A multiscale approximation of a Cahn–Larché system with phase separation on the microscale

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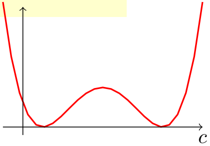
The Cahn–Larché system

$$\partial_t c = \nabla \cdot \Big(M \nabla \Big(f'(c) - \lambda \Delta c - e\,\mathrm{tr}(\mathcal{S}) + \frac{1}{2}(\mathcal{E}(u) - ec\mathbb{1}) : \mathcal{A}'(c)(\mathcal{E}(u) - ec\mathbb{1}) \Big) \Big)$$

$$\varrho\, \partial_{tt} u = \nabla \cdot \big(\mathcal{A}(c)\, (\mathcal{E}(u) - ec\mathbb{1}) \big)$$



$f(c) = \varphi c^2 (1-c)^2$

with:

- linear strain $\mathcal{E}(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^{\mathrm{T}}\right)$
- elasticity tensor $\mathcal{A}(c) = \mathcal{A}_1 + m(c)(\mathcal{A}_2 - \mathcal{A}_1)$
- stress $\mathcal{S} = \mathcal{A}(c)\,(\mathcal{E}(u) - ec\mathbb{1})$
- eigenstrain $e(c)c\mathbb{1}$

Phase separation including elasticity

We consider the Cahn–Larché system

$$\begin{aligned} \partial_t c &= \nabla \cdot \Big(M \nabla \big(f'(c) - \lambda \Delta c - e \, \mathrm{tr}(\mathcal{S}) \\ &\qquad + \frac{1}{2} (\mathcal{E}(u) - ec\mathbb{1}) : \mathcal{A}'(c)(\mathcal{E}(u) - ec\mathbb{1}) \big) \Big) && \text{in } \Omega \times (0,T) \\ \varrho \, \partial_{tt} u &= \nabla \cdot \big(\mathcal{A}(c) \left(\mathcal{E}(u) - ec\mathbb{1} \right) \big) && \text{in } \Omega \times (0,T) \end{aligned}$$

with boundary conditions

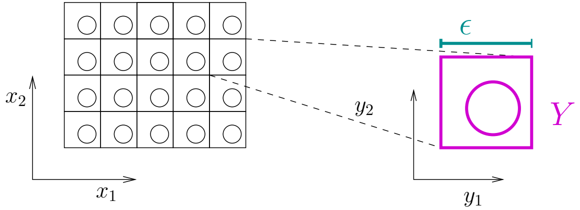
$$\begin{aligned} \nabla c \cdot n &= 0 && \text{on } \Gamma \times (0,T) \\ \nabla \mu \cdot n &= 0 && \text{on } \Gamma \times (0,T) \\ u &= 0 && \text{on } \Gamma_D \times (0,T) \\ \mathcal{S} \cdot n &= g && \text{on } \Gamma_N \times (0,T) \end{aligned}$$

and some suitable initial conditions.

Idea of periodic homogenisation

Consider a periodic domain $\Omega_\epsilon \subset \mathbb{R}^N$ with period ϵ and a scaled unit cell $Y = (0,1)^N$.

example: composite material, porous media

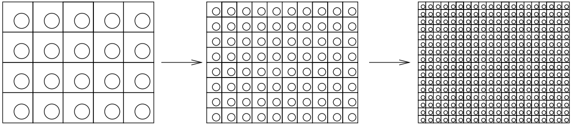


macroscopic scale
$x \in \Omega_\epsilon$

microscopic scale
$y := \frac{x}{\epsilon} \in Y$

Idea of periodic homogenisation

- Consider problems in materials with periodic microstructure with period ϵ
- Consider a sequence of such problems indexed by ϵ
- Find the limit problem or homogenised problem as $\epsilon \to 0$



Phase separation including elasticity

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$$\begin{aligned}\partial_t c_\epsilon &= \nabla \cdot \Big(M \nabla \left(f'(c_\epsilon) - \lambda \Delta c_\epsilon - e \operatorname{tr}(\mathcal{S}_\epsilon) \right. \\ &\qquad \left. + \frac{1}{2} (\mathcal{E}(u_\epsilon) - e c_\epsilon \mathbb{1}) : \mathcal{A}'(c_\epsilon)(\mathcal{E}(u_\epsilon) - e c_\epsilon \mathbb{1}) \right) \Big) && \text{in } \Omega \times (0,T) \\ \varrho\, \partial_{tt} u_\epsilon &= \nabla \cdot \left(\mathcal{A}(c_\epsilon) \left(\mathcal{E}(u_\epsilon) - e c_\epsilon \mathbb{1} \right) \right) && \text{in } \Omega \times (0,T)\end{aligned}$$

