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Optimal Triangulations: Existence, Approximation, and Double Differentiation of P_1 Finite Element Functions

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Abstract—Some theoretical issues associated with optimal unstructured triangulations are considered. Published results are overviewed, and an existence theorem is proved for optimal triangulations.

1. INTRODUCTION

In a number of papers published in the last decade (see, for example, [1, 2]), it was shown that obtuse triangles stretched in the direction of the minimal second derivative of a certain function can be the elements best suited for minimizing an interpolation error. It is for this reason that optimal adaptive grids frequently contain anisotropic elements, i.e., obtuse triangles. Theoretical analysis of anisotropic meshing is a challenging problem. In this paper, we review several theoretical issues related to optimal (possibly anisotropic) triangulations. The results published in our previous papers [3–5] are summarized. All results, except for the existence of an optimal triangulation, are extended to the case of tetrahedral 3D meshes [3, 4].

The paper is organized as follows. In Section 2, we define the optimal triangulation and prove its existence under certain assumptions. In Section 3, we formulate the main property of optimal triangulations and give an L_∞ error estimate for a piecewise linear interpolation operator. In Section 4, we give a constructive definition of quasi-optimal triangulations and show that they approximate optimal ones. The methodology used in this paper is based on the Hessian recovered from a discrete P_1 solution. In Section 5, we discuss some methods for Hessian recovery.

2. EXISTENCE OF OPTIMAL TRIANGULATIONS

Let $\Omega \in \mathbb{R}^2$ be a polygon and Ω_h be its conformal partition into triangles,

$$\Omega_h = \bigcup_{i=1}^{\mathcal{N}(\Omega_h)} e_i,$$

where $\mathcal{N}(\Omega_h)$ is the number of elements in Ω_h. Let $C^k(D)$ be the space of functions with continuous partial derivatives up to order k in $D \subset \bar{\Omega}$. Denote by $\|\cdot\|_{\infty,D}$ and $\|\cdot\|_{2,D}$ the $L_\infty(D)$ and $C^2(D)$ norms, respectively, and define $\|\cdot\|_\infty \equiv \|\cdot\|_{\infty,\Omega}$. We also define the space $P_1(\Omega_h)$ of functions that are continuous on Ω and linear on each element in Ω_h. Furthermore, let $\mathcal{P}^h_{\Omega_h} : C^0(\bar{\Omega}) \longrightarrow P_1(\Omega_h)$ be a projector onto the discrete space $P_1(\Omega_h)$ and $\mathcal{I}^h_{\Omega_h} : C^0(\bar{\Omega}) \longrightarrow P_1(\Omega_h)$ be a linear interpolation operator. We omit mesh-related subscripts whenever this does not result in ambiguity.

Some theoretical results formulated in this paper are based on the assumption that the solution of a continuous second-order boundary value problem belongs to $C^2(\bar{\Omega})$. However, the constants contained in our error estimates are independent of the actual value of the C^2 norm of the solution. Since $C^2(\bar{\Omega})$ is dense in $C^0(\bar{\Omega})$, one can try to analyze regularized problems having smooth solutions and obtain error estimates for the original problem, making use of the density mentioned above. We will address this challenging problem in a future study.

Definition 1. Let $u \in C^0(\bar{\Omega})$ and $\mathcal{P}^h_{\Omega_h}$ be given. A triangulation $\Omega_h(N_T, u)$ consisting of at most N_T elements is said to be optimal if it solves the optimization problem

$$\Omega_h(N_T, u) = \arg \min_{\Omega_h \,:\, \mathcal{N}(\Omega_h) \le N_T} \|u - \mathcal{P}^h_{\Omega_h} u\|_\infty. \tag{1}$$

Another optimization problem can be formulated when the number of nodes is restricted. Denote the number of nodes in Ω_h by $\mathcal{M}(\Omega_h)$.

Definition 2. Let $u \in C^0(\bar{\Omega})$ and $\mathcal{P}^h_{\Omega_h}$ be given. A triangulation $\Omega_h(N_P, u)$ consisting of at most N_P nodes is said to be optimal if it solves the optimization problem

$$\Omega_h(N_P, u) = \arg \min_{\Omega_h \,:\, \mathcal{M}(\Omega_h) \le N_P} \|u - \mathcal{P}^h_{\Omega_h} u\|_\infty. \tag{2}$$

In the general case, optimization problems (1) and (2) may be ill posed, and the optimal triangulation may not exist. However, the definitions of optimal triangulations imply that there exists a triangulation that is arbitrarily close to the optimal one. Under certain conditions, optimal triangulations can be proved to exist. Since the number of triangles is not greater than twice the number of nodes in any conformal grid, optimization problems (1) and (2) are equivalent.

