Wave Attenuation Along a Rough Floating Elastic Beam

Sebastian Rupprecht [1]

Malte A. Peter [1] Luke G. Bennetts [2]

[1]Institute of Mathematics, University of Augsburg, Germany

[2]School of Mathematics, University of Adelaide, Australia

Marginal Ice Zone

Source: www.seaice.acecrc.org.au

Motivation

- Interested in attenuation of ocean waves (linear water waves) by many ice floes (floating elastic plates)
- Significant effects of irregularities in beam properties
- Consider a rough floating elastic beam as a model problem
- Exponential attenuation of wave energy expected
- Goal: Extraction of an attenuation coefficient Q

$$Energy \approx \mathrm{e}^{-Qx}$$

- Roughness modelled assuming knowledge of average properties
- Semi-analytical approach for effective wave field

Thin-elastic beam floating on water model

z

x

$$\left[\partial_x^2(b(x)\partial_x^2) - \alpha g(x) + 1\right]\partial_z\phi(x,z) - \alpha\phi(x,z) = 0 \qquad z = 0$$

Wave

$$\Delta\phi = 0$$

$$z = -H$$

$$\partial_z\phi = 0$$

Beam floating on water (Assumptions of Linear Theory)

Linear motions in fluid and beam:

Water

$$\left.\begin{array}{l}\bullet\ \text{Incompressible}\\ \bullet\ \text{Irrotational}\\ \bullet\ \text{Inviscid}\end{array}\right\}\ \text{Potential Theory}$$

- Linear, time-harmonic water waves with velocity potential

$$\Phi(x,z,t)=\Re\left(\phi(x,z)\mathrm{e}^{-\mathrm{i}\omega t}\right)$$

Beam floating on water (Assumptions of Linear Theory)

Linear motions in fluid and beam:

Water

- Incompressible
- Irrotational
- Inviscid

$\Big\}$ Potential Theory

- Linear, time-harmonic water waves with velocity potential

$$\Phi(x, z, t) = \Re\left(\phi(x, z)\mathrm{e}^{-\mathrm{i}\omega t}\right)$$

Beam

- Thin, elastic beam
- No submergence
- Linear deformations

$\Big\}$ Beam Equation

- No horizontal motion

Thin-elastic beam floating on water model

$$\left[\partial_x^2(b(x)\partial_x^2) - \alpha g(x) + 1\right] \partial_z \phi(x,z) - \alpha\phi(x,z) = 0$$

- Frequency parameter $\alpha = \omega^2/g_\mathrm{r}$
- unit-amplitude wave incident from $x \to -\infty$
- varying beam rigidity $b(x)$
- varying beam mass $g(x)$

Thin-elastic beam floating on water model

$$\left[\partial_x^2(b\,\partial_x^2) - \alpha g(x) + 1\right] \partial_z \phi(x,z) - \alpha\phi(x,z) = 0$$

- Frequency parameter $\alpha = \omega^2/g_{\mathrm{r}}$
- unit-amplitude wave incident from $x \to -\infty$
- constant beam rigidity b
- varying beam mass $g(x)$

$\left.\begin{array}{l} \text{constant beam rigidity } b \\ \text{varying beam mass } g(x) \end{array}\right\}$ varying mass problem

Thin-elastic beam floating on water model

$$\left[\partial_x^2(b(x)\partial_x^2) - \alpha g + 1\right]\partial_z\phi(x,z) - \alpha\phi(x,z) = 0$$

- Frequency parameter $\alpha = \omega^2/g_{\mathrm{r}}$
- unit-amplitude wave incident from $x \to -\infty$

$$\left.\begin{array}{l}\bullet\ \text{varying beam rigidity } b(x)\\ \bullet\ \text{constant beam mass } g\end{array}\right\}\ \text{varying rigidity problem}$$

Thin-elastic beam floating on water model

$$\left[\partial_x^2(b(x)\partial_x^2) - \alpha g + 1\right]\partial_z\phi(x,z) - \alpha\phi(x,z) = 0$$

- Frequency parameter $\alpha = \omega^2/g_{\mathrm{r}}$
- unit-amplitude wave incident from $x \to -\infty$
- varying beam rigidity $b(x)$
- constant beam mass g