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$$\begin{aligned} \nabla c_\epsilon \cdot n &= 0 && \text{on } \Gamma \times (0,T) \\ \nabla \mu_\epsilon \cdot n &= 0 && \text{on } \Gamma \times (0,T) \\ u_\epsilon &= 0 && \text{on } \Gamma_D \times (0,T) \\ \mathcal{S}_\epsilon \cdot n &= g && \text{on } \Gamma_N \times (0,T) \end{aligned}$$

and some suitable initial conditions.

Nondimensionalisation

We consider phase separation of a binary lipid mixture:

- choose characteristic microscopic and macrosopic length scales l and L and set $\epsilon := l/L$
- consider characteristic times $T_d = \frac{l^k\, L^{2-k}}{D_{\text{ref}}}$ and $T_m^2 = \frac{l^m\, L^{2-m}\, \varrho_{\text{ref}}}{\mathcal{A}_{\text{ref}}}$, with $k, m \in [0, 2]$

$$\partial_t c_\epsilon = \epsilon^k\, \Delta\Big(f'(c_\epsilon) - \epsilon^2 \lambda \Delta c_\epsilon - e\, \mathrm{tr}(\mathcal{S}_\epsilon) + \frac{1}{2}(\mathcal{E}(u_\epsilon) - ec_\epsilon \mathbb{1}) : \mathcal{A}'(c_\epsilon)(\mathcal{E}(u_\epsilon) - ec_\epsilon \mathbb{1})\Big)$$

$$\partial_{tt} u_\epsilon = \epsilon^m\, \nabla \cdot \Big(\mathcal{A}(c_\epsilon)\, (\mathcal{E}(u_\epsilon) - ec_\epsilon \mathbb{1})\Big)$$

- match the different characteristic times: $T_d = T_m \approx 1$

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$$\begin{aligned} \partial_t c_\epsilon &= \epsilon^2 \, \Delta\Big(f'(c_\epsilon) - \epsilon^2 \lambda \Delta c_\epsilon - e \, \mathrm{tr}(\mathcal{S}_\epsilon) \\ &\qquad\qquad + \frac{1}{2}(\mathcal{E}(u_\epsilon) - e c_\epsilon \mathbb{1}) : \mathcal{A}'(c_\epsilon)(\mathcal{E}(u_\epsilon) - e c_\epsilon \mathbb{1})\Big) \\ \partial_{tt} u_\epsilon &= \epsilon^0 \, \nabla \cdot \Big(\mathcal{A}(c_\epsilon)\,(\mathcal{E}(u_\epsilon) - e c_\epsilon \mathbb{1})\Big) \end{aligned}$$

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Assumptions on the elasticity tensor $\mathcal{A}(c) = \mathcal{A}^1 + m(c)(\mathcal{A}^2 - \mathcal{A}^1)$:

- $\mathcal{A}^1, \mathcal{A}^2$ symmetric:

$$(\mathcal{A}^l)_{ijkh} = (\mathcal{A}^l)_{jikh} = (\mathcal{A}^l)_{ijhk} = (\mathcal{A}_l)_{khij}$$

- $\exists \alpha_l, \beta_l \in \mathbb{R}, 0 < \alpha_l < \beta_l$:

$$\alpha_l |X|^2 \leq \mathcal{A}^l X X \leq \beta_l |X|^2,$$

  for any symmetric matrix X
- $\mathcal{A}_{ijkh} \in L^\infty(\Omega)$

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Further assumptions:

- $e(c) = e^1 + m(c)(e^2 - e^1)$
- $e \in L^\infty(\Omega)$

The method of asymptotic expansion

- Asymptotic expansion in ϵ of the unknowns

$$c_\epsilon(x,t) = \sum_{i=0}^{\infty} \epsilon^i c_i(x,y,t), \quad u_\epsilon(x,t) = \sum_{i=0}^{\infty} \epsilon^i u_i(x,y,t),$$

  with coefficent functions $c_i(x,y,t)$ $H^2(Y)$-periodic and $u_i(x,y,t)$ $H^1(Y)$-periodic w.r.t. y

