Overshoots and maximum principles in porous media flow models

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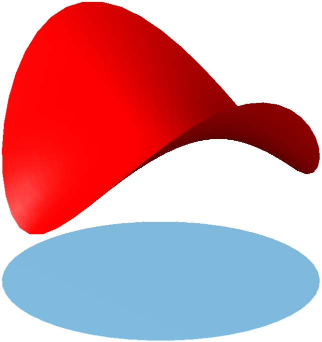
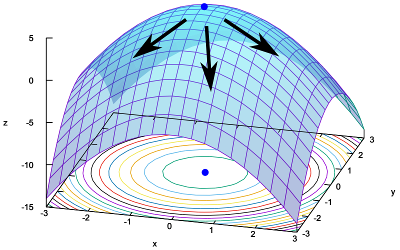
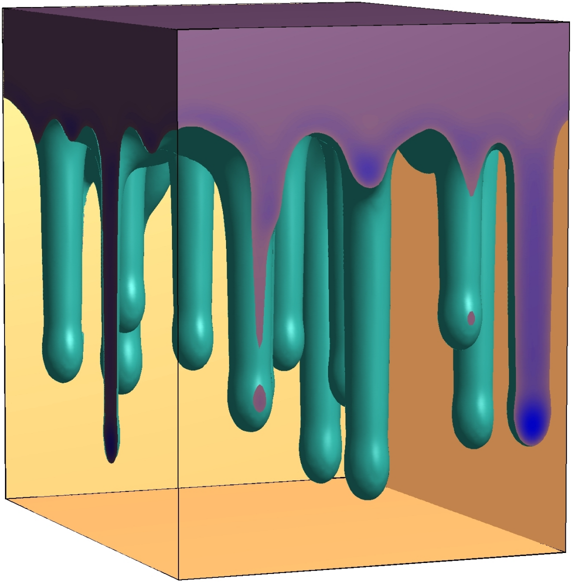
Outline

Maximum principle

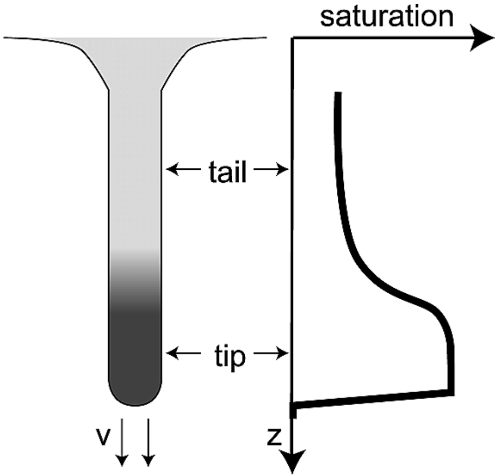




Gravity-driven fingering



Saturation and pressure overshoots



[*Xiong Y.* Flow of water in porous media with saturation overshoot: A review // J. Hydrol. 2014. Vol. 510. Pp. 353–362.]

Pressure overshoot

Saturation overshoot leads to capillary pressure and water pressure overshoots.
[*Xiong Y.* Flow of water in porous media with saturation overshoot: A review // J. Hydrol. 2014. Vol. 510. Pp. 353–362.]

Two-phase flow equations

$$\frac{\partial}{\partial t}\left(\frac{\phi s_\alpha}{b_\alpha}\right) + div\left(\frac{1}{b_\alpha}u_\alpha\right) = q_\alpha,$$

$$u_\alpha = -\frac{k_{r\alpha}}{\mu_\alpha}\mathbb{K}(\nabla p_\alpha - \rho_\alpha g \nabla z), \alpha = w, o$$

$$p_o - p_w = p_c(s_w), s_o + s_w = 1$$

- $\alpha = w, o$ – water, oil
- s_α – saturation
- p_α – pressure
- u_α – Darsy's velocity
- b_α – volume formation
- q_α – sources

