A two-scale Stefan problem arising in a model for tree-sap exudation

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I. Konrad, M. A. Peter, J. M. Stockie (2017) A two-scale Stefan problem arising in a model for tree sap exudation. IMA J. Appl. Math. (in press).

Motivation

In late winter with freezing nights and warm days maple sap is harvested.

Source: Wikipedia

Question

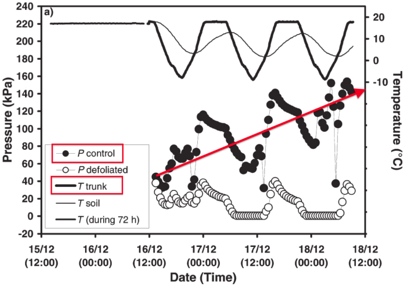
Why is the maple sap driven out of the tree if tapped?

Exudation in sugar maple (Acer saccharum)

Key observations:

- Maple sap begins to *exude* in late February or early March.
- Temperatures must oscillate above/below $0°C$ to generate exudation pressures.
- In winter, there is no transpiration and little uptake from roots.

Experiments on walnut trees
(Améglio et al., 2001)



Basic question:

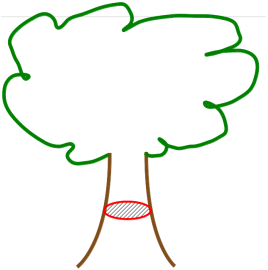
What causes the build-up in stem pressure that drives sap exudation during winter months when the tree is seemingly dormant?

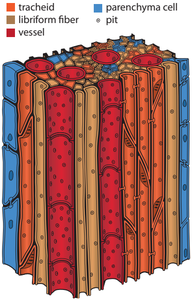
How does the high pressure in the tree arise?

$\begin{cases} \text{Vessels: main sap carriers} \\ \text{(Fibre) Tracheids: secondary role in sap transport} \end{cases}$

(Libriform) Fibres: structural role

Unique in Maple: (Lib.) Fibres filled with air.

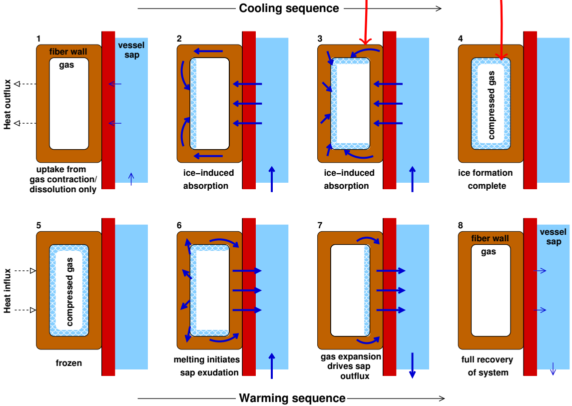




Modified Milburn–O'Malley prop.: freezing process

- As temperature drops, ice forms on inner wall of the fibre.
- Ice formation drives absorption from the vessel.
- Ice growth compresses gas trapped in the fibre.

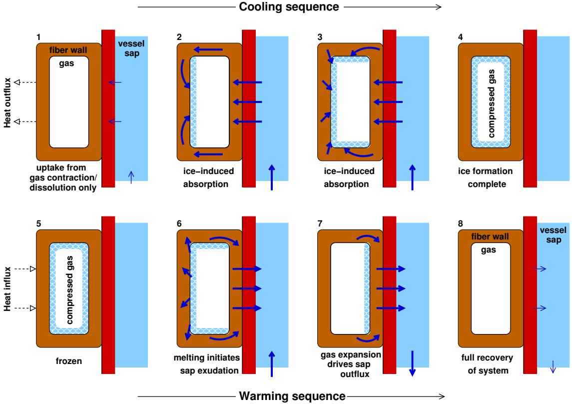
Freeze



Modified Milburn–O'Malley prop.: thawing process

- As temperature rises, the process runs in reverse.
- Compressed gas generates positive pressure in vessels and drives sap flow if there is an exit (taphole).