- Derivation rule

$$\partial_{x_i} = \partial_{x_i} + \epsilon^{-1}\partial_{y_i}$$

- Insert expansions into the C–L system and identify the coefficients of the different ϵ-powers

The method of asymptotic expansion

The ϵ^{-2}-term gives

$$\nabla_y \cdot \Big(\mathcal{A}(c_0)\,\mathcal{E}_y(u_0(x,y,t))\Big) = 0 \quad \text{in } Y,$$
$$u_0 \text{ is } Y\text{-periodic.}$$

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Due to Fredholm alternative, there exists a unique (up to an additive constant) solution u_0 and furthermore it does not dependent on y, i.e.

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The ϵ^{-1}-term gives

$$\nabla_y \cdot \big(A(c_0)\,(\mathcal{E}_y(u_1) - ec_0\mathbb{1})\big) = -\nabla_y \cdot \big(\mathcal{A}(c_0)\, \mathcal{E}_x(u_0)\big) \quad \text{in } Y,$$
$$u_1 \text{ is } Y\text{-periodic.}$$

Cell problems

For $l, m = 1, \dots, n$, we define the cell problems: Find $\omega^{lm} \in (H^1_{\text{per}}(Y)/\mathbb{R})^n$, such that

$$\nabla \cdot \big(\mathcal{A}(c_0)\,\mathcal{E}_y(\omega^{lm})\big) = -\nabla \cdot \big(\mathcal{A}(c_0)\,\mathcal{E}_y(\lambda^{lm})\big) \quad \text{in } Y,$$
$$\omega^{lm}(y) \ \text{ is } Y\text{-periodic,}$$

where the k-th compontent of λ^{lm} is defined by $(\lambda^{lm})_k = y_m \delta_{kl}$, for $l, m, k = 1, \dots, n$.

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$$\Rightarrow \big(\mathcal{E}_y(u_1)\big)_{ij} = \sum_{l,m} \big(\mathcal{E}_x(u_0)\big)_{ij} \big(\mathcal{E}_y(\omega^{lm})\big)_{ij} + \delta_{ij} e c_0$$

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$$=: \left(\mathcal{E}_\omega(u_0)\right)_{ij} + \delta_{ij} e c_0$$

Asymptotic expansion

The ϵ^0-term gives

$$\partial_t c_0 = \Delta_y \Big(f'(c_0) - \lambda \, \Delta_y c_0 - e \, \mathrm{tr}(\mathcal{S}_0) \\ + \frac{1}{2} \left(\mathcal{E}_x(u_0) + \mathcal{E}_y(u_1) - e c_0 \mathbb{1} \right) : \mathcal{A}'(c_0) \left(\mathcal{E}(u_0) + \mathcal{E}_y(u_1) - e c_0 \mathbb{1} \right) \Big),$$

$$\partial_{tt} u_0 = \nabla_y \cdot \Big(\mathcal{A}(c_0) \big(\mathcal{E}_x(u_1) + \mathcal{E}_y(u_2) - e(c_0) c_1 \mathbb{1} - e'(c_0) c_0 c_1 \mathbb{1} \big) \\ + \mathcal{A}'(c_0) c_1 \big(\mathcal{E}_x(u_0) + \mathcal{E}_y(u_1) - e(c_0) c_0 \mathbb{1} \big) \Big) \\ + \nabla_x \cdot \Big(\mathcal{A}(c_0) \left(\mathcal{E}_x(u_0) + \mathcal{E}_y(u_1) - e(c_0) c_0 \mathbb{1} \right) \Big).$$