Theorem 1. *Let $u \in C^0(\bar{\Omega})$ and $\|u - \mathcal{P}^h_{\Omega_h} u\|_\infty$ be a continuous functional of the node coordinates, i.e.,*

$$\left| \|u - \mathcal{P}^h_{\Omega_h} u\|_\infty - \|u - \mathcal{P}^h_{\Omega^\varepsilon_h} u\|_\infty \right| \le C(u)\varepsilon,$$

where Ω^ε_h is the triangulation resulting from an arbitrary ε-perturbation of nodes in a conformal triangulation Ω_h. Furthermore, let the projector $\mathcal{P}^h_{\Omega_h}$ satisfy

$$\|u - \mathcal{P}^h_{\Omega^2_h} u\|_\infty \le \|u - \mathcal{P}^h_{\Omega^1_h} u\|_\infty$$

for any triangulation Ω^2_h obtained as a hierarchical partition of a triangulation Ω^1_h. Then, the optimization problem (4) *has a solution.*

Proof. Since $\|u - \mathcal{P}^h_{\Omega_h} u\|_\infty \ge 0$, there exists a sequence of triangulations $\{\Omega^k_h\}_{k=1}^\infty$ such that

$$\lim_{k \to \infty} \|u - \mathcal{P}^h_{\Omega^k_h} u\|_\infty = \inf_{\Omega_h \,:\, \mathcal{M}(\Omega_h) \le N_P} \|u - \mathcal{P}^h_{\Omega_h} u\|_\infty. \tag{3}$$

A triangulation Ω^k_h can be defined by a set of nodes X^k_h and a connectivity table T^k_h (a list of triangles with reference to nodes). Since the Cartesian product $\Omega \times \ldots \times \Omega$ of N_P compact and bounded sets is compact and bounded, the sequence X^k_h contains a convergent (in the product metric) subsequence. For the sake of simplicity, we assume that this subsequence is $\{X^k_h\}_{k=1}^\infty$. Let

$$X^\infty_h = \lim_{k \to \infty} X^k_h \tag{4}$$

and N^∞_P be the number of distinct elements (distinct points) in the set X^∞_h. It is obvious that $N^\infty_P \le N_P$.

Let x^k_j $(j = 1, 2, \ldots, N^k_P)$ be the distinct elements of X^k_h $(k = 1, 2, \ldots, \infty)$ (distinct points in $\bar{\Omega}$) and denote the minimal distance between these points by δ^k. The convergence in (4) means that any small $\varepsilon > 0$ can be associated with k_ε such that any x^k_i with $k \ge k_\varepsilon$ belongs to the disk $(x^\infty_j, \varepsilon)$ of radius ε centered at x^∞_j. The indexes i and j may be different since the elements of X^∞_h are not arranged in any particular order.

Denote by α^∞ the minimal angle in all possible nondegenerate triangles with nodes from X^∞_h and define $\varepsilon_m = 1/m$ for an integer $m \ge m_0 > 0$, where m_0 is a sufficiently large integer such that $\delta^\infty \sin\alpha^\infty/10 > \varepsilon_m$. By virtue of (3), there exists a conformal triangulation $\Omega^{k_m}_h = \{X^{k_m}_h, T^{k_m}_h\}$ with nodes in the ε_m-neighborhood of X_h^∞. Let the nodes in X_h^∞ be numbered similarly to those in $X_h^{k_m}$: the existence of l nodes of $X_h^{k_m}$ in the ε_m-neighborhood of a node x_j^∞ entails counting x_j^∞ l times. Therefore, we can formally define the triangulation $\hat{\Omega}_h^{k_m} = \{X_h^\infty, T_h^{k_m}\}$ as resulting from an ε_m-perturbation of $\Omega_h^{k_m}$. Then, by the continuity assumption, we can write

$$\left\|u - \mathcal{P}^h_{\hat{\Omega}_h^{k_m}} u\right\|_\infty \le \left\|u - \mathcal{P}^h_{\Omega_h^{k_m}} u\right\|_\infty + C(u)\varepsilon_m,$$