- p_c – capillary pressure
- μ_α – viscosity
- $k_{r\alpha}$ – rel. permeability
- $\mathbb{K}$ – abs. permeability
- ρ_α – density

Three-phase flow equations

$$\frac{\partial}{\partial t}\left(\frac{\phi s_\alpha}{b_\alpha}\right) + div\left(\frac{1}{b_\alpha}u_\alpha\right) = q_\alpha, \alpha = w, o,$$

$$\frac{\partial}{\partial t}\left(\frac{\phi s_g}{b_g} - \frac{r_s s_o}{b_o}\right) + div(\frac{1}{b_g}u_g + \frac{r_s}{b_o}u_o) = q_g,$$

$$u_\alpha = -\frac{k_{r\alpha}}{\mu_\alpha}\mathbb{K}(\nabla p_\alpha - \rho_\alpha g \nabla z), \alpha = w, o, g$$

$$s_w + s_o + s_g = 1, p_\alpha - p_o = p_{c\alpha}, \alpha = w, o, g,$$

- $\alpha = w, o, g$ – water, oil, gas
- $p_{c\alpha}$ – capillary pressure
- r_s – gas solubility

- s_α – saturation
- p_α – pressure
- u_α – Darcy's velocity
- b_α – volume formation
- q_α – sources

- μ_α – viscosity
- $k_{r\alpha}$ – rel. permeability
- $\mathbb{K}$ – abs. permeability
- ρ_α – density

Models with saturation overshoot (1)

$$
\begin{aligned}
\phi \frac{\partial s_w}{\partial t} + \Xi(S) \frac{\partial}{\partial z} \left[f_{im}(s_w) - D_{im}(s_w) \frac{\partial S}{\partial z} \right] & \\
[1 - \Xi(s_w)] \frac{\partial}{\partial z} \left[f_{dr}(s_w) - D_{dr}(s_w) \frac{\partial S}{\partial z} \right] & = 0, \\
\Xi(s_w) = \lim_{\varepsilon \to 0} \Theta \left[\frac{\partial s_w}{\partial t}(z, t - \varepsilon) \right], & \\
\Theta(y) = \begin{cases} 1, z/t > c^* \\ 0, z/t \leq c^* \end{cases} &
\end{aligned}
$$

[*Hilfer R., Steinle R.* Saturation overshoot and hysteresis for two-phase flow in porous media // Eur. Phys. J. Special Topics. 2014. Vol. 223. Pp. 2323–2338.]

Models with saturation overshoot (2). Dynamic capillary pressure model.

$$F(s_w, p_c, \frac{\partial s_w}{\partial t}, \frac{\partial p_w}{\partial t}) = 0$$

For example

$$p_c = \psi(s_w) + \tau(s_w, p_w)\frac{\partial s_w}{\partial t}$$

[*Van Duijn C. G., Cao X., Pop I. S.* Two-phase flow in porous media: dynamic capillarity and heterogeneous media // Transport Porous Med. 2016. Vol. 114, no. 2. Pp. 283–308.]

Differential maximum principles: summary

	two-phase			three-phase	
	p_α	$p_o + \widetilde{p}$	s_α	p_α	$p_o + \widetilde{p}$
$p_{cwo} \equiv 0, p_{cgo} \equiv 0$	green	red	red	green	red
$\exists \widetilde{p} : \nabla \widetilde{p} = f_w \nabla p_{cwo} + f_g \nabla p_{cgo}$	green	green	red	green	green
$\mu_\alpha \equiv const$	red	green	green	red	green
$b_\alpha \equiv 1$	red	red	green	red	red
$\phi \equiv const$	red	red	green	red	red

- green: required by a theorem
- red: not required by a theorem

Finite volume schemes

- **Nonlinear multipoint** – satisfies the discrete maximum principle for diffusion equation
- Nonlinear two-point – violates the dicrete maximum principle for diffusion equations
- Linear two-point – satisfies the dicrete maximum principle but do not approximate diffusion equation

