

# An unconditionally stable scheme for simulation of incompressible flows in time-dependent domains

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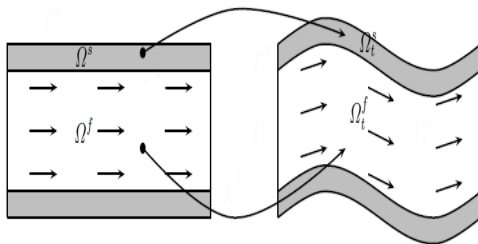
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Mathematical Modelling in Geophysical and Biomedical  
Sciences*

June 13, 2017, Moscow

# Fluid-Structure Interaction problem

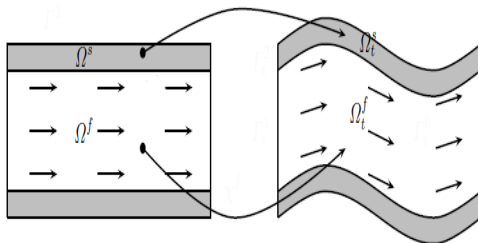
## Prerequisites for FSI



- ▶ reference subdomains  $\Omega_f, \Omega_s$

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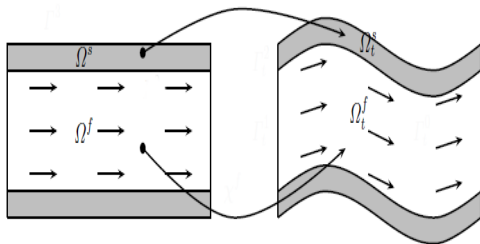
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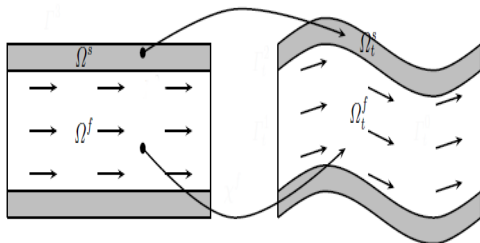
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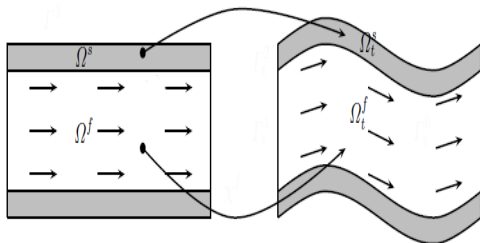
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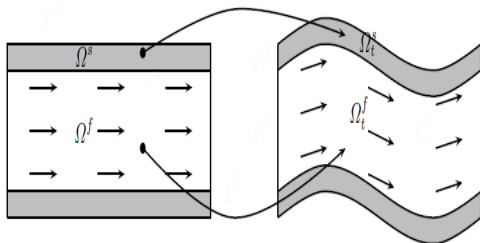
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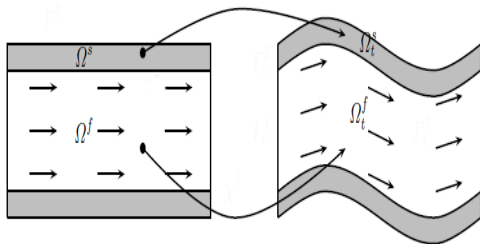
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- ▶ Cauchy stress tensors  $\boldsymbol{\sigma}_f, \boldsymbol{\sigma}_s$
- ▶ pressures  $p_f, p_s$
- ▶ densities  $\rho_s, \rho_f$  are constant



# Fluid-Structure Interaction problem

Universal equations in reference subdomains

Dynamic equations

$$\frac{\partial \mathbf{v}}{\partial t} = \begin{cases} \rho_s^{-1} \operatorname{div} (J \boldsymbol{\sigma}_s \mathbf{F}^{-T}) & \text{in } \Omega_s, \\ (J \rho_f)^{-1} \operatorname{div} (J \boldsymbol{\sigma}_f \mathbf{F}^{-T}) - \nabla_{\mathbf{v}} \left( \mathbf{F}^{-1} \left( \mathbf{v} - \frac{\partial \mathbf{u}}{\partial t} \right) \right) & \text{in } \Omega_f \end{cases}$$

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$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{v} \quad \text{in } \Omega_s$$

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Constitutive relation for the fluid stress tensor

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# FSI problem

User-dependent equations in reference subdomains

Constitutive relation for the solid stress tensor

$$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s(J, \mathbf{F}, p_s, \lambda_s, \mu_s, \dots) \quad \text{in } \Omega_s$$

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Monolithic approach: Extension of the displacement field to the fluid domain

$$\begin{aligned} G(\mathbf{u}) &= 0 && \text{in } \Omega_f, \\ \mathbf{u} &= \mathbf{u}^* && \text{on } \partial\Omega_f \end{aligned}$$

