Model order reduction of parameterized systems

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Outline

- Parameterized control systems
- Model order reduction problem
- Model reduction of parameter independent systems
- Parametric model reduction
  - interpolation in the frequency domain
  - interpolation in the time domain
  - interpolation of the projection subspaces
- Numerical examples
- Conclusion

Parameterized linear control systems

Consider a parameterized linear control system

$$u \longrightarrow \boxed{\begin{aligned} E(p)\,\dot{x}(t,p) &= A(p)\,x(t,p) + B(p)\,u(t) \\ y(t,p) &= C(p)\,x(t,p) \end{aligned}} \longrightarrow y$$

where $u : \mathbb{R} \to \mathbb{R}^m$ – input (excitations),

$x : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^n$ – state (internal variables),

$y : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^q$ – output (measurements, observations),

$E(p), A(p) \in \mathbb{R}^{n\times n}$, $B(p) \in \mathbb{R}^{n\times m}$, $C(p) \in \mathbb{R}^{q\times n}$,

$p \in \mathbb{P} \subset \mathbb{R}^d$ – parameter vector.

Model reduction in time domain

Given a large-scale system

u

$$\begin{aligned} E(p)\,\dot{x}(t,p) &= A(p)\,x(t,p) + B(p)\,u(t) \\ y(t,p) &= C(p)\,x(t,p) \end{aligned}$$

y

with $E(p), A(p) \in \mathbb{R}^{n\times n}$, $B(p) \in \mathbb{R}^{n\times m}$, $C(p) \in \mathbb{R}^{q\times n}$,

find a reduced-order model

u

$$\begin{aligned} \widetilde{E}(p)\,\dot{\widetilde{x}}(t,p) &= \widetilde{A}(p)\,\widetilde{x}(t,p) + \widetilde{B}(p)\,u(t) \\ \widetilde{y}(t,p) &= \widetilde{C}(p)\,\widetilde{x}(t,p) \end{aligned}$$

y

with $\widetilde{E}(p), \widetilde{A}(p) \in \mathbb{R}^{r\times r}$, $\widetilde{B}(p) \in \mathbb{R}^{r\times m}$, $\widetilde{C}(p) \in \mathbb{R}^{q\times r}$ and $r \ll n$.

Model reduction in frequency domain

Laplace transform: $u(t) \mapsto \boldsymbol{u}(s)$, $x(t,p) \mapsto \boldsymbol{x}(s,p)$, $y(t,p) \mapsto \boldsymbol{y}(s,p)$

$\hookrightarrow$ $\boldsymbol{y}(s,p) = \boldsymbol{G}(s,p)\boldsymbol{u}(s) + C(p)(sE(p) - A(p))^{-1}E(p)x(0)$

with the transfer function $\boldsymbol{G}(s,p) = C(p)\big(sE(p) - A(p)\big)^{-1}B(p)$

Given $\boldsymbol{G}(s,p) = C(p)\big(sE(p) - A(p)\big)^{-1}B(p)$ with $E(p), A(p) \in \mathbb{R}^{n\times n}$,

find $\widetilde{\boldsymbol{G}}(s,p) = \widetilde{C}(p)\big(s\widetilde{E}(p) - \widetilde{A}(p)\big)^{-1}\widetilde{B}(p)$ with $\widetilde{E}(p), \widetilde{A}(p) \in \mathbb{R}^{r\times r}$

such that $\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|$ is small.

$\mathbb{H}_\infty \otimes \mathbb{L}_\infty(\mathbb{P})$-norm: $\|\boldsymbol{G}\|_{\mathbb{H}_\infty\otimes\mathbb{L}_\infty(\mathbb{P})} = \sup_{p\in\mathbb{P}} \sup_{\omega\in\mathbb{R}} \|\boldsymbol{G}(i\omega,p)\|_2$

$\hookrightarrow$ $\|\widetilde{y} - y\|_{\mathbb{L}_2\otimes\mathbb{L}_\infty(\mathbb{P})} \le \|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty\otimes\mathbb{L}_\infty(\mathbb{P})}\, \|u\|_{\mathbb{L}_2}$

Model reduction: goals

- Preserve the parametric dependence
- Preserve system properties
  - stability ($\lambda_j(E(p), A(p)) \in \mathbb{C}^-$)
  - passivity (= system does not generate energy)
  - contractivity ($\|y(\cdot, p)\|_{\mathbb{L}_2} \leq \|u\|_{\mathbb{L}_2}$)

  . . .
- Satisfy desired error tolerance

  $\|\tilde{\boldsymbol{G}} - \boldsymbol{G}\| \leq tol$ or $\|\tilde{y} - y\| \leq tol \cdot \|u\|$ for all $u \in \mathcal{U}$

  $\hookrightarrow$ need for computable error bounds
- Automatic generation of reduced-order models
- Use numerically stable and efficient methods

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  - Balanced truncation
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Balanced truncation

- Ляпунов equations: ($\lambda_j(E, A) \in \mathbb{C}^-$, E is nonsingular)

$$AXE^T + EXA^T = -BB^T \quad \rightsquigarrow \quad X - \text{controllability Gramian}$$

$$A^TYE + E^TYA = -C^TC \quad \rightsquigarrow \quad Y - \text{observability Gramian}$$

- Balance the dynamical system

$$(\hat{E},\ \hat{A},\ \hat{B},\ \hat{C}\,) = (\,W_b^TEV_b,\ W_b^TAV_b,\ W_b^TB,\ CV_b\,)$$

$$= \left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, [\,C_1, C_2\,] \right)$$

$$\hookrightarrow \quad V_b^{-1}XV_b^{-T} = W_b^{-1}YW_b^{-T} = \mathrm{diag}(\,\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_n\,)$$

- Truncate the states corresponding to small σ_j

$$\hookrightarrow \quad (\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C}\,) = (\,E_{11},\ A_{11},\ B_1,\ C_1\,)$$

Balanced truncation algorithm

1. Solve the Lyapunov equations for $X = RR^T$ and $Y = LL^T$;
2. Compute the SVD $L^TER = [\,U_1,\ U_2\,]\begin{bmatrix}\Sigma_1 & \\ & \Sigma_2\end{bmatrix}[\,V_1,\ V_2\,]^T$

   with $\Sigma_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$, $\Sigma_2 = \mathrm{diag}(\sigma_{r+1}, \ldots, \sigma_n)$;
3. Compute $(\,\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C}\,) = (\,W^TEV,\ W^TAV,\ W^TB,\ CV\,)$

   with $W = LU_1\Sigma_1^{-1/2} \in \mathbb{R}^{n\times r}$, $V = RV_1\Sigma_1^{-1/2} \in \mathbb{R}^{n\times r}$.