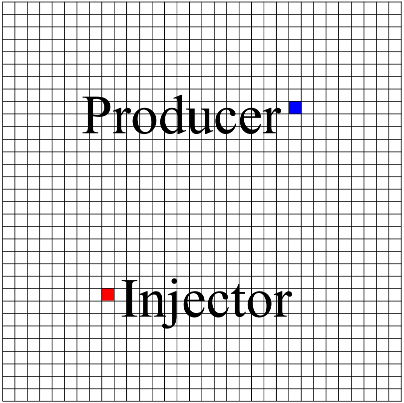
[2] *Lipnikov K., Svyatskiy D., Vassilevski Y.* Minimal stencil finite volume scheme with the discrete maximum principle // Russ. J. Numer. Anal. Math. Modelling. – 2012. T. 27, № 4. – Pp. 369–385.

Differential and discrete maximum principles for pressure in two-phase flow model

	Differential	Discrete
$p_c \equiv 0$	■ (green)	■ (green)
$\mu_\alpha \equiv const$	■ (red)	■ (red)
$b_\alpha \equiv 1$	■ (red)	■ (green)
$\phi \equiv const$	■ (red)	■ (green)

- ■ (green) required by a theorem
- ■ (red) not required by a theorem

Numerical experiment #1



1. zero capillary pressure
   $p_c \equiv 0$
2. constant viscosities
   $\mu_\alpha = const$
3. incompressibility $b_\alpha = 1$
4. constant porosity $\phi \equiv const$
5. Absolute permeability $\mathbb{K} = R_z(-\theta_z) diag(k_1, k_2, k_3) R_z(\theta_z)$, where
   - $k_1 = k_3 = 100$, $k_2 = 0.1$,
   - $\theta_z = 112.5^\circ$,
   - $R_z(\alpha)$ is the matrix of rotation in xy-plane.

Numerical pressures

	multi-point	nonlinear two-point	linear two-point
max	4099.81	4112.64	4097.03
min	3901.10	3888.20	3928.91

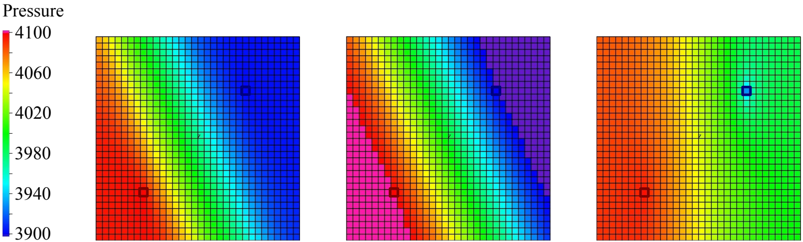
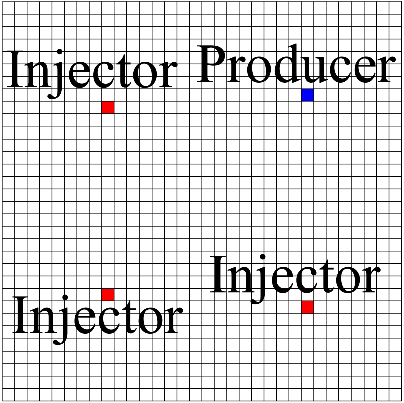


Рис.: Pressure after 2000 model days for different flux discretization schemes.

Numerical experiment #2



1. zero capillary pressure
   $p_c \equiv 0$
2. constant viscosities
   $\mu_\alpha = const$
3. incompressibility $b_\alpha = 1$
4. constant porosity $\phi \equiv const$
5. Absolute permeability $\mathbb{K} = R_z(-\theta_z) diag(k_1, k_2, k_3) R_z(\theta_z)$, where
   - $k_1 = k_3 = 100$, $k_2 = 0.1$,
   - $\theta_z = 112.5^\circ$,
   - $R_z(\alpha)$ is the matrix of rotation in xy-plane.