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+ Initial, boundary, interface conditions ( $\boldsymbol{\sigma}_f \mathbf{F}^{-T} \mathbf{n} = \boldsymbol{\sigma}_s \mathbf{F}^{-T} \mathbf{n}$ )



# Numerical scheme

- ▶ Conformal triangular or tetrahedral mesh  $\Omega_h$  in  $\widehat{\Omega}$
- ▶ LBB-stable pairs for velocity and pressure  $P_2/P_1$  or  $P_2/P_0$
- ▶ Fortran open source software Ani2D, Ani3D (Advanced numerical instruments 2D/3D)

<http://sf.net/p/ani2d/>    <http://sf.net/p/ani3d/>:

- ▶ mesh generation
- ▶ FEM systems
- ▶ algebraic solvers

## Numerical scheme

Find  $\{\mathbf{u}^{k+1}, \mathbf{v}^{k+1}, p^{k+1}\} \in \mathbb{V}_h^0 \times \mathbb{V}_h \times \mathbb{Q}_h$  s.t.

$$\mathbf{v}^{k+1} = \mathbf{g}_h(\cdot, (k+1)\Delta t) \text{ on } \Gamma_{f0}, \quad \left[ \frac{\partial \mathbf{u}}{\partial t} \right]_{k+1} = \mathbf{v}^{k+1} \text{ on } \Gamma_{fs}$$

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where

$$\mathbb{V}_h \subset H^1(\widehat{\Omega})^3, \mathbb{Q}_h \subset L^2(\widehat{\Omega}), \mathbb{V}_h^0 = \{\mathbf{v} \in \mathbb{V}_h : \mathbf{v}|_{\Gamma_{s_0} \cup \Gamma_{f_0}} = \mathbf{0}\}, \mathbb{V}_h^{00} = \{\mathbf{v} \in \mathbb{V}_h^0 : \mathbf{v}|_{\Gamma_{f_s}} = \mathbf{0}\}$$

$$\left[ \frac{\partial \mathbf{f}}{\partial t} \right]_{k+1} := \frac{3\mathbf{f}^{k+1} - 4\mathbf{f}^k + \mathbf{f}^{k-1}}{2\Delta t}$$

# Numerical scheme

$$\begin{aligned} & \int_{\Omega_s} \rho_s \left[ \frac{\partial \mathbf{v}}{\partial t} \right]_{k+1} \psi \, d\Omega + \int_{\Omega_s} J_k \mathbf{F}(\tilde{\mathbf{u}}^k) \mathbf{S}(\mathbf{u}^{k+1}, \tilde{\mathbf{u}}^k) : \nabla \psi \, d\Omega + \\ & \int_{\Omega_f} \rho_f J_k \left[ \frac{\partial \mathbf{v}}{\partial t} \right]_{k+1} \psi \, d\Omega + \int_{\Omega_f} \rho_f J_k \nabla \mathbf{v}^{k+1} \mathbf{F}^{-1}(\tilde{\mathbf{u}}^k) \left( \tilde{\mathbf{v}}^k - \left[ \frac{\partial \mathbf{u}}{\partial t} \right]_k \right) \psi \, d\Omega + \\ & \int_{\Omega_f} 2\mu_f J_k \mathbf{D}_{\tilde{\mathbf{u}}^k} \mathbf{v}^{k+1} : \mathbf{D}_{\tilde{\mathbf{u}}^k} \psi \, d\Omega - \int_{\Omega} p^{k+1} J_k \mathbf{F}^{-T}(\tilde{\mathbf{u}}^k) : \nabla \psi \, d\Omega = 0 \quad \forall \psi \in \mathbb{V}_h^0 \end{aligned}$$

$$J_k := J(\tilde{\mathbf{u}}^k), \quad \tilde{\mathbf{f}}^k := 2\mathbf{f}^k - \mathbf{f}^{k-1}, \quad \mathbf{D}_{\mathbf{u}\mathbf{v}} := \{\nabla \mathbf{v} \mathbf{F}^{-1}(\mathbf{u})\}_s, \quad \{\mathbf{A}\}_s := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

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$$\int_{\Omega_s} \left[ \frac{\partial \mathbf{u}}{\partial t} \right]_{k+1} \phi \, d\Omega - \int_{\Omega_s} \mathbf{v}^{k+1} \phi \, d\Omega + \int_{\Omega_f} G(\mathbf{u}^{k+1}) \phi \, d\Omega = 0 \quad \forall \phi \in \mathbb{V}_h^{00}$$