where $C(u)$ depends only on u. However, $\hat{\Omega}_h^{k_m}$ may not be a conformal triangulation, since some triangles in $T_h^{k_m}$ may be degenerate (have zero areas). The assumption that $\delta^\infty \sin\alpha^\infty/10 > \varepsilon_m$ implies that the triangles in $\hat{\Omega}_h^{k_m}$ cannot tangle. Indeed, assume that there exists a triangle in $\hat{\Omega}_h^{k_m}$ with vertices $(x_{j_1}^\infty, x_{j_2}^\infty, x_{j_3}^\infty)$ that is tangled with a neighboring triangle. Since ε_m is small with respect to the sides and angles in these triangles, the triangle in $\Omega_h^{k_m}$ with vertices $(x_{j_1}^{k_m}, x_{j_2}^{k_m}, x_{j_3}^{k_m})$ is tangled with its neighbor as well. This contradicts the conformality of $\Omega_h^{k_m}$. We modify $T_h^{k_m}$ to obtain a conformal triangulation $\tilde{\Omega}_h^{k_m}$ with the nodes in X_h^∞. To this end, we eliminate from $T_h^{k_m}$ all triangles that degenerate to a point, a side, or two sides (Fig. 1). This does not alter the norm of the error. The hanging nodes left by the triangles that reduce to edges are transformed into nodes of a conformal mesh by hierarchical partition of triangles adjoining the degenerate ones without inserting any additional nodes (Fig. 2). The partition cannot increase the norm of the error. By the assumptions of Theorem 1,

$$\left\|u - \mathcal{P}^h_{\tilde{\Omega}_h^{k_m}} u\right\|_\infty \le \left\|u - \mathcal{P}^h_{\hat{\Omega}_h^{k_m}} u\right\|_\infty \le \left\|u - \mathcal{P}^h_{\Omega_h^{k_m}} u\right\|_\infty + C(u)\varepsilon_m.$$

Therefore,

$$\lim_{m\to\infty}\left\|u - \mathcal{P}^h_{\tilde{\Omega}_h^{k_m}} u\right\|_\infty \le \lim_{m\to\infty}\left\|u - \mathcal{P}^h_{\Omega_h^{k_m}} u\right\|_\infty = \inf_{\Omega_h\,:\,\mathcal{M}(\Omega_h)\le N_p}\left\|u - \mathcal{P}^h_{\Omega_h} u\right\|_\infty$$

and there exists a sequence of conformal triangulations $\tilde{\Omega}_h^{k_m}$ with nodes in X_h^∞ such that

$$\lim_{m\to\infty}\left\|u - \mathcal{P}^h_{\tilde{\Omega}_h^{k_m}} u\right\|_\infty = \inf_{\Omega_h\,:\,\mathcal{M}(\Omega_h)\le N_p}\left\|u - \mathcal{P}^h_{\Omega_h} u\right\|_\infty.$$

Since the total number of connectivity tables corresponding to a particular X_h^∞ is finite, there exists a conformal triangulation Ω_h^∞ with nodes in X_h^∞ minimizing $\left\|u - \mathcal{P}^h_{\tilde{\Omega}_h} u\right\|_\infty$ such that

$$\left\|u - \mathcal{P}^h_{\Omega_h^\infty} u\right\|_\infty = \inf_{\Omega_h\,:\,\mathcal{M}(\Omega_h)\le N_p}\left\|u - \mathcal{P}^h_{\Omega_h} u\right\|_\infty.$$

The theorem is proved.

Note that the interpolation operator $\mathcal{I}^h_{\Omega_h}$ satisfies the assumptions of Theorem 1. Therefore optimization problem (2) with $\mathcal{P}^h_{\Omega_h} = \mathcal{I}^h_{\Omega_h}$ has a solution that is not necessarily unique.

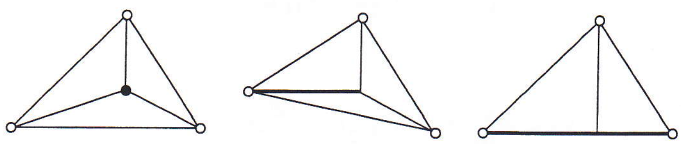

Fig. 1. Types of triangle degeneration. Open circles correspond to original triangles; bold segments and closed circle, to degenerate triangles.

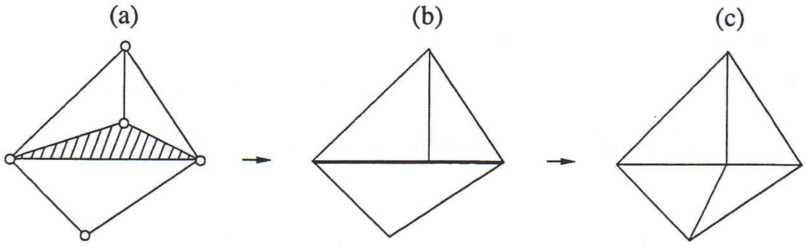

Fig. 2. Breakdown and recovery of mesh conformality: (a) original (hatched) triangle; (b) degenerate triangle (two solid segments); (c) hierarchical partition.