Properties

- $(\,\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C}\,)$ is asymptotically stable and balanced
- error bound $\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} \leq 2(\sigma_{r+1} + \ldots + \sigma_n)$
- need to solve large-scale Lyapunov equations
  (ADI method, (rational) Krylov method, Riemannian method, ...)
- extension to DAE systems with singular E is possible [St'04]
- other balancing-related techniques (e.g., positive real BT)

Interpolatory model reduction

Goal: Given $\mathbf{G}(s)=C(sE-A)^{-1}B$, interpolation points $\{\mu_j\}_{j=1}^{r}$, $\mu_j\in\mathbb{C}$, tangential directions $\{b_j\}_{j=1}^{r}$, $b_j\in\mathbb{C}^m$ and $\{c_j\}_{j=1}^{r}$, $c_j\in\mathbb{C}^q$,

find $\widetilde{\mathbf{G}}(s)=\widetilde{C}(s\widetilde{E}-\widetilde{A})^{-1}\widetilde{B}$ such that

$$c_j^T\mathbf{G}(\mu_j)=c_j^T\widetilde{\mathbf{G}}(\mu_j),\quad \mathbf{G}(\mu_j)b_j=\widetilde{\mathbf{G}}(\mu_j)b_j,\quad c_j^T\mathbf{G}'(\mu_j)b_j=c_j^T\widetilde{\mathbf{G}}'(\mu_j)b_j.$$

- For $V=\big[\,(\mu_1E-A)^{-1}B\,b_1,\;\ldots,\;(\mu_rE-A)^{-1}B\,b_r\,\big]$,

  $W=\big[\,(\mu_1E-A)^{-T}C^Tc_1,\;\ldots,\;(\mu_rE-A)^{-T}C^Tc_r\,\big]$,

  compute $\widetilde{E}=W^TEV$, $\widetilde{A}=W^TAV$, $\widetilde{B}=W^TB$, $\widetilde{C}=CV$

- Optimal choice of $\{\mu_j\}_{j=1}^{r}$, $\{b_j\}_{j=1}^{r}$ and $\{c_j\}_{j=1}^{r}$

  $\hookrightarrow$ $\mathbb{H}_2$-optimal approximation $\widetilde{\boldsymbol{G}}(s)=\underset{\dim(\hat{\boldsymbol{G}})=r}{\arg\min}\,\|\boldsymbol{G}-\hat{\boldsymbol{G}}\|_{\mathbb{H}_2}$

  $\hookrightarrow$ iterative algorithms IRKA and MIRIAm

  [Gugercin et al.'06, Bunse-Gerstner et al.'07, Van Dooren et al.'08]

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Parametric model reduction

- Multivariate moment matching approximation for $\boldsymbol{G}(s,p)$
  [Daniel et al'04, Feng et al.'05, Farle et al.'08]
- Interpolatory based reduction and $\mathbb{H}_2$-optimal approximation
  [Baur/Beattie/Benner/Gugercin'11]
- Model reduction of the local systems $\boldsymbol{G}(s,p_j)$ and interpolation
  [Amsallem/Farhat'08, Baur/Benner'09, Panzer et al.'10, Son/St.'15]
- Data-driven approach based on Loewner matrices
  [Ionita/Antoulas'14]
- Reduced basis method [Patera/Rozza'07, Haasdonk/Ohlberger'11]
- Reduced basis combined with balanced truncation [Son/St.'17]

Interpolation based model reduction

- Model reduction of the *local* systems: for $p_1, \ldots, p_k \in \mathbb{P}$

$$\begin{aligned} E(p_j)\,\dot{x}_j &= A(p_j)\,x_j + B(p_j)\,u \\ y_j &= C(p_j)\,x_j \end{aligned} \quad \overset{\text{MOR}}{\Longrightarrow} \quad \begin{aligned} \widetilde{E}_j\,\dot{\widetilde{x}}_j &= \widetilde{A}_j\,\widetilde{x}_j + \widetilde{B}_j\,u \\ \widetilde{y}_j &= \widetilde{C}_j\,\widetilde{x}_j \end{aligned}$$

  - System matrices: $\widetilde{E}_j = W_j^T E(p_j) V_j, \quad \widetilde{A}_j = W_j^T A(p_j) V_j$
    $\widetilde{B}_j = W_j^T B(p_j), \quad \widetilde{C}_j = C(p_j) V_j$
  - Projection subspaces: $\mathcal{W}_j = \operatorname{im} W_j, \quad \mathcal{V}_j = \operatorname{im} V_j$
  - Transfer functions: $\widetilde{\mathbf{G}}_j(s) = \widetilde{C}_j(s\widetilde{E}_j - \widetilde{A}_j)^{-1}\widetilde{B}_j$

- Interpolation options:
  - interpolation in the time domain
  - interpolation on Grassmann manifolds
  - interpolation in the frequency domain

Multivariate interpolation techniques

Goal: given (p_j, M_j) with $p_j \in \mathbb{P}$ and $M_j \in \mathbb{R}^{n\times m}$, $j = 1, \ldots, k$,

find $M(p) \in \mathbb{R}^{n\times m}$ such that $M(p_j) = M_j$ for $j = 1, \ldots, k$.

- Lagrange interpolation

$$M(p) = \sum_{j=1}^{k} f_j(p) M_j, \quad f_j(p) \text{ - Lagrange basis functions}$$

- Linear spline interpolation

$$M(p) = \sum_{j=1}^{k} f_j(p) M_j, \quad f_j(p) \text{ - linear splines}$$

- Cubic spline interpolation

- Inverse distance weighting

$$M(p) = \sum_{j=1}^{k} f_j(p) M_j, \quad f_j(p) = \|p - p_j\|_2^{-2} / \sum_{i=1}^{k} \|p - p_j\|_2^{-2}$$

- . . .