$\left.\begin{array}{l}\text{varying beam rigidity } b(x)\\ \text{constant beam mass } g\end{array}\right\}$ varying rigidity problem

- Retrieving beam displacement via

$$\left.\frac{\partial\phi(x,z)}{\partial z}\right|_{z=0} = -\mathrm{i}\omega\eta(x)$$

Thin-elastic beam floating on water model

$$\left[\partial_x^2({\color{red}b} \quad \partial_x^2) - \alpha g(x) + 1\right] \partial_z \phi(x, z) - \alpha \phi(x, z) = 0$$

- randomness incorporated via varying mass

$$g(x) = \bar{g} + \epsilon\,\gamma(x) \quad \text{with } \gamma(x) \sim \mathcal{O}(1) \text{ and } \langle \gamma \rangle = 0$$

Thin-elastic beam floating on water model

$$\left[\partial_x^2(b(x)\,\partial_x^2) - \alpha g \quad + 1\right]\partial_z\phi(x,z) - \alpha\phi(x,z) = 0$$

- randomness incorporated via varying rigidity

$$b(x) = \bar{b} + \epsilon\,\beta(x) \quad \text{with } \beta(x) \sim \mathcal{O}(1) \text{ and } \langle\beta\rangle = 0$$

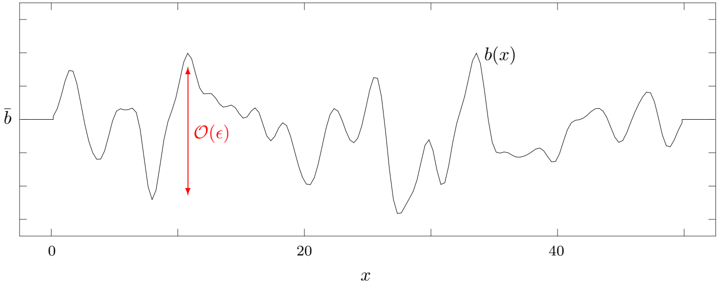
Thin-elastic beam floating on water model

$$\left[\partial_x^2(b(x)\,\partial_x^2) - \alpha g \quad + 1\right]\partial_z\phi(x,z) - \alpha\phi(x,z) = 0$$

- randomness incorporated via varying rigidity

$$b(x) = \bar{b} + \epsilon\,\beta(x) \quad \text{with } \beta(x) \sim \mathcal{O}(1) \text{ and } \langle\beta\rangle = 0$$

- Step approximation of $b(x)$ with M intervals of equal length



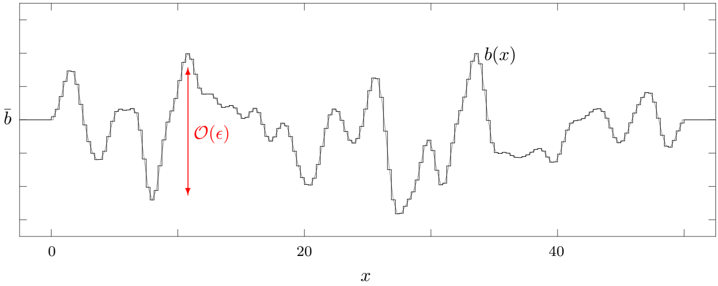
Thin-elastic beam floating on water model

$$\left[\partial_x^2(b(x)\,\partial_x^2) - \alpha g \quad + 1\right]\partial_z\phi(x,z) - \alpha\phi(x,z) = 0$$

- randomness incorporated via varying rigidity

$$b(x) = \bar{b} + \epsilon\,\beta(x) \quad \text{with } \beta(x) \sim \mathcal{O}(1) \text{ and } \langle\beta\rangle = 0$$

- Step approximation of $b(x)$ with M intervals of equal length



Full Solution & Ice Cover Dispersion Relation

Full linear solution

$$\phi(x,z) = \sum_{n=-2}^{\infty} \phi_n(x) \frac{cosh(\kappa_n(z+H))}{cosh(\kappa_n H)}$$