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$$\dots + \int_{\Omega_s} J_k \mathbf{F}(\tilde{\mathbf{u}}^k) \mathbf{S}(\mathbf{u}^{k+1}, \tilde{\mathbf{u}}^k) : \nabla \psi \, d\Omega + \dots$$

- ▶ St. Venant–Kirchhoff model (geometrically nonlinear):

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \lambda_s \operatorname{tr}(\mathbf{E}(\mathbf{u}_1, \mathbf{u}_2)) \mathbf{I} + 2\mu_s \mathbf{E}(\mathbf{u}_1, \mathbf{u}_2);$$

$$\mathbf{E}(\mathbf{u}_1, \mathbf{u}_2) = \{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2) - \mathbf{I}\}_s$$

- ▶ inc. Blatz–Ko model:

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \mu_s (\operatorname{tr}(\{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2)\}_s)) \mathbf{I} - \{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2)\}_s$$

- ▶ inc. Neo-Hookean model:

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \mu_s \mathbf{I}; \quad \mathbf{F}(\tilde{\mathbf{u}}^k) \rightarrow \mathbf{F}(\mathbf{u}^{k+1})$$

$$\{\mathbf{A}\}_s := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

# Numerical scheme

The scheme

- ▶ provides strong coupling on interface
- ▶ semi-implicit
- ▶ produces one linear system per time step
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# Numerical scheme

## The scheme

- ▶ provides strong coupling on interface
- ▶ semi-implicit
- ▶ produces one linear system per time step
- ▶ second order in time
- ▶ unconditionally stable (no CFL restriction), proved with assumptions:
  - ▶ 1st order in time
  - ▶ St. Venant–Kirchhoff inc./comp. (experiment: Neo-Hookean inc./comp.)
  - ▶ extension of  $\mathbf{u}$  to  $\Omega_f$  guarantees  $J_k > 0$
  - ▶  $\Delta t$  is not large

A.Lofovskiy, M.Olshanskii, V.Salamatova, Yu.Vassilevski. An unconditionally stable semi-implicit FSI finite element method. *Comput.Methods Appl.Mech.Engrg.*, 297, 2015

# Incompressible fluid flow in a moving domain

Navier-Stokes equations in reference domain  $\Omega_f$

Let  $\boldsymbol{\xi}$  mapping  $\Omega_f$  to  $\Omega_f(t)$ ,  $\mathbf{F} = \nabla \boldsymbol{\xi} = \mathbf{I} + \nabla \mathbf{u}$ ,  $J = \det(\mathbf{F})$  be given

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Dynamic equations

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Fluid incompressibility

$$\operatorname{div} (J\mathbf{F}^{-1}\mathbf{v}) = 0 \quad \text{in } \Omega_f \quad \text{or} \quad J\nabla_{\mathbf{v}} : \mathbf{F}^{-T} = 0 \quad \text{in } \Omega_f$$

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Mapping  $\xi$  does not define material trajectories  $\rightarrow$  quasi-Lagrangian formulation

## Finite element scheme

Find  $\{\mathbf{v}_h^k, p_h^k\} \in \mathbb{V}_h \times \mathbb{Q}_h$  satisfying b.c.

("do nothing"  $\boldsymbol{\sigma} \mathbf{F}^{-T} \mathbf{n} = 0$  or no-penetration no-slip  $\mathbf{v}^k = (\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1})/\Delta t$ )

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Find  $\{\mathbf{v}_h^k, p_h^k\} \in \mathbb{V}_h \times \mathbb{Q}_h$  satisfying b.c.

("do nothing"  $\sigma \mathbf{F}^{-T} \mathbf{n} = 0$  or no-penetration no-slip  $\mathbf{v}^k = (\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1})/\Delta t$ )

$$\begin{aligned} & \int_{\Omega_f} J_k \frac{\mathbf{v}_h^k - \mathbf{v}_h^{k-1}}{\Delta t} \cdot \boldsymbol{\psi} \, d\mathbf{x} + \int_{\Omega_f} J_k \nabla \mathbf{v}_h^k \mathbf{F}_k^{-1} \left( \mathbf{v}_h^{k-1} - \frac{\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1}}{\Delta t} \right) \cdot \boldsymbol{\psi} \, d\mathbf{x} - \\ & \int_{\Omega_f} J_k p_h^k \mathbf{F}_k^{-T} : \nabla \boldsymbol{\psi} \, d\mathbf{x} + \int_{\Omega_f} J_k q \mathbf{F}_k^{-T} : \nabla \mathbf{v}_h^k \, d\mathbf{x} + \\ & \int_{\Omega_f} \nu J_k (\nabla \mathbf{v}_h^k \mathbf{F}_k^{-1} \mathbf{F}_k^{-T} + \mathbf{F}_k^{-T} (\nabla \mathbf{v}_h^k)^T \mathbf{F}_k^{-T}) : \nabla \boldsymbol{\psi} \, d\mathbf{x} = 0 \end{aligned}$$

for all  $\boldsymbol{\psi}$  and  $q$  from the appropriate FE spaces



# Finite element scheme

The scheme

- ▶ semi-implicit
- ▶ produces one linear system per time step
- ▶ first order in time (may be generalized to the second order)