Recall that the existence of an optimal triangulation was conditioned in [1, 6] on the existence of a coordinate transformation resulting in a canonical Hessian. The sufficient conditions for the existence of such a transformation were found to be very stiff [1] and could be weakened only in a different error norm [6].

3. ERROR ESTIMATES FOR OPTIMAL GRIDS

Let a function $u \in C^2(\bar{\Omega})$ have a nonsingular Hessian $H(x) = \{H_{ps}(x)\}_{p,s=1}^2$, i.e., $\det H(x) \neq 0$ for $\forall x \in \bar{\Omega}$. Since the Hessian is symmetric, there exists its spectral decomposition at any $x \in \bar{\Omega}$,

$$H = W^t \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} W,$$

where W is an orthonormal matrix and $|\lambda_1| < |\lambda_2|$. It is clear that $\lambda_1 \neq 0$ and

$$|H| = W^t \begin{pmatrix} |\lambda_1| & 0 \\ 0 & |\lambda_2| \end{pmatrix} W$$

defines a continuous metric on Ω. Let $|\Omega|_{|H|}$ be the volume of Ω in this metric. Then, the following a priori error estimates are valid for the P_1 interpolation operator.

Theorem 2. *Let $N_T > 0$, $u \in C^2(\bar{\Omega})$, and $|H|$ be the metric induced by the Hessian of u. Furthermore, let $\tilde{\Omega}_h$ be an optimal mesh, and the following estimate holds for any triangle $e \in \tilde{\Omega}_h$:*

$$\|H_{ps} - H_{e,ps}\|_{\infty,e} < q|\lambda_1(H_e)|/2, \quad 0 < q < 1, \quad p, s = 1, 2, \tag{5}$$

where q is a constant, $x_e = \underset{x \in e}{\operatorname{argmax}} |\det(H(x))|$, and $H_e = H(x_e)$. Then,

$$C_1(q)\frac{|\Omega|_{|H|}}{N_T} \le \|u - \mathcal{I}^h_{\tilde{\Omega}_h} u\|_\infty \le C_2(q)\frac{|\Omega|_{|H|}}{N_T}, \tag{6}$$

where $C_1(q)$ and $C_2(q)$ depend only on q.

The proof can be found in [3, 5]. One straightforward corollary to this theorem is as follows: if a projector $\mathcal{P}^h_{\Omega_h}$ satisfies the relation

$$\|u - \mathcal{P}^h_{\Omega_h} u\|_\infty \le \hat{C}\|u - \mathcal{I}^h_{\Omega_h} u\|_\infty, \tag{7}$$

then inequality (6) implies

$$\|u - \mathcal{P}^h_{\tilde{\Omega}_h} u\|_\infty \le \hat{C} C_2(q)\frac{|\Omega|_{|H|}}{N_T}. \tag{8}$$

We should note here that error estimates (6) are in good agreement with Tikhomirov's result [7]: for any discrete space V_h and $\Omega \in \mathbb{R}^2$, it holds that

$$\inf_{V_h\,:\,\dim V_h \le N_T} \sup_{\|u\|_{2,\Omega} = 1} \inf_{v_h \in V_h} \|u - v_h\|_\infty \simeq N_T^{-1}.$$

The assumption that the Hessian is nonsingular is used for the sake of simplicity. Actually, all results presented in this paper can be extended to functions with singular Hessians. To this end, we replace the singular Hessians with a nonsingular approximation and use it to construct the metric [3].

4. QUASI-OPTIMAL MESHES AS APPROXIMATIONS TO THE OPTIMAL MESH

Since the exact solution is not known, the error $\|u - \mathcal{P}^h_{\Omega_h}u\|_\infty$ cannot be estimated. For this reason, optimization problem (1) should be replaced by a different optimization problem whose solution approximates the solution of (1). To this end, we introduce the concepts of mesh quality and mesh quasi-optimality.