The limit problem

The limit or homogenised problem is given by

$$\begin{aligned}\partial_t c_0 = &\Delta_y \Big(f'(c_0) - \lambda\, \Delta_y c_0 - e(c_0)\, \mathrm{tr}[\mathcal{A}(c_0)\,(\mathcal{E}_x(u_0) + \mathcal{E}_\omega(u_0))] \\ &+ \frac{1}{2}\,(\mathcal{E}_x(u_0) + \mathcal{E}_\omega(u_0)) : \mathcal{A}'(c_0)\,(\mathcal{E}(u_0) + \mathcal{E}_\omega(u_0))\Big), \\ \partial_{tt} u_0 = &\nabla_x \cdot \big(\mathcal{A}^{\mathrm{hom}}\, \mathcal{E}(u_0)\big),\end{aligned}$$

with the homogenised or effective elasticity tensor

$$\mathcal{A}^{\mathrm{hom}}_{ijkh} = \int_Y a_{ijlm}(\delta_{lk}\delta_{mh} + e_{lmy}(\omega^{kh}))\, dy.$$

A linear Cahn–Larché system

Let $c_{m,\epsilon}$ and $u_{m,\epsilon}$ be the solutions of the scaled nonlinear C–L-system. Then we add a small perturbation:

$$c_\epsilon(x,t) = c_{m,\epsilon}(x,t) + h\,\tilde{c}_\epsilon(x,t) \quad \text{and} \quad u_\epsilon(x,t) = u_{m,\epsilon}(x,t) + h\,\tilde{u}_\epsilon(x,t)$$

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and derive a linear system:

$$\begin{aligned}
\partial_t \tilde{c}_\epsilon = \epsilon^2\,\Delta \Big(& f''(c_{m,\epsilon})\tilde{c}_\epsilon - \epsilon^2 \lambda\,\Delta \tilde{c}_\epsilon - e\ \mathrm{tr}(\mathcal{S}_\epsilon) \\
& + \big(\mathcal{E}(u_{m,\epsilon}) - e c_{m,\epsilon}\mathbb{1}\big) : \mathcal{A}'(c_{m,\epsilon})\big(\mathcal{E}(\tilde{u}_\epsilon) - e\tilde{c}_\epsilon \mathbb{1}\big) \\
& + \frac{1}{2}\big(\mathcal{E}(u_{m,\epsilon}) - e c_{m,\epsilon}\mathbb{1}\big) : \mathcal{A}''(c_{m,\epsilon})\tilde{c}_\epsilon \,\big(\mathcal{E}(u_{m,\epsilon}) - e c_{m,\epsilon}\mathbb{1}\big)\Big) \quad \text{in } \Omega_T \\
0 = \nabla \cdot \Big(& \mathcal{A}(c_{m,\epsilon})\,\big(\mathcal{E}(\tilde{u}_\epsilon) - e\tilde{c}_\epsilon \mathbb{1}\big) + \mathcal{A}'(c_{m,\epsilon})\,\tilde{c}_\epsilon\,\big(\mathcal{E}(u_{m,\epsilon}) - e c_{m,\epsilon}\mathbb{1}\big)\Big) \quad \text{in } \Omega_T
\end{aligned}$$

Further assumptions

Assumptions on $c_{m,\epsilon}$ and $u_{m,\epsilon}$:

- $c_{m,\epsilon} \in L^\infty(\Omega \times (0,T)),\ u_{m,\epsilon} \in [L^\infty(\Omega \times (0,T))]^n$, $\nabla u_{m,\epsilon} \in [L^\infty(\Omega \times (0,T))]^{n\times n}$
- $c_{m,\epsilon} \to c_{m,0}$ in $L^6(\Omega \times (0,T))$
- $u_{m,\epsilon} \to u_{m,0}$ in $[L^6(\Omega \times (0,T))]^n$
- $\nabla u_{m,\epsilon} \to \nabla u_{m,0}$ in $[L^6(\Omega \times (0,T))]^{n\times n}$
- $\mathcal{E}(u_{m,\epsilon}) = \mathcal{E}_x(u_{m,\epsilon})$

Boundedness

There exists a constant $C > 0$ independent of ϵ, such that

$$\|c_\epsilon\|^2_{L^2(\Omega)} + \int_0^t \|\epsilon \nabla c_\epsilon\|^2_{L^2(\Omega)} \, d\tau$$
$$+ \int_0^t \left\|\epsilon^2 \Delta c_\epsilon\right\|^2_{L^2(\Omega)} \, d\tau + \int_0^t \|u_\epsilon\|^2_{H^1(\Omega)} \, d\tau \leq C,$$

for almost every $t \in [0, T]$.