Interpolation in the frequency domain

$$\widetilde{\boldsymbol{G}}(s,p) = \sum_{j=1}^{k} f_j(p)\tilde{C}_j(s\tilde{E}_j - \tilde{A}_j)^{-1}\tilde{B}_j = \tilde{C}(p)\big(s\tilde{E} - \tilde{A}\big)^{-1}\tilde{B}$$

with $\tilde{E} = \mathrm{diag}\big(\tilde{E}_1, \ldots, \tilde{E}_k\big), \quad \tilde{A} = \mathrm{diag}\big(\tilde{A}_1, \ldots, \tilde{A}_k\big),$

$$\tilde{B} = [\,\tilde{B}_1^T, \ldots, \tilde{B}_k^T\,]^T, \quad \tilde{C}(p) = [\,f_1(p)\tilde{C}_1, \ldots, f_k(p)\tilde{C}_k\,]$$

Properties: [Baur/Benner'09, Ferranti et al.'10, Son'12]

Interpolation	Stability/ passivity	Dimension	Error bound
Lagrange	yes / no	kr	yes
linear splines	yes / yes	cr ($c = d+1$ or $c = 2^d$)	yes
cubic splines	yes / no	kr	yes ($d = 1$, SISO)
distance weights	yes / yes	kr	yes

Interpolation in the frequency domain

Error bound

$$\|\boldsymbol{G}-\widetilde{\boldsymbol{G}}\|_{\mathbb{H}_\infty\otimes\mathbb{L}_\infty(\mathbb{P})} \le \|\mathcal{R}_k\|_{\mathbb{H}_\infty\otimes\mathbb{L}_\infty(\mathbb{P})} + \left(\sup_{p\in\mathbb{P}} \sum_{j=1}^{k} |f_j(p)|\right) \max_{1\le j\le k} \|\boldsymbol{G}_j-\widetilde{\boldsymbol{G}}_j\|_{\mathbb{H}_\infty}$$

with $\mathcal{R}_k(s,p) = G(s,p) - \sum_{j=1}^{k} f_j(p) G_j(s)$

- Balanced truncation + linear spline interpolation:

  $$\|\boldsymbol{G}-\widetilde{\boldsymbol{G}}\|_{\mathbb{H}_\infty\otimes\mathbb{L}_\infty(\mathbb{P})} \le Lh + 2 \max_{1\le j\le k} \left(\sigma_{j,r_j+1} + \ldots + \sigma_{j,n}\right)$$

  provided $\|\boldsymbol{G}(\cdot,p_1) - \boldsymbol{G}(\cdot,p_2)\| \le L\|p_1-p_2\|$ for all $p_1, p_2 \in \mathbb{P}$

[Son/Bunse-Gerstner'11]

Interpolation in the time domain

- Projection: $\widetilde{E}_j = W_j^T E(p_j) V_j$, $\widetilde{A}_j = W_j^T A(p_j) V_j$, $\widetilde{B}_j = W_j^T B$, $\widetilde{C}_j = B^T V_j$
- Interpolation: $$\widetilde{E}(p) = \sum_{j=1}^{k} f_j(p)\, \widetilde{E}_j, \qquad \widetilde{A}(p) = \sum_{j=1}^{k} f_j(p)\, \widetilde{A}_j,$$ $$\widetilde{B}(p) = \sum_{j=1}^{k} f_j(p)\, \widetilde{B}_j, \qquad \widetilde{C}(p) = \sum_{j=1}^{k} f_j(p)\, \widetilde{C}_j$$

Note: Reduced local systems may be incompatible!

Goal: find state space transformations to make the reduced local systems compatible

$$\hookrightarrow \quad \widetilde{E}(p) = \sum_{j=1}^{k} f_j(p)\, S_j^T \widetilde{E}_j T_j, \qquad \widetilde{A}(p) = \sum_{j=1}^{k} f_j(p)\, S_j^T \widetilde{A}_j T_j,$$
$$\widetilde{B}(p) = \sum_{j=1}^{k} f_j(p)\, S_j^T \widetilde{B}_j, \qquad \widetilde{C}(p) = \sum_{j=1}^{k} f_j(p)\, \widetilde{C}_j T_j$$

Interpolation in the time domain

- Find common projection subspaces [Panzer/Mohring/Eid/Lohmann'10]

  $\hookrightarrow \quad S_j = (U_W^T W_j)^{-1}, \qquad T_j = (U_V^T V_j)^{-1},$

  where $[\,V_1,\ \ldots,\ V_k\,] = [U_V,\ U_V'\,]\,\text{diag}(\Sigma_V,\ \Sigma_V')\,[Z_V,\ Z_V'\,]^T,$

  $[W_1, \ldots, W_k] = [U_W,\ U_W']\text{diag}(\Sigma_W, \Sigma_W')[Z_W,\ Z_W']^T$

- Minimize the distance between the projection matrices [Amsallem/Farhat'11]
  - choose the reference projection matrices, e.g., W_1 and V_1
  - solve $S_j = \underset{S\in\mathbb{O}(r)}{\text{argmin}}\|W_j\,S - W_1\|_F^2, \quad T_j = \underset{T\in\mathbb{O}(r)}{\text{argmin}}\|V_j\,T - V_1\|_F^2$

    by computing the SVDs $W_j^T W_1 = U_{W_j}\Sigma_{W_j}Z_{W_j}^T,\ V_j^T V_1 = U_{V_j}\Sigma_{V_j}Z_{V_j}^T$

    $\hookrightarrow \quad S_j = U_{W_j}Z_{W_j}^T, \qquad T_j = U_{V_j}Z_{V_j}^T, \quad j = 2, \ldots, k$

