,\Omega} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p}} [u - \mathcal{I}^2_\Delta u]_{p,\Omega} \\ &= |\Omega|^{\frac{1}{q}-\frac{1}{p}} \|u - \mathcal{I}^2_\Delta u\|_{L^p(\Omega)} \leq \frac{12p}{2p-d} |\Omega|^{\frac{1}{q}-\frac{1}{p}} h_N^2 [\omega(\mathbf{x}, \mathbf{y})]_{p,\Omega}. \quad \Box \end{aligned}$$

The properties of functions of two arguments, the estimate (12) for the interpolation error and the upper bound for the P_2-interpolation (14) do not impose any restrictions on triangulation $\mathcal{T}_N$ but conformity. The lower bound for the P_1-interpolation imposes a restriction on the sequence of meshes $\mathcal{T}_N$, $N \to \infty$, which we shall refer to as *condition A*.

Definition 1. The mesh sequence $\mathcal{T}_N$, $N \to \infty$, satisfies condition A if there exist constants $\sigma > 0$ and $0 < \gamma < 1/2$ such that

$$h_N \epsilon_N^\gamma \leq \sigma N^{-\frac{1}{d}}. \tag{15}$$

We recall that $\lim_{h_N \to 0} \epsilon_N = 0$ for any $u \in W^{2,p}(\Omega)$. For a sequence of meshes satisfying *condition A*, $\lim_{N\to\infty} \epsilon_N = 0$ although h_N does not necessarily tend to 0 as $N \to \infty$. Indeed, assume that there exists a subsequence denoted by $\{(\epsilon_N, h_N)\}_N$ such that for $N \geq N_0$ one has $\epsilon_N \geq a > 0$. Then from (15) $h_N \leq \sigma N^{-\frac{1}{d}} a^{-\gamma}$ and $\lim_{N\to\infty} h_N = 0$. Therefore, for this subsequence $\lim_{N\to\infty} \epsilon_N = 0$ which is a contradiction.

We note that the class of meshes satisfying *condition A* is wide enough. It includes all quasiuniform meshes and $\mathfrak{M}$-quasiuniform meshes where $\mathfrak{M}$ is a given tensor metric field. In particular, the meshes may be adaptive and possibly anisotropic [3,4,9,11].

The lower bound for the P_1-interpolation error is derived for mesh sequences satisfying *condition A*.

Lemma 5. *Let $u \in W^{2,p}(\Omega)$, $p > d/2$ and a sequence of conformal triangulations $\mathcal{T}_N$ satisfying condition A be given. Then for $0 < q \leq p$ it holds:*

$$C(d,q)^{\frac{1}{q}} \|\det H\|^{\frac{1}{d}}_{L^{\frac{q}{2q+d}}(\Omega)} \leq \lim_{N\to\infty} N^{\frac{2}{d}} \|u - \mathcal{I}^1_N u\|_{L^q(\Omega)}. \tag{16}$$

Proof. From Corollary 3 and (15) one has

$$C(d,q)^{\frac{1}{q}} \|\det H\|^{\frac{1}{d}}_{L^{\frac{q}{2q+d}}(\Omega)} \leq N^{\frac{2}{d}} \|u - \mathcal{I}^1_N u\|_{L^q(\Omega)} + \frac{6p\sigma^2}{2p-d} |\Omega|^{\frac{1}{q}-\frac{1}{p}} \epsilon_N^{1-2\gamma}.$$

Since $\gamma < 1/2$ and $\lim_{N\to\infty} \epsilon_N = 0$, we obtain (16). $\Box$

4. The L^q-saturation property

Theorem 6. *Let $u \in W^{2,p}(\Omega)$, $p > d/2$ and a sequence of conformal triangulations $\mathcal{T}_N$ satisfying condition A be given. Then for $0 < q \leq p$ it holds:*

$$\lim_{N\to\infty} \frac{\|u - \mathcal{I}^2_N u\|_{L^q(\Omega)}}{\|u - \mathcal{I}^1_N u\|_{L^q(\Omega)}} = 0. \tag{17}$$

Proof. Due to Lemma 4 and (15) there exists a positive constant C_2 depending on p, q, Ω, d only such that

$$\|u - \mathcal{I}^2_N u\|_{L^q(\Omega)} \leq C_2(p,q,\Omega,d) N^{-\frac{2}{d}} \epsilon_N^{1-2\gamma}.$$

Due to Lemma 5 there exist a positive constant C_1 depending on u, q, Ω, d only and an integer N_0 such that for $N \geq N_0$ it holds:

$$C_1(u,q,\Omega,d) N^{-\frac{2}{d}} \leq \|u - \mathcal{I}^1_N u\|_{L^q(\Omega)}.$$

Therefore,

$$\|u - \mathcal{I}^2_N u\|_{L^q(\Omega)} \leq C_2(p,q,\Omega,d) C_1(u,q,\Omega,d)^{-1} \epsilon_N^{1-2\gamma} \|u - \mathcal{I}^1_N u\|_{L^q(\Omega)}.$$

Since $\lim_{N\to\infty} \epsilon_N^{1-2\gamma} = 0$, we prove (17). $\Box$

For mesh sequences satisfying *condition A*, Theorem 6 states the L^q-saturation property (17) even in the stronger form than (1): as $N \to \infty$, the parameter α can be taken arbitrarily small. For other mesh sequences the L^q-saturation property (1) remains the assumption.

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