# Finite element scheme

## The scheme

- ▶ semi-implicit
- ▶ produces one linear system per time step
- ▶ first order in time (may be generalized to the second order)
- ▶ unconditionally stable (no CFL restriction), proved with assumptions:
  - ▶  $\inf_Q J \geq c_J > 0$ ,  $\sup_Q (\|\mathbf{F}\|_F + \|\mathbf{F}^{-1}\|_F) \leq C_F$
  - ▶ LBB-stable pairs (e.g.  $P_2/P_1$  or  $P_2/P_0$ )
  - ▶  $\Delta t$  is not large

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## Energy equality for the weak solution

Let  $\partial\Omega(t) = \partial\Omega^{ns}(t)$  and  $\xi_t$  be given on  $\partial\Omega^{ns}(t)$ . Then there exists  $\mathbf{v}_1 \in C^1(Q)^d$ ,  $\mathbf{v}_1 = \xi_t$ ,  $\operatorname{div}(J\mathbf{F}^{-1}\mathbf{v}_1) = 0$  [Miyakawa1982]

and we can decompose the solution  $\mathbf{v} = \mathbf{w} + \mathbf{v}_1$ ,  $\mathbf{w} = 0$  on  $\partial\Omega^{ns}$

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**Energy balance for  $\mathbf{w}$ :**

$$\underbrace{\frac{1}{2} \frac{d}{dt} \|J^{\frac{1}{2}} \mathbf{w}\|^2}_{\text{variation of kinetic energy}} + \underbrace{2\nu \|J^{\frac{1}{2}} \mathbf{D}_\xi(\mathbf{w})\|^2}_{\text{energy of viscous dissipation}} + \underbrace{(J(\nabla \mathbf{v}_1 \mathbf{F}^{-1} \mathbf{w}), \mathbf{w})}_{\text{intensification due to b.c.}} = \underbrace{(\tilde{\mathbf{f}}, \mathbf{w})}_{\text{work of ext. forces}}$$

$$\mathbf{D}_\xi(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} \mathbf{F}^{-1} + \mathbf{F}^{-T}(\nabla \mathbf{v})^T)$$

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# Stability estimate for the FE solution

Energy equality for  $\mathbf{w}_h = \mathbf{v}_h - \mathbf{v}_{1,h}$ :

$$\underbrace{\frac{1}{2\Delta t} \left( \|J_k^{\frac{1}{2}} \mathbf{w}_h^k\|^2 - \|J_{k-1}^{\frac{1}{2}} \mathbf{w}_h^{k-1}\|^2 \right)}_{\text{variation of kinetic energy}}
 \quad + \underbrace{2\nu \left\| J_k^{\frac{1}{2}} \mathbf{D}_k(\mathbf{w}_h^k) \right\|^2}_{\text{energy of viscous dissipation}}
 \quad + \underbrace{\frac{(\Delta t)}{2} \left\| J_{k-1}^{\frac{1}{2}} [\mathbf{w}_h]_t^k \right\|^2}_{O(\Delta t) \text{ dissipative term}}$$

$$\underbrace{+(J_k(\nabla \mathbf{v}_1^k \mathbf{F}_k^{-1}) \mathbf{w}_h^k, \mathbf{w}_h^k)}_{\text{intensification due to b.c.}}
 \quad = \quad
 \underbrace{(\tilde{\mathbf{f}}^k, \mathbf{w}_h^k)}_{\text{work of ext. forces}}$$

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# Stability estimate for the FE solution

Stability estimate:

$$C_1 \|\nabla \mathbf{v}_1^k\| \leq \nu/2:$$

$$\frac{1}{2} \|\mathbf{w}_h^n\|_n^2 + \nu \sum_{k=1}^n \Delta t \|\mathbf{D}_k(\mathbf{w}_h^k)\|_k^2 \leq \frac{1}{2} \|\mathbf{w}_0\|_0^2 + C \sum_{k=1}^n \Delta t \|\tilde{\mathbf{f}}^k\|^2$$

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$$\text{if } (1 - 2C_2\Delta t) = \alpha > 0$$

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# Conclusions

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- ▶ The FSI scheme can incorporate diverse elasticity models
- ▶ Only one linear system is solved per time step
- ▶ Benchmark tests and biomedical applications will be presented in the next talk