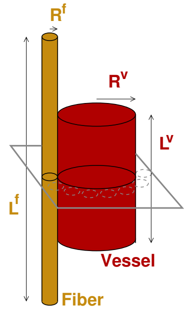
Thaw



Mathematical model of thawing process

Start by considering *only* thawing: (Ceseri & Stockie, 2013)

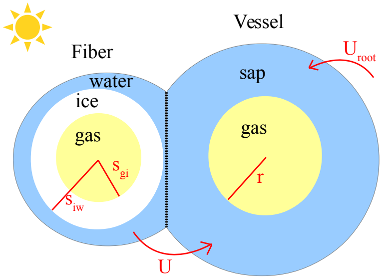
- One vessel, N identical fibres, all cylindrical.



Mathematical model of thawing process

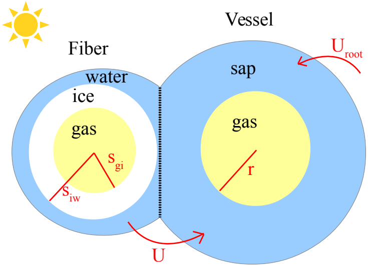
Start by considering *only* thawing: (Ceseri & Stockie, 2013)

- One vessel, N identical fibres, all cylindrical.
- Three phases: gas, ice, water.
- Ice melts in response to external heat source.
- Assume vessel thaws first due to *freezing point depression* (sugar in vessel, none in fibre).
- Melt-water is driven through porous fibre/vessel wall.
- Gas in the vessel (*new!*) is in turn compressed.
- Bubbles dissolve/grow in response to pressure changes.



Mathematical model

Mathematical model by Ceseri & Stockie (2013).



$$\dot{T} = D\Delta T$$

$$\dot{s}_{iw} = -\frac{k_w}{\lambda \rho_w}\partial_x T + \frac{\dot{U}}{2\pi s_{iw} L^f}$$

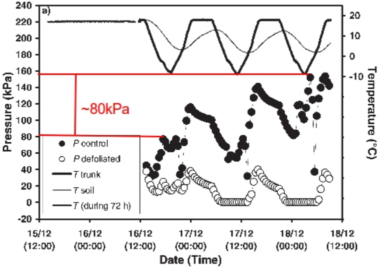
$$0 = -\rho_i s_{gi}\dot{s}_{gi} + (\rho_i - \rho_w)s_{iw}\dot{s}_{iw} + \frac{\dot{U}}{2\pi L^f}$$

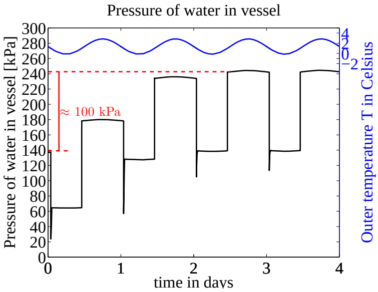
$$\dot{r} = \frac{\dot{U}N + \dot{U}_{root}}{2\pi r L^f}$$

$$\dot{U} = -\frac{KA}{N\rho_w g W}(p_w^v - p_w^f - p_{osm} + p_{cap})$$

$$\dot{U}_{root} = -L_p A_r (p_w^v - p_{root})$$

Results of the microscopic model





This model matches:

1. overall increase
2. step-wise increase
3. pressure approaches threshold

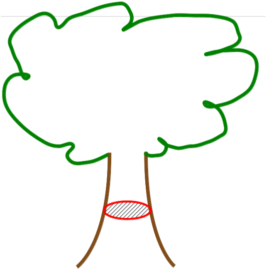
Not yet:

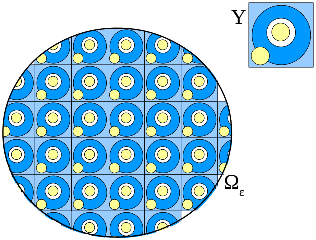
1. gradual increase and decrease
2. pressure relaxation
3. $\Delta P \approx 80 kPa$

Upscaling of the microscopic model to the whole tree stem

Cross-section of the maple-tree stem

Use periodic homogenisation for temperature equation!