Numerical pressures

	multi-point	nonlinear two-point	linear two-point
max	4099.91	4122.31	4099.07
min	3906.32	3852.74	3952.38

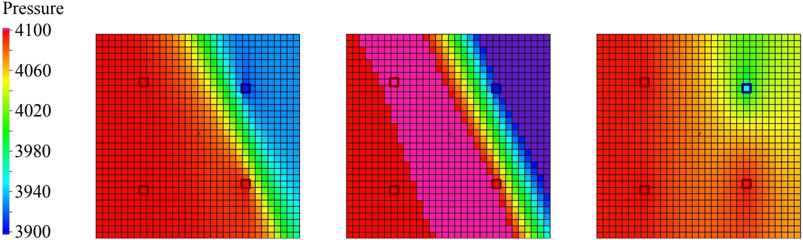


Рис.: Pressure after 100 model days for different flux discretization schemes.

Numerical saturations

	multi-point	nonlinear two-point	linear two-point
max	0.517	0.510	0.574
min	0.150	-0.995	0.150

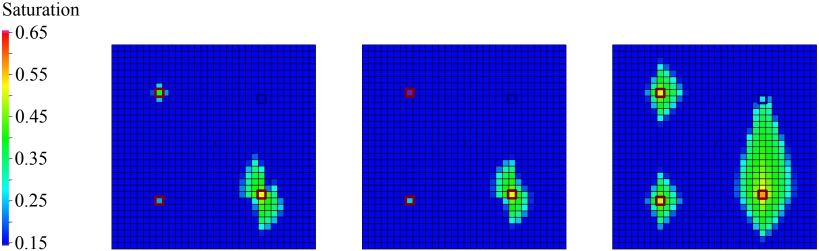


Рис.: Water saturation after 100 model days for different flux discretization schemes. Initial saturation is $s(0) = 0.15$.

Experimental discrete maximum principle for nonconstant parameters

	Differential	Discrete
$p_c \equiv 0$	🟩	🟩
$\mu_\alpha \equiv const$	🟥	🟥
$b_\alpha \equiv 1$	🟥	🟥
$\phi \equiv const$	🟥	🟥

- 🟩 required by a theorem
- 🟥 not required by a theorem

Summary

- 3 differential maximum principles for two-phase flow model and 2 for three-phase flow model have been proven.
- The discrete maximum principle for numerical pressure obtained using nonlinear multipoint scheme has been proven.
- The discrete maximum principle require additional assumption on model coefficients.
- Numerical experiments support possible existence of the discrete maximum principle for fewer assumptions.

Two-phase flow model equations (no gravity)

$$
\begin{cases}
\frac{\partial}{\partial t}\left(\frac{\phi \rho_\alpha s_\alpha}{b_\alpha}\right) - div\left(\frac{\rho_\alpha}{b_\alpha}\frac{k_{r\alpha}}{\mu_\alpha}\mathbb{K}\nabla p_\alpha\right) = q_\alpha \text{ in } \Omega \times (0, T) \\
p_o - p_w = p_c(s_w) \\
s_o + s_w = 1
\end{cases}
$$

Differential maximum principle. Assumptions(1).

- zero capillary pressure $p_c \equiv 0$
- strictly elliptic absolute permeability $\mathbb{K}$
- smooth enough $b_\alpha, \lambda_\alpha, \alpha = w, o$

- no incompressibility assumption: $b_\alpha \not\equiv const$
- no constant porosity assumption: $\phi \not\equiv const$
- no constant viscosity assumption: $\mu_\alpha \not\equiv const$

Differential maximum principle(1). Pressure.