Existence of a weak solution

For every $\epsilon > 0$ exists a unique weak solution

$$(c_\epsilon(x,t), u_\epsilon(x,t)) \in W^{1,2}([0,T], V) \times L^2(([0,T], W)$$

of the linear Cahn–Larché-system with

$V := \{v \in H^2(\Omega) | \nabla v \cdot n = 0 \text{ on } \Gamma\}, \;\; W := \{w \in (H^1(\Omega)/\mathbb{R})^n | w = 0 \text{ on } \Gamma_D\}.$

Asymptotic Expansion

The ϵ^{-2}-terms gives

$$\nabla_y \cdot \left(\mathcal{A}(c_{m,\epsilon})\,\mathcal{E}_y(u_0)\right) = 0 \quad \text{in } Y,$$
$$u_0 \text{ is } Y\text{-periodic.}$$

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The ϵ^{-1}-terms gives

$$-\nabla_y \cdot \Big(\mathcal{A}(c_{m,\epsilon})\,\big(\mathcal{E}_y(u_1) - e c_0 \mathbb{1}\big) + \mathcal{A}'(c_{m,\epsilon})\,c_0\,\big(\mathcal{E}_x(u_{m,\epsilon}) - e c_{m,\epsilon}\mathbb{1}\big)\Big) = \nabla_y \cdot \Big(\mathcal{A}(c_{m,\epsilon})\,\mathcal{E}_x(u_0)\Big) \quad \text{in } Y,$$

Cell problems

For $l, m = 1, \dots, N$ we define our cell problems:

$$-\nabla_y \cdot \Big(\mathcal{A}(c_{m,\epsilon})\, \mathcal{E}_y(\omega^{lm})\Big) = \nabla_y \cdot \Big(\mathcal{A}(c_{m,\epsilon})\, \mathcal{E}_y(\lambda^{lm})\Big) \quad \text{in } Y,$$

$$\omega^{lm}(y) \quad Y\text{-periodic.}$$

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$$\begin{aligned}\mathcal{A}(c_{m,\epsilon})\, \mathcal{E}_y(u_1) = \mathcal{A}(c_{m,\epsilon})\, \mathcal{E}_\omega(u_0) + \mathcal{A}(c_{m,\epsilon})\, e c_0 \mathbb{1} \\ - \mathcal{A}'(c_{m,\epsilon}) \Big(\mathcal{E}_x(u_{m,\epsilon}) - e c_{m,\epsilon} \mathbb{1} \Big) c_0,\end{aligned}$$

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$$\mathcal{A}(c_{m,\epsilon})\,\mathcal{E}_y(u_1) = \mathcal{A}(c_{m,\epsilon})\,\mathcal{E}_\omega(u_0) + \mathcal{A}(c_{m,\epsilon})\,ec_0\mathbb{1} - \mathcal{A}'(c_{m,\epsilon})\Big(\mathcal{E}_x(u_{m,\epsilon}) - ec_{m,\epsilon}\mathbb{1}\Big)c_0,$$

or

$$\mathcal{E}_y(u_1) = \mathcal{E}_\omega(u_0) + ec_0\mathbb{1} - \mathcal{A}^{-1}\mathcal{A}'(c_{m,\epsilon})\Big(\mathcal{E}_x(u_{m,\epsilon}) - ec_{m,\epsilon}\mathbb{1}\Big)c_0,$$

The limit problem

The homogenised problem is given by

$$\begin{aligned}\partial_t c_0 =& \Delta_y \Big(f''(c_{m,\epsilon})c_0 - \lambda \Delta_y c_0 - e \ \mathrm{tr} \left[\mathcal{A}(c_{m,\epsilon}) \left(\mathcal{E}_x(u_0) + \mathcal{E}_\omega(u_0) \right) \right] \\ & + \left(\mathcal{E}_x(u_{m,\epsilon}) - e c_{m,\epsilon} \mathbb{1} \right) : \mathcal{A}'(c_{m,\epsilon}) \big(\mathcal{E}_x(u_0) + \mathcal{E}_\omega(u_0) \big), \\ & - \left(\mathcal{E}_x(u_{m,\epsilon}) - e c_{m,\epsilon} \mathbb{1} \right) : \mathcal{A}'(c_{m,\epsilon}) \big(\mathcal{A}^{-1} \mathcal{A}' \, (c_{m,\epsilon}) (\mathcal{E}_x(u_{m,\epsilon}) - e c_{m,\epsilon} \mathbb{1}) c_0 \big), \\ & + \frac{1}{2} \big(\mathcal{E}_x(u_{m,\epsilon}) - e c_{m,\epsilon} \mathbb{1} \big) : \mathcal{A}''(c_{m,\epsilon}) c_0 \left(\mathcal{E}_x(u_{m,\epsilon}) - e c_{m,\epsilon} \mathbb{1} \right) \Big),\end{aligned}$$