Full Solution & Ice Cover Dispersion Relation

Full linear solution

$$\phi(x,z) = \sum_{n=-2}^{\infty} \phi_n(x) \frac{cosh(\kappa_n(z+H))}{cosh(\kappa_n H)}$$

Wave number κ has to satisfy dispersion relation for elastic plates (with constant rigidity, b, and mass, g):

$$\kappa \tanh(\kappa H) = \frac{\alpha}{b\kappa^4 - \alpha g + 1}.$$

Full Solution & Ice Cover Dispersion Relation

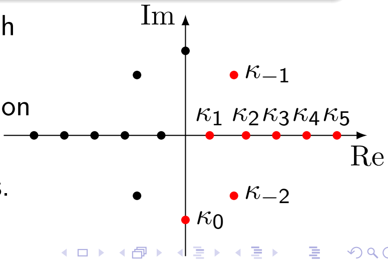
Full linear solution

$$\phi(x,z) = \sum_{n=-2}^{\infty} \phi_n(x) \frac{cosh(\kappa_n(z+H))}{cosh(\kappa_n H)}$$

Wave number κ has to satisfy dispersion relation for elastic plates (with constant rigidity, b, and mass, g):

$$\kappa \tanh(\kappa H) = \frac{\alpha}{b\kappa^4 - \alpha g + 1}.$$

- κ_{-2} and κ_{-1} complex solutions with positive real part,
- κ_0 purely imaginary, negative solution $\rightarrow \quad \kappa = \mathrm{i}\kappa_0$,
- κ_n for $n > 0$ positive, real solutions.



Full Solution & Ice Cover Dispersion Relation

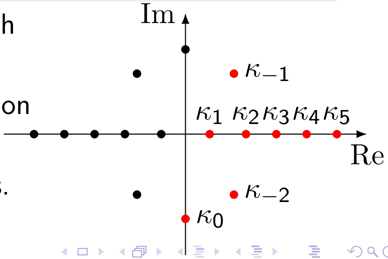
Full linear solution

$$\phi(x,z) = \sum_{n=-2}^{\infty} \left(a^{(n)} \mathrm{e}^{-\mathrm{i}\kappa_n} + b^{(n)} \mathrm{e}^{\mathrm{i}\kappa_n} \right) \frac{cosh(\kappa_n(z+H))}{cosh(\kappa_n H)}$$

Wave number κ has to satisfy dispersion relation for elastic plates (with constant rigidity, b, and mass, g):

$$\kappa \tanh(\kappa H) = \frac{\alpha}{b\kappa^4 - \alpha g + 1}.$$

- κ_{-2} and κ_{-1} complex solutions with positive real part,
- κ_0 purely imaginary, negative solution $\rightarrow \quad \kappa = \mathrm{i}\kappa_0$,
- κ_n for $n > 0$ positive, real solutions.



Full Solution & Ice Cover Dispersion Relation

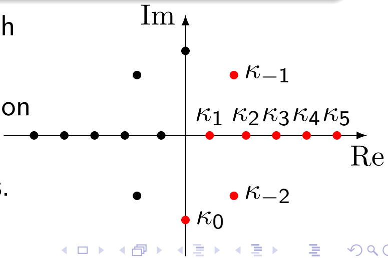
Multi-mode approximation of potential

$$\phi(x,z) \approx \sum_{n=-2}^{N} \left(a^{(n)} \mathrm{e}^{-\mathrm{i}\kappa_n} + b^{(n)} \mathrm{e}^{\mathrm{i}\kappa_n} \right) \frac{\cosh(\kappa_n(z+H))}{\cosh(\kappa_n H)}$$

Wave number κ has to satisfy dispersion relation for elastic plates (with constant rigidity, b, and mass, g):

$$\kappa \tanh(\kappa H) = \frac{\alpha}{b\kappa^4 - \alpha g + 1}.$$

- κ_{-2} and κ_{-1} complex solutions with positive real part,
- κ_0 purely imaginary, negative solution $\rightarrow \quad \kappa = \mathrm{i}\kappa_0$,
- κ_n for $n > 0$ positive, real solutions.