- $b_o q_o + b_w q_w \leq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\sup_{\Omega\times[0,T]} p_\alpha \leq \sup_{\partial\Omega\times[0,T)} p_\alpha$$

- $b_o q_o + b_w q_w \geq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\inf_{\Omega\times[0,T]} p_\alpha \geq \inf_{\partial\Omega\times[0,T)} p_\alpha, \alpha = w, o$$

Differential maximum principle. Assumptions(2)

- fractional flows $f_\alpha = \frac{\lambda_\alpha}{\lambda_w + \lambda_o}$, $\alpha = w, o$ depend solely on s_w
  - (implies constant viscosities, since $\lambda_\alpha = \frac{k_{r\alpha}}{\mu_\alpha}$ and $\mu_\alpha = \mu_\alpha(p_\alpha)$)
- there exists function $\widetilde{p}$ such that $\nabla \widetilde{p} = f_w \nabla p_c$[1],
- strictly elliptic absolute permeability $\mathbb{K}$
- smooth enough $b_\alpha, \lambda_\alpha, \alpha = w, o$

- no incompressibility assumption: $b_\alpha \not\equiv const$
- no constant porosity assumption: $\phi \not\equiv const$

[1] Chen Z. Formulations and Numerical Methods of the Black Oil Model in Porous Media. SIAM J. Numer. Anal., 2000; 38(2):489–514.

Differential maximum principle(2). Pressure.

- $b_o q_o + b_w q_w \leq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\sup_{\Omega\times[0,T]} p \leq \sup_{\partial\Omega\times[0,T)} p$$

- $b_o q_o + b_w q_w \geq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\inf_{\Omega\times[0,T]} p \geq \inf_{\partial\Omega\times[0,T)} p$$

where $p = p_o - \widetilde{p}$.

Differential maximum principle. Assumptions(3)

- constant viscosities $\mu_\alpha = const$, $\alpha = w, o$
- incompressibility: $b_\alpha = 1, \alpha = w, o$
- constant porosity: $\phi = const$,
- relative permeabilities $k_{r\alpha}$ are monotonic functions of s_w
- p_c is monotonically decreasing function of s_w
- strictly elliptic absolute permeability $\mathbb{K}$
- smooth enough $k_{r\alpha}, \alpha = w, o$

- no constant capillary pressure assumption: $p_c \not\equiv 0$,

Differential maximum principle(3). Saturations.

- $q_w \leq 0, q_o \geq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\sup_{\Omega\times[0,T]} s_w \leq \sup_{\partial\Omega\times[0,T)} s_w, \qquad \inf_{\Omega\times[0,t]} s_o \geq \inf_{\partial\Omega\times[0,T)} s_o$$

- $q_w \geq 0, q_o \leq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\sup_{\Omega\times[0,T]} s_o \leq \sup_{\partial\Omega\times[0,T)} s_o, \qquad \inf_{\Omega\times[0,t]} s_w \geq \inf_{\partial\Omega\times[0,T)} s_w$$

Three-phase flow (no gravity)

$$
\begin{cases}
\frac{\partial}{\partial t}\left(\frac{\phi \rho_w s_w}{b_w}\right) - div\left(\frac{\rho_w}{b_w}\frac{k_{rw}}{\mu_w}\mathbb{K}\nabla p_w\right) = q_w \\
\frac{\partial}{\partial t}\left(\frac{\phi \rho_o s_o}{b_o}\right) - div\left(\frac{\rho_o}{b_o}\frac{k_{ro}}{\mu_o}\mathbb{K}\nabla p_o\right) = q_o \\
\frac{\partial}{\partial t}\left(\frac{\phi \rho_g s_g}{b_g} + \frac{r_{so}\rho_o s_o}{b_o}\right) - div\big(\frac{\rho_g}{b_g}\frac{k_{rg}}{\mu_g}\mathbb{K}\nabla p_g + \\
\qquad + \frac{r_{so}\rho_o}{b_o}\frac{k_{ro}}{\mu_o}\mathbb{K}\nabla p_o\big) = q_g \\
s_o + s_w + s_g = 1 \\
p_\alpha - p_o = p_{c\alpha o}, \alpha = g, w
\end{cases}
$$