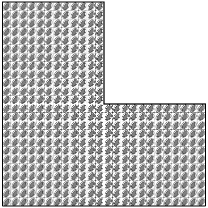




Averaging

General problem:

- Shape of the microstructure:
  complex or not known in detail.
- Rapidly changing coefficients.
- Coefficients do not give (direct) information about observable properties.



Schematic cross-section of a multi-phase medium

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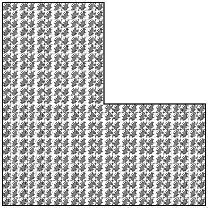
General idea:

Averaging of the unknowns → unknowns defined on the whole domain and observable parameters
→ macromodel



Schematic cross-section of a multi-phase medium

The idea of periodic homogenisation

Periodic homogenisation has been successfully employed to upscale reaction–diffusion problems in porous media.

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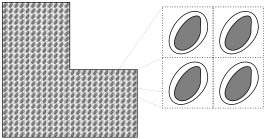
Assume that there exists a

- a representative (unit) cell, $Y = (0,1)^N$, containing all components where all important processes occur, and
- a scale parameter $\varepsilon > 0$

such that the multi-phase material is the union of many scaled cells.



Left: schematic setup of a three-phase material.
Right: enlarged view of the microstructure.

The idea of periodic homogenisation

- Microscale: equations with ε-periodic coefficients valid in each part of the multi-phase medium

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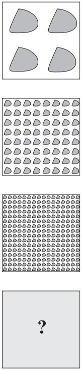
- Considering ε to be a free parameter, a whole family of PDEs,

$$L_\varepsilon u_\varepsilon = f_\varepsilon \quad \text{in } \Omega_\varepsilon,$$

  with appropriate boundary conditions is considered.

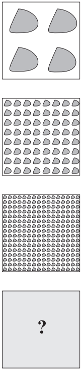


The idea of periodic homogenisation

- Microscale: equations with ε-periodic coefficients valid in each part of the multi-phase medium
- Considering ε to be a free parameter, a whole family of PDEs,

$$L_\varepsilon u_\varepsilon = f_\varepsilon \quad \text{in } \Omega_\varepsilon,$$

  with appropriate boundary conditions is considered.
- Assuming that f_ε and the sequence of (unique) solutions u_ε converge to limit functions f and u as $\varepsilon \to 0$, the question is: What is the homogenised operator L such that u solves $Lu = f$ in Ω?



Homogenisation limits

In general:

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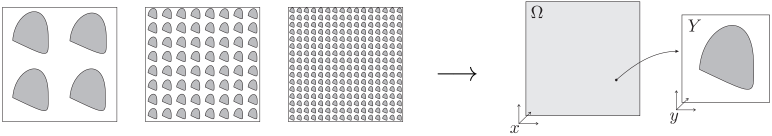

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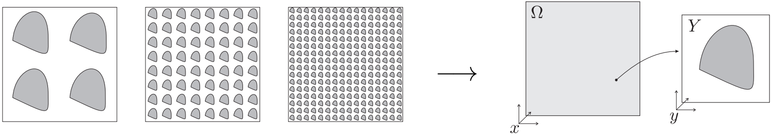

Fortunately, the problem can often be reduced, $u = u(x)$ or $u = u_x(y)$, e.g.