Individual vs. effective wave field

Individual wave field	vs.	Effective wave field
Wave field for single realisation of roughness profile • 1 profile realisation • single wave field		Ensemble average of individual wave fields for many realisations • 1500 profile realisations • mean wave field

Individual vs. effective wave field

Individual wave field	vs.	Effective wave field
Wave field for single realisation of roughness profile • 1 profile realisation • single wave field		Ensemble average of individual wave fields for many realisations • 1500 profile realisations • mean wave field

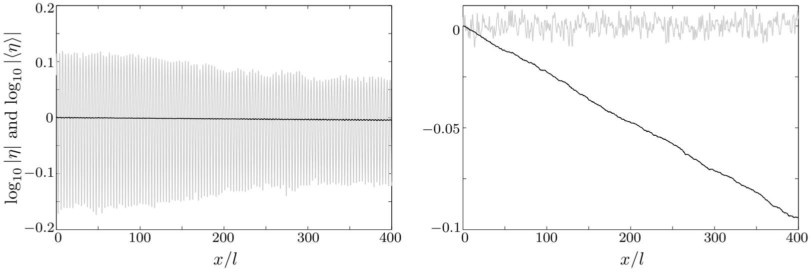


Figure: Example individual wave field (grey) and corresponding effective wave field (black), for $\epsilon = 10^{-2}$, and $l = 0.9$ (left) and 5.0 (right)

Individual vs. effective wave field

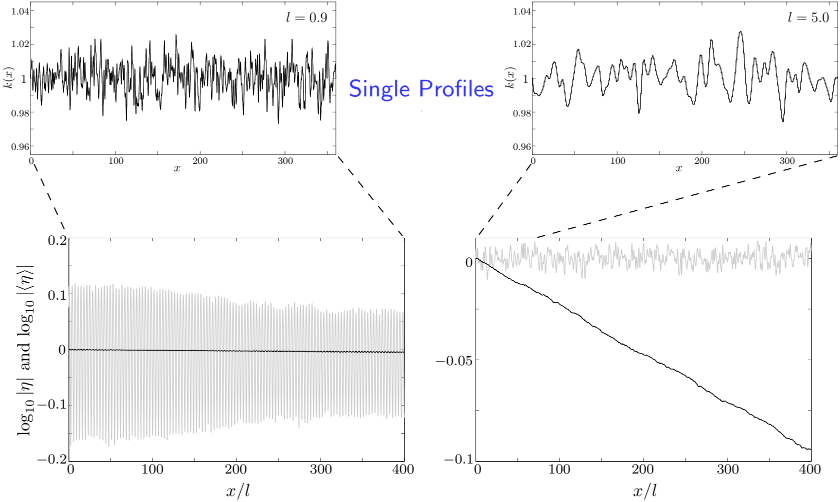


Figure: Example individual wave field (grey) and corresponding effective wave field (black), for $\epsilon = 10^{-2}$, and $l = 0.9$ (left) and 5.0 (right)

Attenuation Results (RS)

Attenuation rate from effective wave field $\langle \eta \rangle$

$$|\langle \eta \rangle| \underset{\sim}{\propto} \mathrm{e}^{-Q_{\mathrm{eff}} x} \quad (0 < x < L)$$

Attenuation Results (RS)

Attenuation rate from effective wave field $\langle \eta \rangle$

$$|\langle \eta \rangle| \underset{\sim}{\propto} \mathrm{e}^{-Q_{\mathit{eff}}\, x} \quad (0 < x < L)$$