Let $Q(\Omega_h)$ be an easily calculable quantitative characteristic of a mesh Ω_h such that $0 < Q(\Omega_h) \le 1$. We invoke the definition of $Q(\Omega_h)$ proposed in [8]. Let the number N_T of elements be prescribed, define a continuous metric $G(x) = \{G_{ps}(x)\}^2_{p,s=1}$, $x \in \mathbb{R}^2$ in Ω_h, and denote by $x_e \in e$ the point in the triangle e at which $|\det(G(x))|$ attains its maximal value. We introduce $G_e = G(x_e)$ and define the area of the triangle and the length $\mathbf{l}_e \in \mathbb{R}^2$ of its side (in metric G) as

$$|e|_G = |e|(\det(G_e))^{1/2} \quad \text{and} \quad |\mathbf{l}_e|_G = (G_e\mathbf{l}_e, \mathbf{l}_e)^{1/2},$$

respectively, where $|e|$ is the triangle's area in the Cartesian coordinate system. Denote the perimeter of the triangle under metric G by $|\partial e|_G$. Let $|\Omega_h|_G$ be the area of the computational domain measured in the metric G:

$$|\Omega_h|_G = \sum_{e \in \Omega_h} |e|_G.$$

Following [3], we define $Q(\Omega_h)$ as

$$Q(\Omega_h) = \min_{e \in \Omega_h} Q(e), \quad \text{with} \quad Q(e) = 12\sqrt{3}\frac{|e|_G}{|\partial e|_G^2}F\left(\frac{|\partial e|_G}{3h^*}\right), \tag{9}$$

where the function $F(\cdot)$ and the average length h^* of the triangle's side (in metric G) are

$$F(x) = \left(\min\left\{x, \frac{1}{x}\right\}\left(2 - \min\left\{x, \frac{1}{x}\right\}\right)\right)^3, \quad h^* = \sqrt{\frac{4|\Omega_h|_G}{\sqrt{3}N_T}},$$

respectively. Hereinafter, we write $Q(G, N_T, \Omega_h)$ instead of $Q(\Omega_h)$ to emphasize its dependence on the metric G and the prescribed number N_T. It is easy to verify that $0 < Q(G, N_T, \Omega_h) \le 1$ and the maximal $Q(G, N_T, \Omega_h)$ is attained when all mesh elements are equilateral (in metric G) triangles of diameter h^*. We say that $Q(G, N_T, \Omega_h)$ is the mesh quality with respect to the metric G and the number of elements N_T.

Definition 3. Let G be a continuous metric and N_T be a given integer. A mesh Ω_h is said to be G-quasi-optimal if there exists a positive constant Q_0 such that $Q_0 = O(1)$ and

$$Q(G, N_T, \Omega_h) > Q_0.$$

Definition 4. Let $u \in C^2(\bar{\Omega})$ and $|H|$ be the metric induced by the Hessian of u. The triangulation $\Omega_h(N_T, u)$ corresponding to the given function u and a given integer N_T is said to be quasi-optimal if it is $|H|$-quasi-optimal.

A quasi-optimal mesh characterized by $Q(H, N_T, \Omega_h) = 1$ may not exist because of restrictions imposed by the boundary of Ω. When $Q_0 < 1$, the above constraint becomes weaker. On the other hand, when $Q(H, N_T, \Omega_h) < 1$, the number $\mathcal{N}(\Omega_h)$ of triangles in the $|H|$-quasi-optimal mesh may differ from N_T, approaching N_T as $Q_0 \longrightarrow 1$.

Quasi-optimal meshes (QOMs) were studied in [3, 5]. It was found that, in certain cases, the QOM is an approximate solution of optimization problem (1).

Theorem 3. *Let $N_T > 0$, $u \in C^2(\bar{\Omega})$ and $|H|$ be the metric induced by the Hessian of u. Furthermore, let $\Omega_h(N_T, u)$ and $\tilde{\Omega}_h(N_T, u)$ be quasi-optimal and optimal meshes, respectively, and $e^* \in \Omega_h$ be the element where $\|u - \mathcal{I}^h_{\Omega_h} u\|_\infty$ is attained. Suppose that the following estimate holds for $e^* \in \Omega_h$ and any element $\tilde{e} \in \tilde{\Omega}_h$:*

$$\|H_{ps} - H_{e,ps}\|_{\infty, e} < q|\lambda_1(H_e)|/2, \quad 0 < q < 1, \quad p, s = 1, 2, \tag{10}$$

where q is a constant, $x_e = \underset{x \in e}{\operatorname{argmax}} |\det(H(x))|$, and $H_e = H(x_e)$. Then,

$$\|u - \mathcal{I}^h_{\Omega_h} u\|_\infty \le C(Q_0, q)\|u - \mathcal{I}^h_{\tilde{\Omega}_h} u\|_\infty, \tag{11}$$

where $C(Q_0, q)$ is a constant depending only on q and Q_0 from Definition 3.