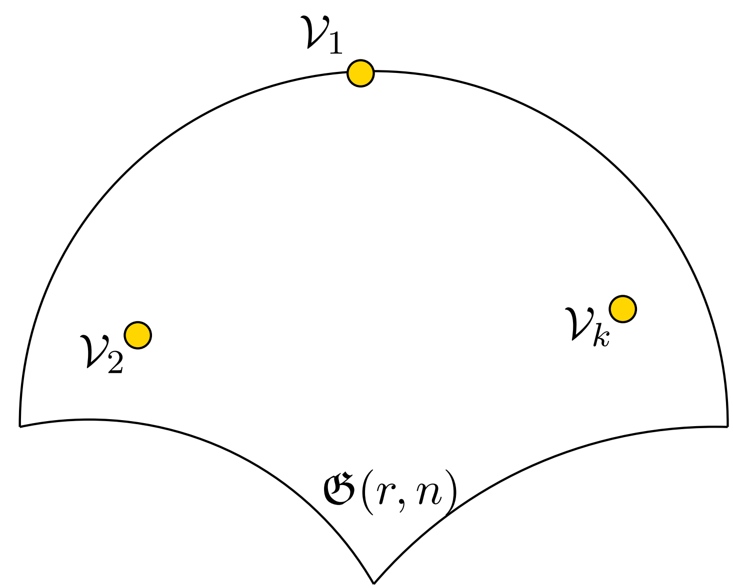
Interpolation in the time domain

Properties

- Stability and passivity are preserved if
  - stability: $E(p) = E^T(p) > 0$, $A(p) + A^T(p) < 0$
    passivity: $E(p) = E^T(p) \geq 0$, $A(p) + A^T(p) \leq 0$, $C(p) = B^T(p)$
  - one-sided projection, i.e., $W_j = V_j$ and $S_j = T_j$
  - the interpolation coefficients $f_j(p)$ are positive
- Reduced local systems should have the same dimension
- No error bounds

Interpolation of the projection subspaces

Let $\mathcal{V}_j = \operatorname{im} V_j \in \mathfrak{G}(r,n)$, where $\mathfrak{G}(r,n)$ is the Grassmann manifold.

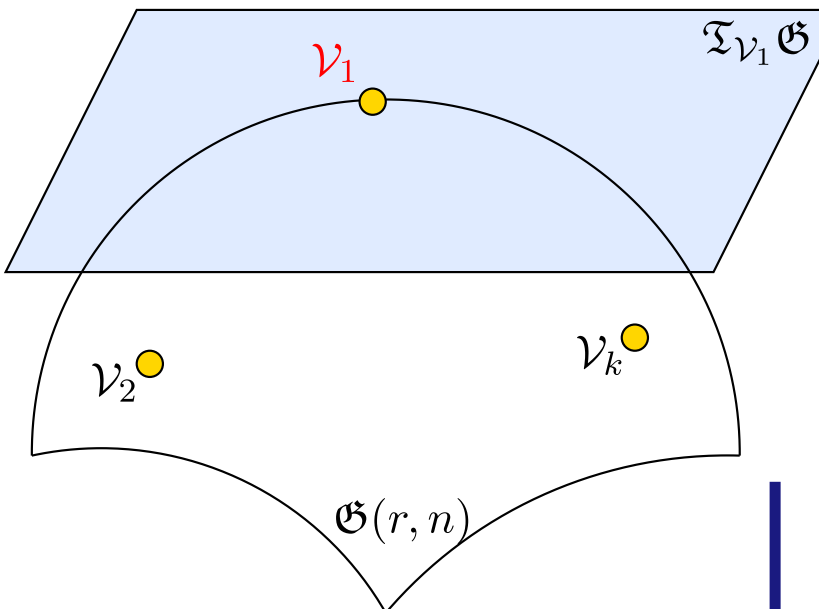


Interpolation of the projection subspaces

Let $\mathcal{V}_j = \operatorname{im} V_j \in \mathfrak{G}(r,n)$, where $\mathfrak{G}(r,n)$ is the Grassmann manifold.

- Choose a reference subspace, e.g., $\mathcal{V}_1 \in \mathfrak{G}(r,n)$ [Amsallem/Farhat'08]

  $\hookrightarrow$ $\mathfrak{T}_{\mathcal{V}_1}\mathfrak{G}(r,n)$ is a tangent space to $\mathfrak{G}(r,n)$ at $\mathcal{V}_1$



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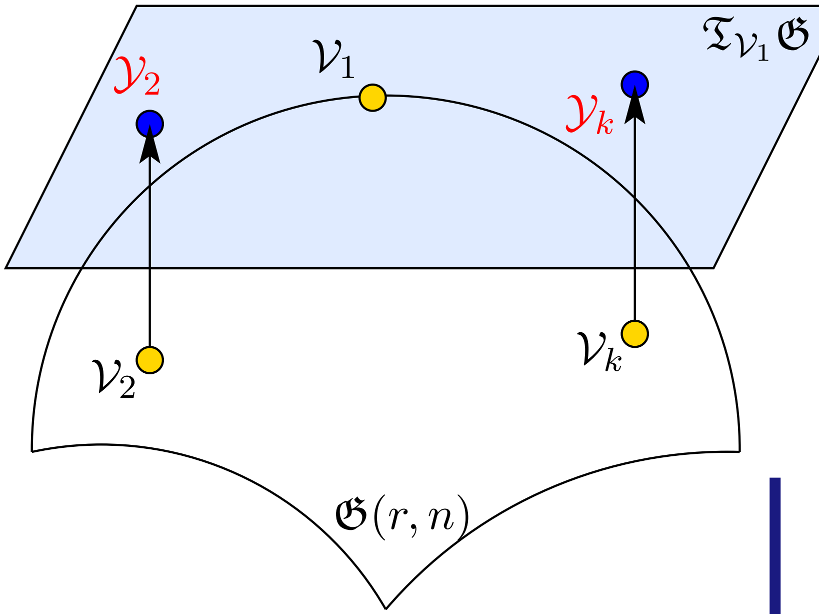
  $\hookrightarrow$ $\mathfrak{T}_{\mathcal{V}_1}\mathfrak{G}(r,n)$ is a tangent space to $\mathfrak{G}(r,n)$ at $\mathcal{V}_1$

- Map $\mathcal{V}_j$ to $\mathfrak{T}_{\mathcal{V}_1}\mathfrak{G}(r,n)$ by $\operatorname{Log}_{\mathcal{V}_1}$