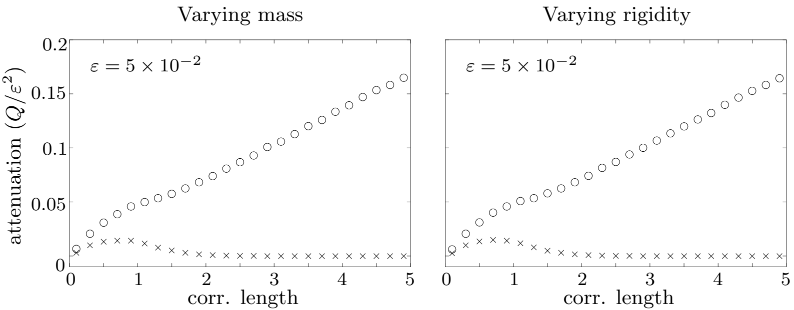


Figure: Attenuation of individual ($\times$) and effective ($\circ$) wave fields

Multiple-scale approach

PDE system for infinitely long, rough floating elastic beam

$$
\begin{aligned}
\Delta\phi &= 0, & z &\in (-H,0), \\
\frac{\partial\phi}{\partial z} &= 0, & z &= -H, \\
\left[\frac{\partial^2}{\partial x^2} b(x) \frac{\partial^2}{\partial x^2} - \alpha g(x) + 1\right] \frac{\partial\phi}{\partial z} &= \alpha\phi, & z &= 0.
\end{aligned}
$$

Goal: Derivation of equation to describe potential

Multiple-scale approach

PDE system for infinitely long, rough floating elastic beam

$$\Delta\phi = 0, \qquad z \in (-H, 0),$$
$$\frac{\partial\phi}{\partial z} = 0, \qquad z = -H,$$
$$\left[\frac{\partial^2}{\partial x^2} b(x) \frac{\partial^2}{\partial x^2} - \alpha g(x) + 1\right] \frac{\partial\phi}{\partial z} = \alpha\phi, \qquad z = 0.$$

Goal: Derivation of equation to describe potential

- Consider two spatial scales:
  - Small scale $s \sim 2\pi/k$ with coordinate x
  - Observation scale $S = \epsilon^{-2} s$ with coordinate $x_2 = \epsilon^2 x$ $(\epsilon \ll 1)$
- Adopt a multiple-scale expansion:
  $\phi(x, z) = \phi_0(x, x_2, z) + \epsilon\,\phi_1(x, x_2, z) + \epsilon^2\,\phi_2(x, x_2, z) + \mathcal{O}(\epsilon^3)$
- Randomness incorporated via same process as in RS
  - varying beam mass: $g(x) = \bar{g} + \epsilon\gamma(x)$,
  - varying beam rigidity: $b(x) = \bar{b} + \epsilon\beta(x)$

Multiple-scale approach

PDE system of order 0

$$\begin{aligned} \Delta\phi &= 0, & z &\in (-H,0),\\ \frac{\partial\phi}{\partial z} &= 0, & z &= -H,\\ \left[\bar{b}\frac{\partial^4}{\partial x^4} - \alpha\bar{g} + 1\right]\frac{\partial\phi}{\partial z} &= \alpha\phi, & z &= 0. \end{aligned}$$

Goal: Derivation of equation to describe potential

- Consider two spatial scales:
  - Small scale $s \sim 2\pi/k$ with coordinate x
  - Observation scale $S = \epsilon^{-2}s$ with coordinate $x_2 = \epsilon^2 x$ $(\epsilon \ll 1)$
- Adopt a multiple-scale expansion:
  $\phi(x,z) = \phi_0(x,x_2,z) + \epsilon\,\phi_1(x,x_2,z) + \epsilon^2\,\phi_2(x,x_2,z) + \mathcal{O}(\epsilon^3)$
- Randomness incorporated via same process as in RS
  - varying beam mass: $g(x) = \bar{g} + \epsilon\gamma(x)$,
  - varying beam rigidity: $b(x) = \bar{b} + \epsilon\beta(x)$

Multiple-scale approach

PDE system of order 0

$$\begin{aligned} \Delta\phi &= 0, & z &\in (-H, 0), \\ \frac{\partial \phi}{\partial z} &= 0, & z &= -H, \\ \left[\bar{b}\frac{\partial^4}{\partial x^4} - \alpha\bar{g} + 1\right]\frac{\partial \phi}{\partial z} &= \alpha\phi, & z &= 0. \end{aligned}$$

- System is deterministic

Multiple-scale approach

PDE system of order 0

$$\begin{aligned} \Delta\phi &= 0, & z &\in (-H, 0), \\ \frac{\partial\phi}{\partial z} &= 0, & z &= -H, \\ \left[\bar{b}\frac{\partial^4}{\partial x^4} - \alpha\bar{g} + 1\right]\frac{\partial\phi}{\partial z} &= \alpha\phi, & z &= 0. \end{aligned}$$