$$0 = \nabla_x \cdot \left(\mathcal{A}^{\mathrm{hom}} \, \mathcal{E}_x(u_0) \right) + \nabla_x \cdot \int_Y \mathcal{A}'(c_{m,\epsilon}) c_0 \left(\mathcal{E}_x(u_{m,\epsilon}) - e c_{m,\epsilon} \mathbb{1} \right) dy.$$

Two-scale convergence

Definition

A sequence of functions u_ϵ in $L^2(\Omega)$ two-scale converge to a limit $u_0(x,y) \in L^2(\Omega \times Y)$ if,

$$\lim_{\epsilon \to 0} \int_\Omega u_\epsilon \, \psi(x, \frac{x}{\epsilon}) \, dx = \int_\Omega \int_Y u_0 \, \psi(x,y) \, dy \, dx,$$

for any function $\psi(x,y) \in C^\infty[\Omega; C^\infty_{\text{per}}(Y)]$.

[G. Nguetseng, G. Allaire]

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for any function $\psi(x,y) \in C^\infty[\Omega; C^\infty_{\text{per}}(Y)]$.

Theorem

From each bounded sequence u_ϵ in $L^2(\Omega)$ one can extract a subsequence, and there exists a limit $u_0(x,y) \in L^2(\Omega \times Y)$ such that this subsequence two-scale converges to u_0.

[G. Nguetseng, G. Allaire]

Proposition

- *Let* u_ϵ *be a bounded sequence in* $H^1(\Omega)$. *Then there exist* $u(x) \in H^1(\Omega)$ *and* $u_1(x,y)$ *in* $L^2[\Omega; H^1_{\text{per}}(Y)/\mathbb{R}]$ *such that (up to a subsequence)*

$$u_\epsilon \stackrel{2s.}{\rightharpoonup} u(x), \quad \partial_{x_i} u_\epsilon \stackrel{2s.}{\rightharpoonup} \partial_{x_i} u + \partial_{y_i} u_1(x,y).$$

- *Let* u_ϵ *and* $\epsilon \partial_{x_i} u_\epsilon$ *be bounded sequences in* $L^2(\Omega)$, *respectively, then there exists a function* $u_0(x,y) \in L^2[\Omega; H^1_{\text{per}}(Y)/\mathbb{R}]$ *such that (up to a subsequence)*

$$u_\epsilon \stackrel{2s.}{\rightharpoonup} u_0(x,y), \quad \epsilon\, \partial_{x_i} u_\epsilon \stackrel{2s.}{\rightharpoonup} \partial_{y_i} u_0(x,y).$$

[G. Nguetseng, G. Allaire]

Proposition

- *Let u_ϵ be a bounded sequence in $H^1(\Omega)$. Then there exist $u(x) \in H^1(\Omega)$ and $u_1(x,y)$ in $L^2[\Omega; H^1_{\text{per}}(Y)/\mathbb{R}]$ such that (up to a subsequence)*

$$u_\epsilon \overset{2s.}{\rightharpoonup} u(x), \quad \partial_{x_i} u_\epsilon \overset{2s.}{\rightharpoonup} \partial_{x_i} u + \partial_{y_i} u_1(x,y).$$

- *Let u_ϵ and $\epsilon \partial_{x_i} u_\epsilon$ be bounded sequences in $L^2(\Omega)$, respectively, then there exists a function $u_0(x,y) \in L^2[\Omega; H^1_{\text{per}}(Y)/\mathbb{R}]$ such that (up to a subsequence)*

$$u_\epsilon \overset{2s.}{\rightharpoonup} u_0(x,y), \quad \epsilon\, \partial_{x_i} u_\epsilon \overset{2s.}{\rightharpoonup} \partial_{y_i} u_0(x,y).$$

If furthermore $\epsilon^2 \partial^2_{x_i x_j} u_\epsilon$ is bounded in $L^2(\Omega)$, then

$$\epsilon^2\, \partial^2_{x_i x_j} u_\epsilon \overset{2s.}{\rightharpoonup} \partial^2_{y_i y_j} u_0(x,y),$$

with $u_0(x,y) \in L^2[\Omega; H^2_{\text{per}}(Y)/(\mathbb{R} + \mathbb{R}^{\mathrm{T}} y)]$.