  $\hookrightarrow$ $\operatorname{Log}_{\mathcal{V}_1}(\mathcal{V}_j) = \mathcal{Y}_j = \operatorname{im} Y_j$, $Y_j \in \mathbb{R}^{n\times r}$



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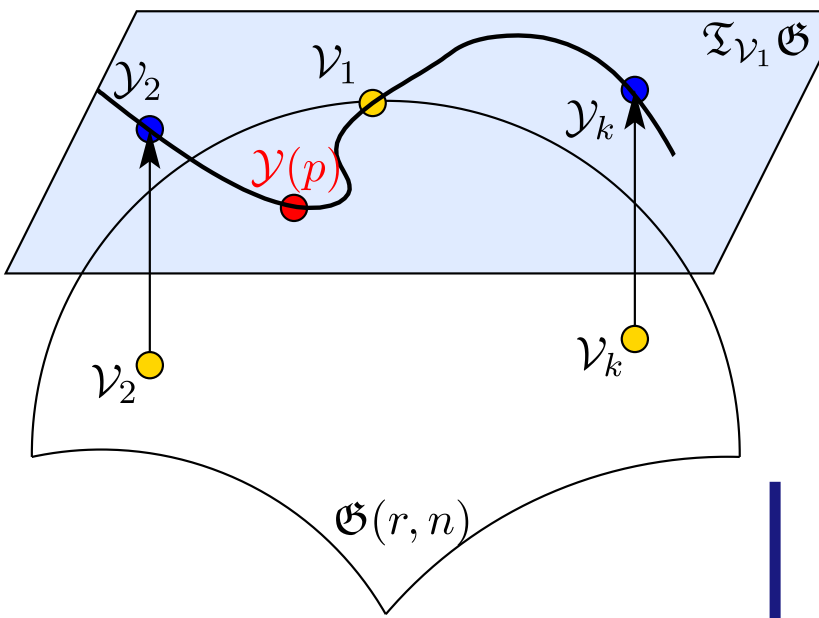
- Map $\mathcal{V}_j$ to $\mathfrak{T}_{\mathcal{V}_1}\mathfrak{G}(r,n)$ by $\mathrm{Log}_{\mathcal{V}_1}$

  $\hookrightarrow$ $\mathrm{Log}_{\mathcal{V}_1}(\mathcal{V}_j) = \mathcal{Y}_j = \operatorname{im} Y_j, \quad Y_j \in \mathbb{R}^{n\times r}$

- Interpolate on $\mathfrak{T}_{\mathcal{V}_1}\mathfrak{G}(r,n)$

  $\hookrightarrow$ $Y(p) = \sum_{j=1}^{k} f_j(p)\, Y_j,$

  $\mathcal{Y}(p) = \operatorname{im} Y(p) \in \mathfrak{T}_{\mathcal{V}_1}\mathfrak{G}(r,n)$



Interpolation of the projection subspaces

Let $\mathcal{V}_j = \operatorname{im} V_j \in \mathfrak{G}(r,n)$, where $\mathfrak{G}(r,n)$ is the Grassmann manifold.

- Choose a reference subspace, e.g., $\mathcal{V}_1 \in \mathfrak{G}(r,n)$ [Amsallem/Farhat'08]

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- Map $\mathcal{V}_j$ to $\mathfrak{T}_{\mathcal{V}_1}\mathfrak{G}(r,n)$ by $\operatorname{Log}_{\mathcal{V}_1}$

  $\hookrightarrow$ $\operatorname{Log}_{\mathcal{V}_1}(\mathcal{V}_j) = \mathcal{Y}_j = \operatorname{im} Y_j, \quad Y_j \in \mathbb{R}^{n\times r}$

- Interpolate on $\mathfrak{T}_{\mathcal{V}_1}\mathfrak{G}(r,n)$

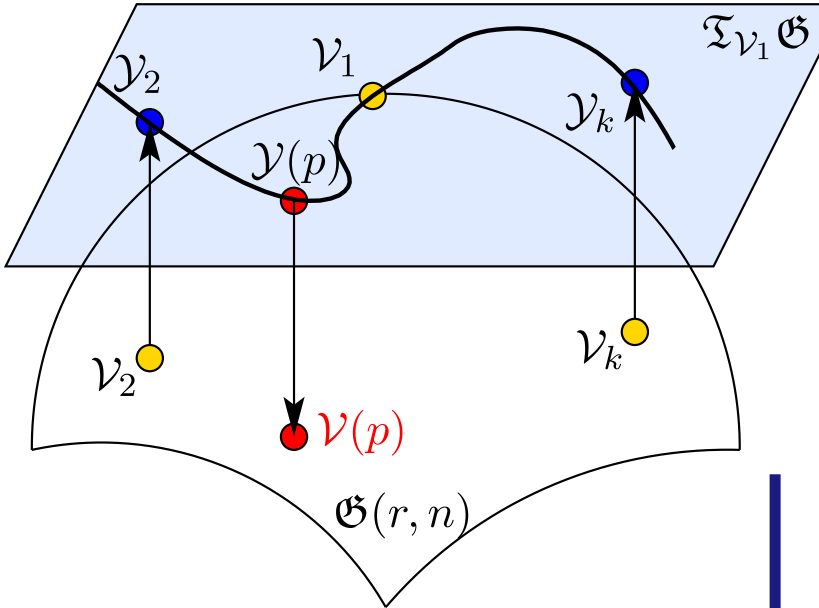
  $\hookrightarrow$ $Y(p) = \sum_{j=1}^{k} f_j(p)\, Y_j,$

  $\mathcal{Y}(p) = \operatorname{im} Y(p) \in \mathfrak{T}_{\mathcal{V}_1}\mathfrak{G}(r,n)$

- Map $\mathcal{Y}(p)$ back to $\mathfrak{G}(r,n)$ by $\operatorname{Exp}_{\mathcal{V}_1}$

  $\hookrightarrow$ $\operatorname{Exp}_{\mathcal{V}_1}(\mathcal{Y}(p)) = \mathcal{V}(p) = \operatorname{im} V(p)$,

  $V(p) \in \mathbb{R}^{n,r}$



Interpolation of the projection subspaces

Let $\mathcal{V}_j = \operatorname{im} V_j \in \mathfrak{G}(r,n)$, where $\mathfrak{G}(r,n)$ is the Grassmann manifold.