- System is deterministic
- Consider a rightward propagating wave in leading order system

Solution

$$\phi_0(x, x_2, z) = A(x_2)\frac{\cosh(\kappa(z+H))}{\cosh(\kappa H)}\mathrm{e}^{\mathrm{i}\kappa x}$$

Multiple-scale approach

PDE system of order 0

$$
\begin{aligned}
\Delta\phi &= 0, & z &\in (-H, 0),\\
\frac{\partial\phi}{\partial z} &= 0, & z &= -H,\\
\left[\bar{b}\frac{\partial^4}{\partial x^4} - \alpha\bar{g} + 1\right]\frac{\partial\phi}{\partial z} &= \alpha\phi, & z &= 0.
\end{aligned}
$$

- System is deterministic
- Consider a rightward propagating wave in leading order system

Solution

$$\phi_0(x, x_2, z) = A(x_2)\frac{\cosh(\kappa(z+H))}{\cosh(\kappa H)}\mathrm{e}^{\mathrm{i}\kappa x}$$

Task: Determine leading-order wave amplitude $A(x_2)$

Equation for $A(x_2)$

- Solving PDE system of 1^{st} and 2^{nd} order involves some algebra (Green's function, ensemble average $\langle\phi_2\rangle$, ...)

Equation for $A(x_2)$

- Solving PDE system of 1^{st} and 2^{nd} order involves some algebra (Green's function, ensemble average $\langle\phi_2\rangle$, ...)

Envelope equation for $\langle A(x_2)\rangle$

$$\frac{\partial}{\partial x_2}\langle A(x_2)\rangle = \cdots = -Q \cdot \langle A(x_2)\rangle.$$

with complex coefficient Q

- Solution of ODE

Solution of envelope equation

$$\langle A(x_2)\rangle = A_0 \cdot \mathrm{e}^{-Qx_2}$$

Equation for $A(x_2)$

- Solving PDE system of 1^{st} and 2^{nd} order involves some algebra (Green's function, ensemble average $\langle \phi_2 \rangle$, ...)

Envelope equation for $\langle A(x_2) \rangle$

$$\frac{\partial}{\partial x_2} \langle A(x_2) \rangle = \cdots = -Q \cdot \langle A(x_2) \rangle.$$

with complex coefficient Q

- Solution of ODE

Solution of envelope equation

$$\langle A(x_2) \rangle = A_0 \cdot \mathrm{e}^{-Q x_2}$$

- **Attenuation coefficient Q describes effective wave field!**

Attenuation Results (RS & MS)

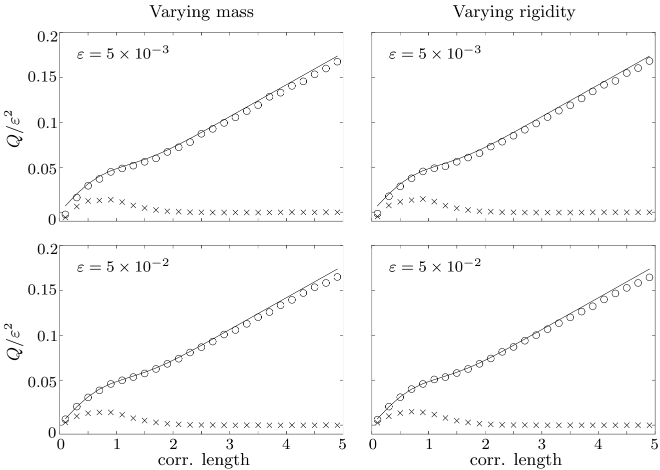


Figure: Attenuation of MS approach (-) compared to attenuation of individual ($\times$) and effective ($\circ$) wave fields

Attenuation Results (RS & MS)