The proof can be found in [3, 5]. One straightforward corollary to this theorem is as follows: if a projector $\mathcal{P}^h_{\Omega_h}$ satisfies (7), then relations (6) and (11) entail

$$\|u - \mathcal{P}^h_{\Omega_h} u\|_\infty \le \tilde{C} C_2(q) C(Q_0, q) \frac{|\Omega|_{|H|}}{N_T}. \tag{12}$$

5. DOUBLE DIFFERENTIATION ON OPTIMAL AND QUASI-OPTIMAL MESHES

As a rule, the Hessian $H(x)$ is an unknown function. Practical computations make use of its approximation H^h recovered from the discrete solution $\mathcal{P}^h_{\Omega_h} u$. In what follows, we briefly describe some methods for Hessian recovery [4, 8, 9] and advocate the replacement of $H(x)$ by its discrete counterpart H^h.

Let $u^h = \mathcal{P}^h_{\Omega_h} u$ be a discrete function in $P_1(\Omega_h)$. The discrete Hessian $H^h = \{H^h_{ps}\}^2_{p,s=1}$ with $H^h_{ps} \in P_1(\Omega_h)$ is defined as follows. At an interior node a_i, its entries are defined by

$$\int_{\sigma_i} H^h_{ps}(a_i) v^h dx = -\int_{\sigma_i} \frac{\partial u^h}{\partial x_p} \frac{\partial v^h}{\partial x_s} dx \quad \forall v^h \in P_1(\sigma_i), \quad v^h = 0 \text{ on } \partial\sigma_i, \tag{13}$$

where σ_i is the union of the triangles sharing the node a_i (superelement). At a boundary node a_i, the values of $H^h_{ps}(a_i)$ $(p, s = 1, 2)$ are obtained by weighted extrapolation from the neighboring interior nodal values [4]:

$$H^h_{ps}(a_i) = \int_{\sigma_i} \varphi(a_i) \mathring{H}^h_{ps} dx \left[\int_{\sigma_i} \varphi(a_i) \left(\sum_{a_j \notin \partial\Omega_h} \varphi(a_j)\right) dx\right]^{-1}, \tag{14}$$

where $\varphi(a_i)$ denotes a nodal basis function from $P_1(\Omega_h)$ and $\mathring{H}^h_{ps}$ stands for the finite element function defined by (13) as vanishing on $\partial\Omega_h$.

Theorem 4. *Let $N_T > 0$, $u \in C^2(\bar{\Omega})$, $u^h = \mathcal{P}^h_{\Omega_h} u$, H be the Hessian of u, and H^h be the discrete Hessian recovered from u^h by using* (13) *and* (14). *Furthermore, let the following estimates hold for any superelement $\sigma \in \Omega_h$ associated with a mesh node a:*

$$\|H_{ps} - H_{\sigma,ps}\|_{\infty,\sigma} < \delta, \tag{15}$$

$$|H^h_{ps}(a) - H_{\sigma,ps}| < \varepsilon, \tag{16}$$

where $H_\sigma = H(x_\sigma)$ and $x_\sigma = \underset{x \in \sigma}{\operatorname{argmax}} |\det(H(x))|$. Then, for ε and δ that are sufficiently small with respect to the minimal eigenvalue of $|H_\sigma|$, the $|H^h|$ is quasi-optimal mesh Ω_h $(Q(|H^h|, N_T, \Omega_h) \ge Q_0)$ is $|H|$-quasi-optimal as well:

$$Q(|H|, N_T, \Omega_h) \ge CQ_0$$

with constant C independent of N_T and $\|u\|_{2,\Omega}$.

The proof can be found in [3]. The theorem states that, under certain assumptions, $|H^h|$-quasi-optimality is a sufficient condition for $|H|$-quasi-optimality. By assumption (15), the variation of the Hessian on any superelement σ is small. Assumption (16) means that the Hessian must be approximated at the nodes. The latter assumption does not always hold true in practice, since it implies that the gradient error in u^h is small. Small gradient error is not typical for functions with singularities. To recover the discrete Hessian for non-smooth functions, we suggest a different definition of the discrete Hessian, which satisfies (16) in a weaker norm [4].

The alternative definition of the discrete Hessian is based on the following identity:

$$\int_\sigma H_{ps} v dx = \int_\sigma u \frac{\partial^2 v}{\partial x_s \partial x_p} dx - \int_{\partial\sigma} u \frac{\partial v}{\partial x_s} n_p dt \quad \forall v \in C^2(\sigma), \quad v = 0 \text{ on } \partial\sigma, \tag{17}$$

where $p, s = 1, 2$. Representation (17) has an important advantage over (13): the Hessian is defined in terms of the function rather than its derivatives. Its main drawback is higher smoothness of the test functions, which imposes restrictions on the geometry of σ and, as a consequence, on the triangulation used in recovering a discrete Hessian.