Two-scale convergence

We choose a test function $\varphi(x, \frac{x}{\epsilon}) \in C^\infty(\Omega; C^\infty_{\text{per}}(Y))$:

$$\int_\Omega \partial_t c_\epsilon(x)\, \varphi(x, \frac{x}{\epsilon})\, dx =$$

$$= \epsilon^2 \int_\Omega \Big(f''(c_{m,\epsilon})\, c_\epsilon(x) - \epsilon^2 \lambda \Delta c_\epsilon(x) - e\, \mathrm{tr}(\mathcal{S}_\epsilon) \Big)\, \Delta\varphi(x, \frac{x}{\epsilon})\, dx$$

$$+ \epsilon^2 \int_\Omega \Big\{ \Big(\mathcal{E}(u_{m,\epsilon}) - e c_{m,\epsilon} \mathbb{1} \Big) : \mathcal{A}'(c_{m,\epsilon}) \Big(\mathcal{E}(u_\epsilon(x)) - e c_\epsilon(x) \mathbb{1} \Big)$$

$$+ \frac{1}{2} \Big(\mathcal{E}(u_{m,\epsilon}) - e c_{m,\epsilon} \mathbb{1} \Big) : \mathcal{A}''(c_{m,\epsilon}) c_\epsilon(x) \Big(\mathcal{E}(u_{m,\epsilon}) - e c_{m,\epsilon} \mathbb{1} \Big) \Big\}\, \Delta\varphi(x, \frac{x}{\epsilon})\, dx.$$

Two-scale convergence

For $\epsilon \to 0$ we obtain

$$\int_\Omega \int_Y \partial_t c_0(x,y)\,\varphi(x,y)\,dydx =$$

$$= \int_\Omega \int_Y \left(f''(c_{m,0})\,c_0(x,y) - \lambda \Delta_y c_0(x,y) - e\,\mathrm{tr}(\mathcal{S}_0)\right) \Delta_y \varphi(x,y)\,dydx$$

$$+ \int_\Omega \int_Y \Big\{ \Big(\mathcal{E}_x(u_{m,0}) - e c_{m,0}\mathbb{1}\Big) : \mathcal{A}'(c_{m,0}) \Big(\mathcal{E}_x(u_0(x)) + \mathcal{E}_y(u_1(x,y)) - e c_0(x,y)\mathbb{1}\Big)$$

$$+ \frac{1}{2}\Big(\mathcal{E}(u_{m,0}) - e c_{m,0}\mathbb{1}\Big) : \mathcal{A}''(c_{m,0}) c_0(x,y) \Big(\mathcal{E}_x(u_{m,0}) - e c_{m,\epsilon}\mathbb{1}\Big)\Big\} \Delta_y \varphi(x,y)\,dydx$$

Two-scale convergence

We choose a test function $\psi(x, \frac{x}{\epsilon}) = \psi_0(x) + \epsilon\psi_1(x, \frac{x}{\epsilon})$ with $\psi_0 \in [C^\infty(\Omega)]^2$ and $\psi_1 \in [C^\infty(\Omega; C^\infty_{\text{per}}(Y))]^2$:

$$\int_\Omega \Big\{ \mathcal{A}(c_{m,\epsilon}) \Big(\mathcal{E}(u_\epsilon(x,t)) - e c_\epsilon(x,t) \mathbb{1} \Big) + \mathcal{A}'(c_{m,\epsilon})\, c_\epsilon(x,t) \Big(\mathcal{E}(u_m) - e c_{m,\epsilon} \mathbb{1} \Big) \Big\}\, \mathcal{E}(\psi(x, \frac{x}{\epsilon}))\, dx - \int_{\Gamma_N} g\, \psi(x, \frac{x}{\epsilon})\, d\sigma = 0$$