Differential maximum principle. Assumptions(1)

- $p_{c\alpha o} \equiv 0, \alpha = w, g$
- strictly elliptic absolute permeability $\mathbb{K}$
- smooth enough $b_\alpha, \lambda_\alpha, \alpha = w, o$

- no incompressibility assumption: $b_\alpha \not\equiv const$
- no constant porosity assumption: $\phi \not\equiv const$
- no constant viscosity assumption: $\mu_\alpha \not\equiv const$

Differential maximum principle(1). Pressure

- $b_o q_o + b_w q_w + b_g q_g - b_g r_{so} q_o \leq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\sup_{\Omega\times[0,T]} p_\alpha \leq \sup_{\partial\Omega\times[0,T)} p_\alpha$$

- $b_o q_o + b_w q_w + b_g q_g - b_g r_{so} q_o \geq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\inf_{\Omega\times[0,T]} p_\alpha \geq \inf_{\partial\Omega\times[0,T)} p_\alpha, \alpha = w, o, g$$

Differential maximum principle. Assumptions(2)

- fractional flows $f_\alpha = \frac{\lambda_\alpha}{\lambda_w+\lambda_o+\lambda_g}$, $\alpha = w, g, o$ depend solely on s_w and s_o
    - (implies constant viscosities, since $\lambda_\alpha = \frac{k_{r\alpha}}{\mu_\alpha}$ and $\mu_\alpha = \mu_\alpha(p_\alpha)$)
- there exits such function $\widetilde{p}$ that $\nabla \widetilde{p} = f_w \nabla p_{cwo} + f_g \nabla p_{cgo}$[1]
- strictly elliptic absolute permeability $\mathbb{K}$
- smooth enough $b_\alpha, \lambda_\alpha, \alpha = w, o$

- no incompressibility assumption: $b_\alpha \not\equiv const$
- no constant porosity assumption: $\phi \not\equiv const$

[1] Chen Z. Formulations and Numerical Methods of the Black Oil Model in Porous Media. SIAM J. Numer. Anal., 2000; 38(2):489–514.

Differential maximum principle(2). Pressure.

- $b_o q_o + b_w q_w + b_g q_g - b_g r_{so} q_o \leq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\sup_{\Omega \times [0,T]} p \leq \sup_{\partial\Omega \times [0,T)} p$$

- $b_o q_o + b_w q_w + b_g q_g - b_g r_{so} q_o \geq 0$ in $\Omega \times [0, T] \Rightarrow$

$$\inf_{\Omega \times [0,T]} p \geq \inf_{\partial\Omega \times [0,T)} p$$

where $p = p_o + \widetilde{p}$.

Discrete maximum principle. Assumptions.

- zero capillary pressure: $p_c \equiv 0$
- constant porosity: $\phi \equiv const$
- incompressibility: $b_\alpha = 1, \alpha = w, o$
- $\mathbb{K}$ is strictly elliptic

- no constant viscosities assumption: $\mu_\alpha \not\equiv 0$

Discrete maximum principle. Pressure.

- Let $\mathcal{T}_{inj}$ be a set of cells where $q_w + q_o \geq 0$ and $\mathcal{T}_B$ be a set of boundary faces. Then

$$\max_{T \in \mathcal{T} \setminus (\mathcal{T}_{inj} \cup \mathcal{T}_B)} p_T \leq p_{max} = \max_{\mathcal{T}_{inj} \cup \mathcal{T}_B} p_T .$$

- Let $\mathcal{T}_{prod}$ be the set of cells where $q_w + q_o \leq 0$ and $\mathcal{T}_B$ be a set of boundary faces. Then

$$\min_{T \in \mathcal{T} \setminus (\mathcal{T}_{prod} \cup \mathcal{T}_B)} p_T \leq p_{min} = \min_{\mathcal{T}_{prod} \cup \mathcal{T}_B} p_T$$