- Choose a reference subspace, e.g., $\mathcal{V}_1 \in \mathfrak{G}(r,n)$ [Amsallem/Farhat'08]
- Map $\mathcal{V}_j$ to $\mathfrak{T}_{\mathcal{V}_1}\mathfrak{G}(r,n)$ by $\operatorname{Log}_{\mathcal{V}_1}(\mathcal{V}_j) = \mathcal{Y}_j = \operatorname{im} Y_j$

  $\hookrightarrow$ $Y_j = U_{V_j} \arctan(\Sigma_{V_j}) Z_{V_j}^T$, where

  $$(I - V_1(V_1^T V_1)^{-1} V_1^T) V_j (V_1^T V_j)^{-1} (V_1^T V_1)^{1/2} = U_{V_j} \Sigma_{V_j} Z_{V_j}^T$$

- Interpolate on $\mathfrak{T}_{\mathcal{V}_1}\mathfrak{G}(r,n)$: $Y(p) = \sum_{j=1}^{k} f_j(p)\, Y_j$, $\mathcal{Y}(p) = \operatorname{im} Y(p)$
- Map $\mathcal{Y}(p)$ back to $\mathfrak{G}(r,n)$ by $\operatorname{Exp}_{\mathcal{V}_1}(\mathcal{Y}(p)) = \mathcal{V}(p) = \operatorname{im} V(p)$

  $\hookrightarrow$ compute the SVD $Y(p) = U(p)\Sigma(p)Z^T(p)$

  $\hookrightarrow$ $V(p) = V_1 Z(p) \cos(\Sigma(p)) + U(p) \sin(\Sigma(p))$, $W(p)$ analogously

Reduced model: $\widetilde{E}(p) = W^T(p)E(p)V(p)$, $\widetilde{A}(p) = W^T(p)A(p)V(p)$,

$$\widetilde{B}(p) = W^T(p)B(p), \quad \widetilde{C}(p) = C(p)V(p)$$

Interpolation of the projection subspaces

Properties

- Stability is preserved in the special cases
  (e.g., if $E(p) = E^T(p) > 0$, $A(p) + A^T(p) < 0$ and $W(p) = V(p)$)
- Passivity is preserved in the special cases
  (e.g., if $E(p) = E^T(p) \geq 0$, $A(p) + A^T(p) \leq 0$, $C(p) = B^T(p)$
  and $W(p) = V(p)$)
- No restriction on the interpolation weights $f_j(p)$
- Reduced local systems should have the same dimension
- Offline-online decomposition is possible for

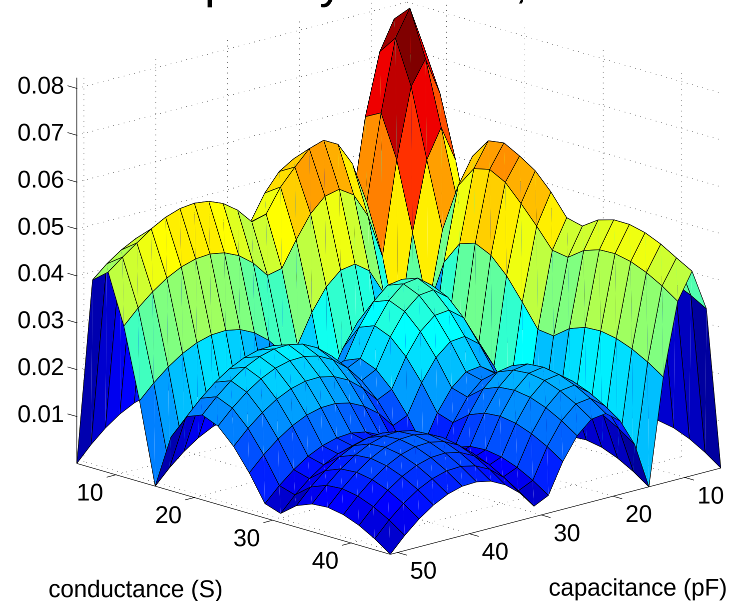
  $$E(p) = \sum_{i=1}^{n_E} \theta_i^E(p)\hat{E}_i \quad \text{and} \quad A(p) = \sum_{i=1}^{n_A} \theta_i^A(p)\hat{A}_i$$

- No error bounds

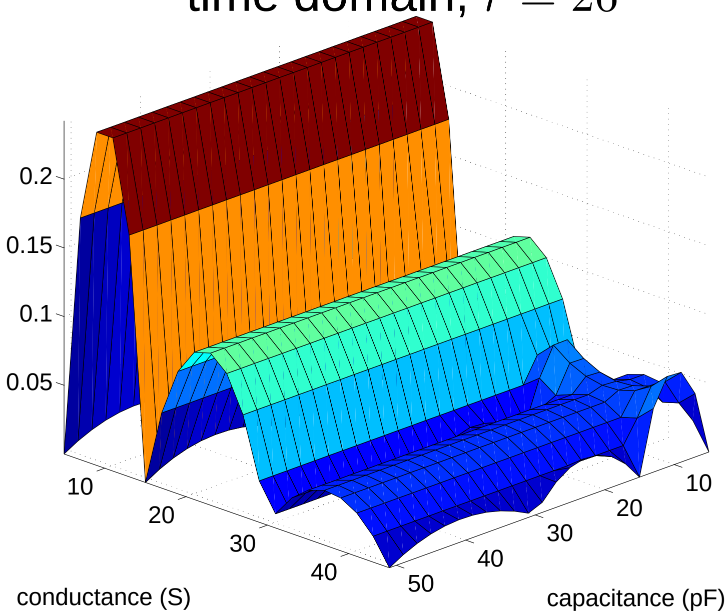
Example: RC circuit

- $E(p) = p^{(1)}E, \;\; C = B^T,$
  $A(p) = \hat{A}_1 + p^{(2)}\hat{A}_2$
- $n = 5000, \;\; m = 1$
- $\mathbb{P} = [10^{-12}, 5 \cdot 10^{-11}] \times [1, 45]$