Two-scale convergence

We choose a test function $\psi(x, \frac{x}{\epsilon}) = \psi_0(x) + \epsilon\psi_1(x, \frac{x}{\epsilon})$ with $\psi_0 \in [C^\infty(\Omega)]^2$ and $\psi_1 \in [C^\infty(\Omega; C^\infty_{\text{per}}(Y))]^2$:

$$\int_\Omega \Big\{ \mathcal{A}(c_{m,\epsilon}) \Big(\mathcal{E}(u_\epsilon(x,t)) - ec_\epsilon(x,t)\mathbb{1} \Big) + \mathcal{A}'(c_{m,\epsilon})\, c_\epsilon(x,t) \Big(\mathcal{E}(u_m) - ec_{m,\epsilon}\mathbb{1} \Big) \Big\} \, \mathcal{E}(\psi(x, \frac{x}{\epsilon}))\, dx - \int_{\Gamma_N} g\, \psi(x, \frac{x}{\epsilon})\, d\sigma = 0$$

For $\epsilon \to 0$ we obtain

$$\int_\Omega \int_Y \Big\{ \mathcal{A}(c_{m,0}) \Big(\mathcal{E}_x(u_0(x,t)) + \mathcal{E}_y(u_1(x,y,t)) - ec_0(x,y,t)\mathbb{1} \Big) + \mathcal{A}'(c_{m,0})\, c_0(x,y,t) \Big(\mathcal{E}(u_{m,0}) - ec_{m,0}\mathbb{1} \Big) \Big\} \Big(\mathcal{E}_x(\psi_0(x)) + \mathcal{E}_y(\psi_1(x,y)) \Big)\, dy\, dx - \int_{\Gamma_N} g\, \psi_0(x)\, d\sigma = 0$$

The two-scale limit problem

$$
\begin{aligned}
\partial_t c_0 = \Delta_y \Big\{ & f''(c_{m,0}) c_0 - \lambda \Delta_y c_0 \\
& - e \operatorname{tr} \Big[\mathcal{A}'(c_{m,0}) c_0 \Big(\mathcal{E}(u_{m,0}) - e c_{m,0} \mathbb{1} \Big) + \mathcal{A}(c_{m,0}) \Big(\mathcal{E}_x(u_0) + \mathcal{E}_y(u_1) - e c_0 \mathbb{1} \Big) \Big] \\
& + \Big(\mathcal{E}(u_{m,0}) - e c_{m,0} \mathbb{1} \Big) : \mathcal{A}'(c_{m,0}) \Big(\mathcal{E}_x(u_0) + \mathcal{E}_y(u_1) - e c_0 \mathbb{1} \Big) \\
& + \frac{1}{2} \Big(\mathcal{E}(u_{m,0}) - e c_{m,0} \mathbb{1} \Big) : \mathcal{A}''(c_{m,0}) c_0 \Big(\mathcal{E}(u_{m,0}) - e c_{m,0} \mathbb{1} \Big) \Big\}, \quad \text{in } \Omega \times Y,
\end{aligned}
$$

$$
\begin{aligned}
0 = \nabla_x \cdot \int_Y \Big\{ & \mathcal{A}'(c_{m,0})\, c_0 \Big(\mathcal{E}(u_{m,0}) - e c_{m,0} \mathbb{1} \Big) \\
& + \mathcal{A}(c_{m,0}) \Big(\mathcal{E}_x(u_0) + \mathcal{E}_y(u_1) - e c_0 \mathbb{1} \Big) \Big\}\, dy, \qquad \text{in } \Omega,
\end{aligned}
$$

$$
\begin{aligned}
0 = \nabla_y \cdot \Big\{ & \mathcal{A}'(c_{m,0})\, c_0 \Big(\mathcal{E}(u_{m,0}) - e c_{m,0} \mathbb{1} \Big) \\
& + \mathcal{A}(c_{m,0}) \Big(\mathcal{E}_x(u_0) + \mathcal{E}_y(u_1) - e c_0 \mathbb{1} \Big) \Big\}, \qquad \text{in } \Omega \times Y.
\end{aligned}
$$

Thank you for your attention!