Two-scale convergence

Definition

A sequence (u_ε) in $L^2(\Omega)$ *two-scale converges* to a limit function $u_0(x,y) \in L^2(\Omega \times Y)$ if

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\, v(x, x/\varepsilon)\, \mathrm{d}x = \int_\Omega \int_Y u_0(x,y)\, v(x,y)\, \mathrm{d}y\, \mathrm{d}x$$

for all $v \in \mathcal{C}_0^\infty(\Omega; \mathcal{C}_\#^\infty(Y))$.

(Nguetseng 1989, Allaire 1992)

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Theorem

Every bounded sequence (u_ε) in $L^2(\Omega)$ contains a subsequence, which two-scale converges to a limit function $u_0(x,y) \in L^2(\Omega \times Y)$.

(Nguetseng 1989, Allaire 1992)

Melting ice bars

For now: Only consider temperature! Array of “melting ice bars”.

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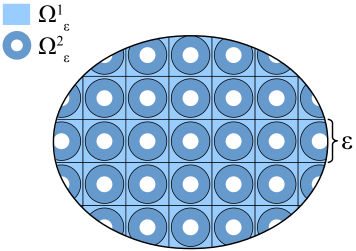

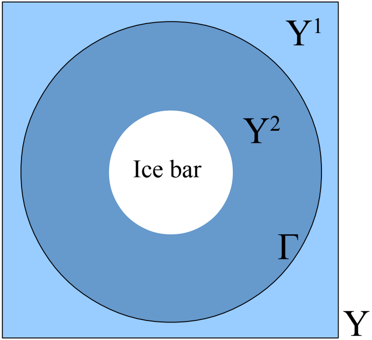

Melting ice bars

Using a regularised (i.e. smooth, strongly monotone) temperature–enthalpy relation $T = \omega(E)$, we may state the strong formulation of the two-phase Stefan problem as

$$\partial_t E_{1,\varepsilon} - \nabla \cdot [D(E_{1,\varepsilon}) \nabla T_{1,\varepsilon}] = 0 \qquad \text{in } \Omega_\varepsilon^1, \tag{1a}$$

$$D(E_{1,\varepsilon}) \nabla T_{1,\varepsilon} \cdot \mathbf{n} = -\varepsilon^2 D(E_{2,\varepsilon}) \nabla T_{2,\varepsilon} \cdot \mathbf{n} \qquad \text{on } \Gamma_\varepsilon, \tag{1b}$$

$$-D(E_{1,\varepsilon}) \nabla T_{1,\varepsilon} \cdot \mathbf{n} = \alpha (T_{1,\varepsilon} - T_{\mathrm{a}}) \qquad \text{on } \partial\Omega \cap \partial\Omega_\varepsilon^1, \tag{1c}$$

$$\partial_t E_{2,\varepsilon} - \varepsilon^2 \nabla \cdot [D(E_{2,\varepsilon}) \nabla T_{2,\varepsilon}] = 0 \qquad \text{in } \Omega_\varepsilon^2, \tag{1d}$$

$$E_{2,\varepsilon} = E_{1,\varepsilon} \qquad \text{on } \Gamma_\varepsilon, \tag{1e}$$

Melting ice bars

Function spaces (for Dirichlet b.c. at the exterior)

$$\mathcal{V}_\varepsilon^1 := \left\{ u \in L^2([0,t_\mathrm{m}], \mathcal{H}^1(\Omega_\varepsilon^1)) \cap \mathcal{H}^1([0,t_\mathrm{m}], \mathcal{H}^1(\Omega_\varepsilon^1)') \mid u = 0 \text{ on } \partial\Omega_\varepsilon^1 \cap \partial\Omega \right\},$$
$$\mathcal{V}_\varepsilon^2 := \left\{ u \in L^2([0,t_\mathrm{m}], \mathcal{H}^1(\Omega_\varepsilon^2)) \cap \mathcal{H}^1([0,t_\mathrm{m}], \mathcal{H}^1(\Omega_\varepsilon^2)') \mid u = 0 \text{ on } \Gamma_\varepsilon \right\},$$
$$\mathcal{V} := L^2([0,t_\mathrm{m}], \mathcal{H}_0^1(\Omega)) \cap \mathcal{H}^1([0,t_\mathrm{m}], \mathcal{H}^1(\Omega)'),$$