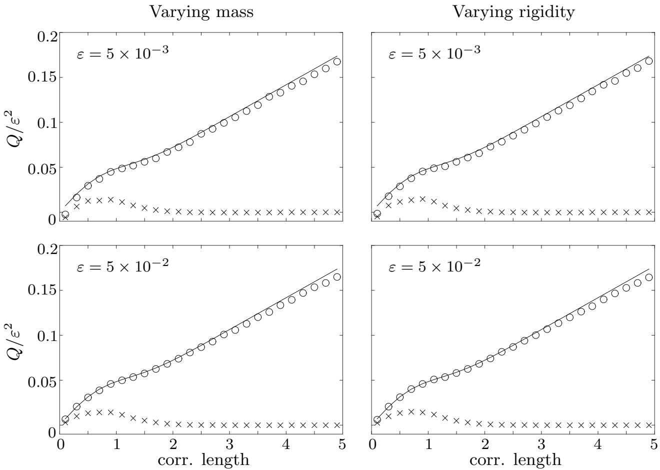


Figure: Attenuation of MS approach (-) compared to attenuation of individual ($\times$) and effective ($\circ$) wave fields

- Attenuation rates predicted by MS agree for both problems
- Agreement between RS and MS up to $\varepsilon \approx 10^{-1}$

Effective vs. individual wave field

Question

When is effective wave field representative for individual wave field?

Effective vs. individual wave field

Question

When is effective wave field representative for individual wave field?

Sample problem

Wave propagation along a rough string with varying density

Effective vs. individual wave field

Question

When is effective wave field representative for individual wave field?

Sample problem

Wave propagation along a rough string with varying density

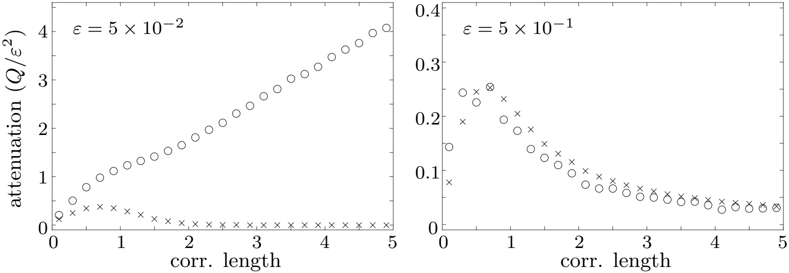


Figure: Attenuation of individual ($\times$) and effective ($\circ$) wave fields

Question

When is effective wave field representative for individual wave field?

Sample problem

Wave propagation along a rough string with varying density

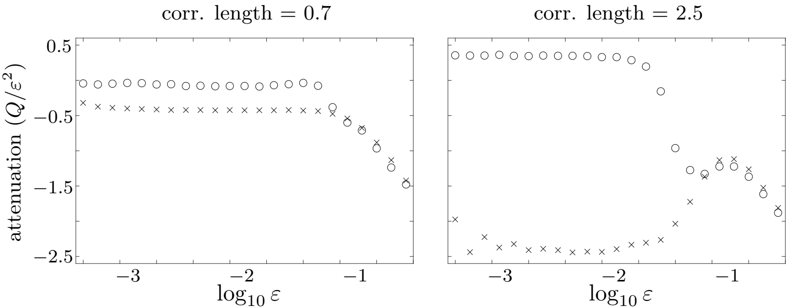


Figure: Attenuation of individual (×) and effective (○) wave fields

Summary & Future work

Summary:

- RS computationally expensive (CPU time: Days vs. 2sec)
- Multiple-scale approach only captures attenuation rate for effective wave field (for small ε)
- Same attenuation rates for varying mass and varying rigidity
- Attenuation rates for effective wave field not the same as attenuation rates for individual wave fields
- Representative attenuation for large profile roughness

Summary & Future work

Summary:

- RS computationally expensive (CPU time: Days vs. 2sec)
- Multiple-scale approach only captures attenuation rate for effective wave field (for small ε)
- Same attenuation rates for varying mass and varying rigidity
- Attenuation rates for effective wave field not the same as attenuation rates for individual wave fields
- Representative attenuation for large profile roughness

Future work:

- Further investigation of attenuation rate regime change
- Experimental validation for in-vacuo beams
- Extension of the method to multiple floating rough elastic plates

Thanks for your attention!