Definition 5. A triangulation Ω_h satisfies Condition A if, for any ith interior superelement, there exists an affine mapping $\mathcal{F}_i = \mathcal{S}_i \circ \mathcal{R}_i$ such that $\mathcal{F}_i(\sigma_i)$ is a shape regular superelement of diameter 1 whose inscribed radius is $O(1)$. Here, $\mathcal{S}_i$ and $\mathcal{R}_i$ denote scaling and rotation matrices, respectively.

Note that a triangulation may not satisfy Condition A. A two-dimensional mesh containing two anisotropic neighboring triangles whose stretching axes are orthogonal provides a simple example. Adaptive triangulations, however, do satisfy Condition A. Condition A does not imply the shape regularity of every triangle but requires local similarity of triangles. Thus, adaptive anisotropic meshes satisfy Condition A.

Let a_i be an interior node of a mesh Ω_h and σ_i be the corresponding superelement. Let $\hat{B}_i$ be the largest circle centered at $\mathcal{F}_i(a_i)$ and inscribed in $\mathcal{F}_i(\sigma_i)$. Due to the shape regularity of $\mathcal{F}_i(\sigma_i)$, the radius $\hat{R}_i$ of $\hat{B}_i$ is $O(1)$. Introducing polar coordinates with the origin at $\mathcal{F}_i(a_i)$, we define the smooth function $\hat{v}_i = 1 - r^2/\hat{R}_i^2$ on $\hat{B}_i$. The span of the functions $v = \alpha\mathcal{F}_i^{-1}(\hat{v}_i)$ with $\alpha \in \mathcal{R}^1$ defines a space V_i of local test functions. Note that $v \in V_i$ implies that the support $|B_i| \gtrsim |\sigma_i|$ satisfies the relation $B_i = \mathcal{F}_i^{-1}(\hat{B}_i)$, and $v \in C^2(B_i)$, $v = 0$ on ∂B_i.

Now, we recover the components $H^h_{ps} \in P_1(\Omega_h)$ of a discrete Hessian at the interior nodes a_i:

$$\int_{B_i} H^h_{ps}(a_i) v^h \partial x = \int_{B_i} u^h \frac{\partial^2 v^h}{\partial x_s \partial x_p} \partial x - \int_{\partial B_i} u^h \frac{\partial v}{\partial x_s} n_p \partial t \quad \forall v^h \in V_i. \tag{18}$$

There may exist such triangulations that some components of the Hessian cannot be recovered by using identity (17) at boundary nodes. For this reason, the values of the discrete Hessian H^h at all boundary nodes are the weighted extrapolations given by (14).

As was mentioned above, Condition A is a natural restriction on the shape of the superelement σ_i. To establish that a discrete Hessian converges to the differential one, we have to impose additional restrictions on the mesh triangles. Recall that Condition A is satisfied if, for any superelement σ_i, there exists a pair of operators (rotation $\mathcal{R}_i$ and scaling $\mathcal{S}_i$) whose combination transforms the superelement into a shape regular one. Therefore, for any triangle $\Delta \subset \Omega^h$ (superelement $\sigma \subset \Omega^h$), there exists a rotation operator $\mathcal{R}_\Delta$ ($\mathcal{R}_\sigma$) such that the image $\Delta_\mathcal{R} = \mathcal{R}_\Delta(\Delta)$ ($\sigma_\mathcal{R} = \mathcal{R}_\sigma(\sigma)$) can be scaled along the coordinate axes into a shape regular element. A rotated triangle $\Delta_\mathcal{R}$ is naturally characterized by

$$h_{\mathcal{R},k} = \max_{x, y \in \Delta_\mathcal{R}} |(x)_k - (y)_k|, \quad k = 1, 2.$$

A similar characterization applies to a rotated superelement $\sigma_\mathcal{R}$. Note that the rotation operator does not affect the best P_1 approximation of the function u:

$$\overline{u_\mathcal{R}} = \mathcal{R}_\Delta(\bar{u}), \quad u_\mathcal{R} = u(\mathcal{R}_\Delta(x)).$$

Recall that the best P_1 approximation is defined as follows:

$$\int_{\Delta}(u-\bar{u})dx = 0, \quad \int_{\Delta}\partial^{\alpha}(u-\bar{u})dx = 0 \quad \forall|\alpha| = 1, \tag{19}$$

where $\alpha = \{\alpha_1, \alpha_2\}$ is a multiindex with $\alpha_k = 0, 1$.