We then define the function $\Theta_\varepsilon \in L^2([0,t_\mathrm{m}], \mathcal{H}^1(\Omega))$ by

$$\Theta_\varepsilon = \begin{cases} E_{1,\varepsilon} & \text{in } \Omega_\varepsilon^1, \\ E_{2,\varepsilon} & \text{in } \Omega_\varepsilon^2, \end{cases} \qquad \text{and} \quad \kappa_\varepsilon = \chi_{\Omega_\varepsilon^1} + \varepsilon^2 \chi_{\Omega_\varepsilon^2}.$$

With $\varrho_\varepsilon = \Theta_\varepsilon - \omega^{-1}(T_\mathrm{a})$, the weak form is finding $\varrho_\varepsilon \in \mathcal{V}$ such that

$$(\partial_t \varrho_\varepsilon, \phi)_\Omega + (\kappa_\varepsilon D \omega'(\varrho_\varepsilon + \omega^{-1}(T_\mathrm{a})) \nabla \varrho_\varepsilon, \nabla \phi)_\Omega = (-\partial_t \omega^{-1}(T_\mathrm{a}), \phi)_\Omega, \quad (2)$$

for all $\phi \in \mathcal{H}_0^1(\Omega)$.

Results

Theorem

For given $\varepsilon > 0$, there exists a solution of Eq. (2).

Proof.

This seems to be non-standard (?). Our proof for this parabolic equation with non-monotone non-linearity is based on writing the problem as

$$\begin{aligned} u' + \mathcal{A}(u,u) &= f \qquad \text{in } V^*, \\ u(0) &= u_0, \end{aligned} \tag{3}$$

with $\mathcal{A}$ being the realisation of
$\langle A(t)(u,v), w\rangle = \sum_{j=1}^{n} \int_\Omega a(x,t,u)\, \partial_{x_j} v\, \partial_{x_j} w \, \mathrm{d}x$
and applying Rothe's method. □

Results

Lemma

There exists a constant C_1, independent of ε, such that the solution Θ_ε of (2) (equivalently, $E_{1,\varepsilon}$ and $E_{2,\varepsilon}$) satisfies

$$\|\Theta_\varepsilon\|_{\Omega}^2 + \|\kappa_\varepsilon \nabla \Theta_\varepsilon\|_{\Omega,t}^2 = \|E_{1,\varepsilon}\|_{\Omega_\varepsilon^1}^2 + \|\nabla E_{1,\varepsilon}\|_{\Omega_\varepsilon^1,t}^2 + \|E_{2,\varepsilon}\|_{\Omega_\varepsilon^2}^2 + \varepsilon^2 \|\nabla E_{2,\varepsilon}\|_{\Omega_\varepsilon^2,t}^2 \leq C_1.$$

Proof.

Pretty much standard estimates. □

Results

The a priori estimates immediately yield the following two-scale convergence results

Lemma

There exist functions $E_{1,0} \in L^2([0,t_{\mathrm{m}}],\mathcal{H}^1(\Omega))$, $\widehat{E}_{1,0} \in L^2([0,t_{\mathrm{m}}],L^2(\Omega,\mathcal{H}^1_{\#}(Y^1)))$ *and* $E_{2,0} \in L^2([0,t_{\mathrm{m}}],L^2(\Omega,\mathcal{H}^1_{\#}(Y^2)))$ *such that, up to subsequences,*

$$
\begin{aligned}
E_{1,\varepsilon} &\xrightarrow{\text{2-scale}} E_{1,0},\\
\nabla E_{1,\varepsilon} &\xrightarrow{\text{2-scale}} \nabla_x E_{1,0} + \nabla_y \widehat{E}_{1,0},\\
E_{2,\varepsilon} &\xrightarrow{\text{2-scale}} E_{2,0},\\
\nabla E_{2,\varepsilon} &\xrightarrow{\text{2-scale}} \nabla_y E_{2,0}.
\end{aligned}
$$