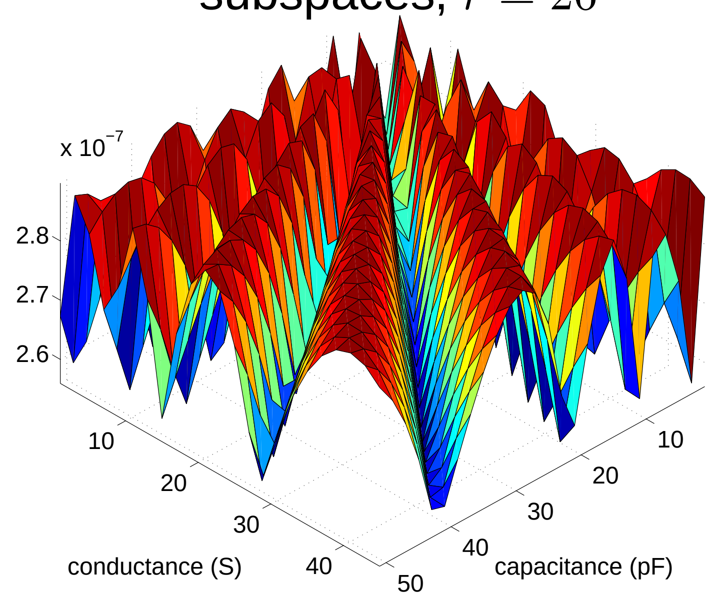
frequency domain, $r = 104$



time domain, $r = 26$

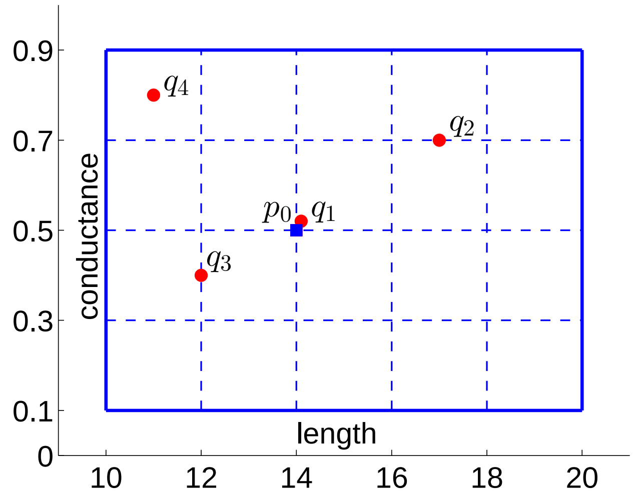


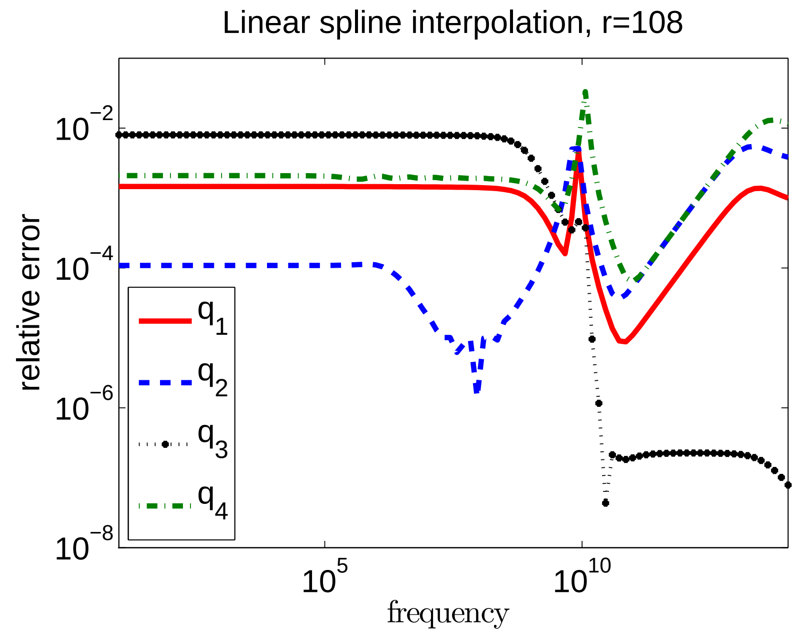
subspaces, $r = 26$

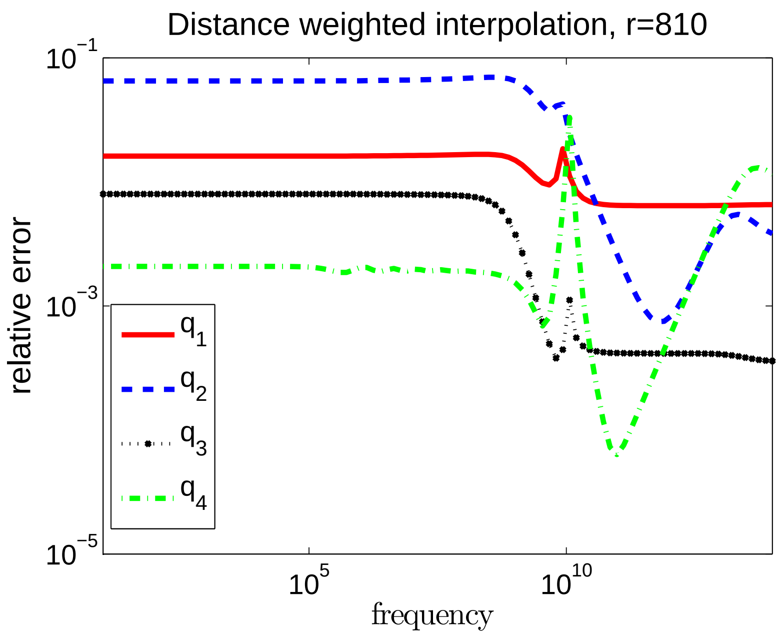


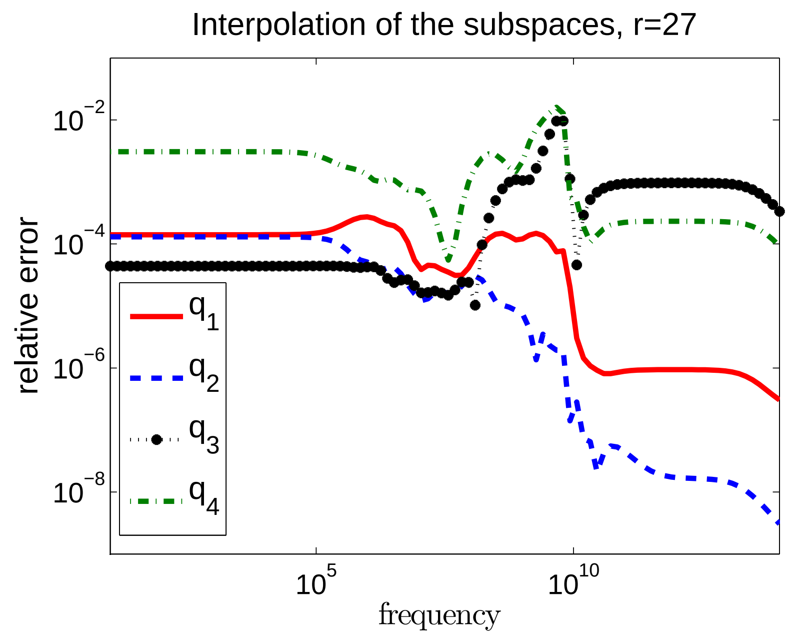
Example: transmission line

Dimensions: $n = 12000$, $m = 1$

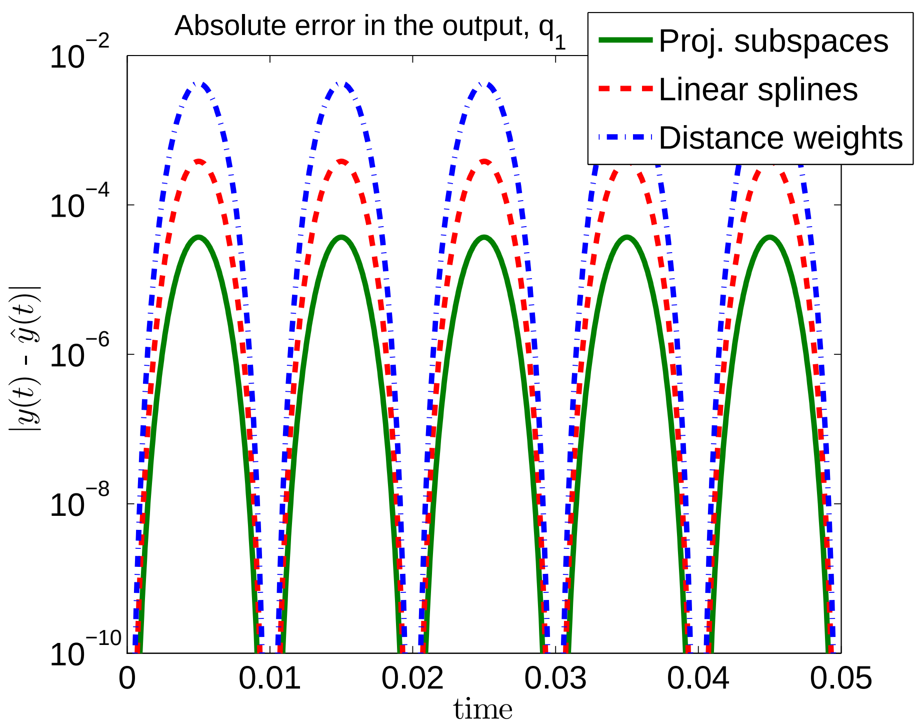


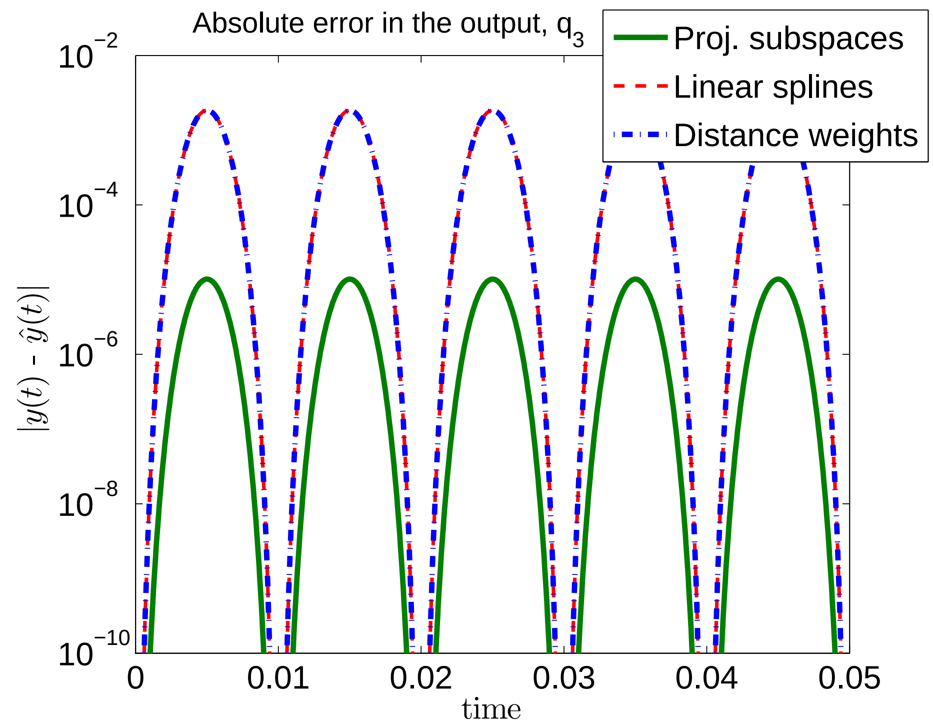






Example: transmission line





Conclusions

- Interpolation based model reduction of parameterized systems (reduction of the local systems + interpolation)

Interpolation	Stability	Dimension	Error bound
frequency domain	yes	$c\,r$	yes
time domain	(yes)	r	no
projection subspaces	(yes)	r	no

- High-dimensional parameter space - ?

- Optimal parameter selection strategies - ?

- Model reduction of systems with time-dependent parameters - ?