Definition 6. For a given function $u \in W^{1,p}(\Omega)$, a triangle in a triangulation satisfying Condition A satisfies Condition B if there exists a constant $C_B > 0$ such that

$$\max_{|\alpha|=1} h^{\alpha}_{\mathfrak{R}}\|\partial^{\alpha}(u_{\mathfrak{R}} - \overline{u_{\mathfrak{R}}})\|_{L_p(\Delta_{\mathfrak{R}})} \le C_B \min_{|\alpha|=1} h^{\alpha}_{\mathfrak{R}}\|\partial^{\alpha}(u_{\mathfrak{R}} - \overline{u_{\mathfrak{R}}})\|_{L_p(\Delta_{\mathfrak{R}})}, \tag{20}$$

where

$$h^{\alpha}_{\mathfrak{R}} := \prod_{k=1}^{2} h^{\alpha_k}_{\mathfrak{R},k}.$$

Condition B means isotropic distribution of the gradient error associated with the best P_1 approximation of $u \in W^{1,p}(\Omega)$ defined by (19). It does not imply any conditions for the angles of a triangle. Rather, it means that the triangle Δ must be adapted to the local behavior of the function u.

Definition 7. A triangle Δ in a triangulation Ω^h satisfying Condition A and a function $u \in W^{1,p}(\Omega)$ satisfy Condition C if there exists a constant $C_C > 0$ such that

$$h^{\alpha}_{\mathfrak{R}}\|\partial^{\alpha}(u_{\mathfrak{R}} - \overline{u_{\mathfrak{R}}})\|_{L_p(\Delta_{\mathfrak{R}})} \le C_C h^{\alpha\beta}_{\mathfrak{R}}, \quad |\alpha| = 1, \quad \beta > 0. \tag{21}$$

Condition C means convergence of the best P_1 approximation $\bar{u}$ to the function u on Δ. Actually, it implies a higher than $W^{1,p}(\Omega)$ smoothness of u. We do not specify any class of smooth functions here since the appropriate class varies widely depending on the application.

Conditions B and C rely on a certain relationship between the mesh and the function. The triangles of the mesh must ensure isotropic distribution of the error associated with the best P_1 approximation in the sense of (20) and convergence of the approximation to u (21) if u has some extra smoothness. It is clear that a mesh complying with Condition A may not satisfy Conditions B and C. However, every mesh adapted to the function does meet both Condition B and Condition C.

Theorem 5. *Let a function $u \in W^{1,p}(\Omega) \cap W^{2,1}(\Omega)$ $(p > 2)$ and the interior superelement σ_i satisfying Conditions* A, B, *and* C *be given. Suppose that the differential Hessian H deviates insignificantly from its mean value on σ_i:*

$$\|H_{ps} - \bar{H}_{ps}\|_{L_1(\sigma_i)} \le \delta, \quad p, s = 1, 2. \tag{22}$$

Then, the discrete Hessian recovered from the piecewise linear interpolant $\mathcal{I}^h_{\sigma_i}u$ by using (18) *converges to the differential Hessian:*

$$\|H_{ps} - H^h_{ps}\|_{L_1(B_i)} \lesssim \delta + C_B C_C|\sigma_i|^{1-1/p} \min_{k=1,2} h^{\beta-1}_{\mathfrak{R}_i,k}. \tag{23}$$

Moreover,

$$\|H_{ps} - H^h_{ps}\|_{L_1(B_i)} \lesssim \delta + C_B C_C \min_{k=1,2} h^{\beta-1/P}_{\mathfrak{R}_i,k}. \tag{24}$$

The proof can be found in [2]. Note that estimate (24) implies the local convergence of the discrete Hessian in the weak norm as the number of triangles in the mesh tends to infinity. Indeed, for any triangulation adapted to a function with a nonsingular Hessian,

$$\max_{\sigma_i}\min_{k=1,2} h_{\mathfrak{R}_i,k} \longrightarrow 0 \text{ при } \mathcal{N}(\Omega_h) \longrightarrow \infty.$$

The importance of the theorem lies in the fact that it is the first (to our knowledge) result where the local convergence of the recovered Hessian is established for anisotropic meshes and for functions with singularities.

CONCLUSIONS

Several theoretical issues related to optimal triangulations have been reviewed. The results presented here provide insight into the asymptotic properties of optimal and quasi-optimal triangulations and methods for recovering the discrete Hessian.

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