Limit problem

If we *assume* that the function $D\omega'$ in (2) is independent of Θ_ε or if we *assume* that Θ_ε converges strongly, we can identify the limit problem:

$$\begin{aligned} |Y^1|(\partial_t E_1,\ \phi_0)_\Omega + (\Pi D(E_1)\omega'(E_1)\nabla_x E_1,\ \nabla_x \phi_0)_\Omega \\ + \langle D(E_2)\omega'(E_2)\nabla_y E_2,\ \phi_0\rangle_{\Gamma\times\Omega} = 0, \quad \text{(4a)} \\ (\partial_t E_2,\ \phi_2)_{\Omega\times Y^2} + (D(E_2)\omega'(E_2)\nabla_y E_2,\ \nabla_y \phi_2)_{\Omega\times Y^2} = 0. \quad \text{(4b)} \end{aligned}$$

for all $\phi_0 \in \mathcal{H}_0^1(\Omega)$ and $\phi_2 \in L^2(\Omega, \mathcal{H}_\#^1(Y^2))$, where Π is determined by (standard) cell problems.

We can also show:

Lemma

The limit problem (i.e. the weak form of (4)) has at most one solution.

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We can also show:

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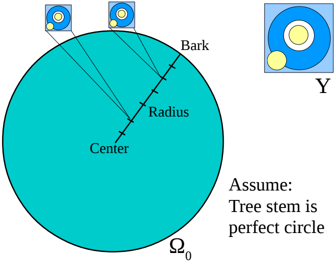
The limit problem (i.e. the weak form of (4)*) has at most one solution.*

Macroscopic model for sap exudation

Add the rest of the model of Ceseri & Stockie to the cell problem.



$$\dot{s}_{iw} = -\frac{k_w}{\lambda \rho_w}\partial_x T + \frac{\dot{U}}{2\pi s_{iw} L^f}$$

$$\dot{r} = \frac{\dot{U}N + \dot{U}_{root}}{2\pi r L^f}$$

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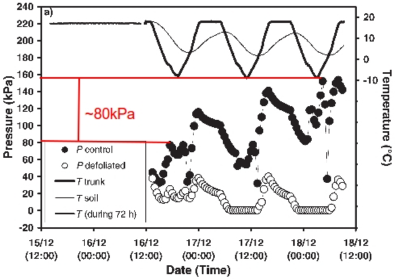
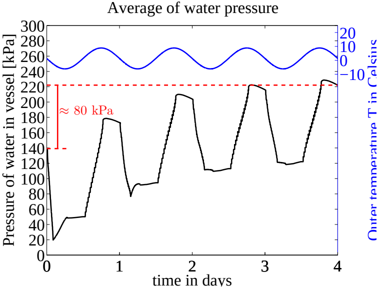
$$\dot{U}_{root} = -L_p A_r \left(p_w^v - p_{root}\right)$$

Result of the macroscopic model





Improved model also matches:

1. gradual increase and decrease
2. pressure relaxation
3. $\Delta P \approx 80kPa$

Summary

- The most important aspects of tree-sap exudation can be explained by a two-scale model.
- Homogenisation of a two-phase Stefan problem in a highly heterogeneous medium with melting/thawing in the slow transport region was performed (limit identification still incomplete for general case) .

Fibre	**Vessel**
Air, no sap	*Air, sap*
←← Water moves from vessel to fibre ←←	
Water freezes	Sugar prevents sap from freezing
	↑↑ Water uptake through roots to vessel ↑↑
Ice melts	
→→ Water moves back from fibre to vessel →→	
Higher pressure in the tree	