Novel analytical approach to the synthesis of optimal multiband electrical filters

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History and background

E.I. Zolotarev (1847-1878, Russia) invented 'Zolotarev fractions', analogy of Chebyshev polynomials in the realm of rational functions.

W.Cauer (1900-1945, Germany) invented low and high pass 'Cauer electrical filters'

Bell Laboratories

Bell Labs (1925 - 2016, USA) implemented filters to practice of electrical engineering

Setting optimization problem

Given: $E = E^+ \cup E^- \subset \mathbb{R}$ being finite set of disjoint segments $E^\pm = \cup_j E_j^\pm$ pass $(+)$ and stop $(-)$ bands.
Competition among real rational functions $R(x) = P(x)/Q(x)$ of bounded degree $\deg R \le n$.
Goal function

1. $$\frac{\max_{x\in E^+}|R(x)|}{\min_{x\in E^-}|R(x)|} \longrightarrow \min =: \theta^2$$

2. $$\max\{\max_{x\in E^+}|R(x)|, \max_{x\in E^-} 1/|R(x)|\} \longrightarrow \min =: \theta$$

3. Minimize θ under conditions
$$\min_{x\in E_-}|Q_n(x)| \ge \theta^{-1}, \quad \max_{x\in E_+}|Q_n(x)| \le \theta.$$

4. Define the ideal transfer function $F(x) = \pm 1$ when $x \in E^\pm$
$$||R_1 - F||_{C(E)} := \max_{x\in E}|R_1(x) - F(x)| \to \min =: \mu.$$

$1/\mu = (\theta + 1/\theta)/2.$

Study of optimization problem

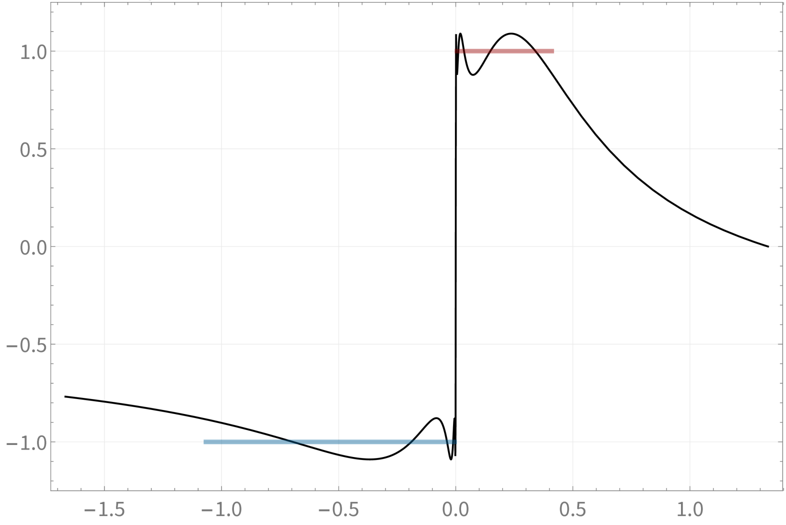
- Projective invariance
- N.I.Akhiezer 1928; R.A.-R.Amer, H.R.Schwarz, 1964; V.N.Malozemov, 1978
- Many minima; classes: fix $Sign\ Q(x)|_{E_j^+}$; $Sign\ P(x)|_{E_j^-}$
- Solution characterized by alternation (=equiripple) property: The error $\delta(x) = R(x) - F(x)$ has $2n+2$ 'alternation' points $x_s \in E$: $\delta(x_s) = \pm||\delta||_E$

In a sense the solution for this problem is simple – just manifest function with suitable equiripple property

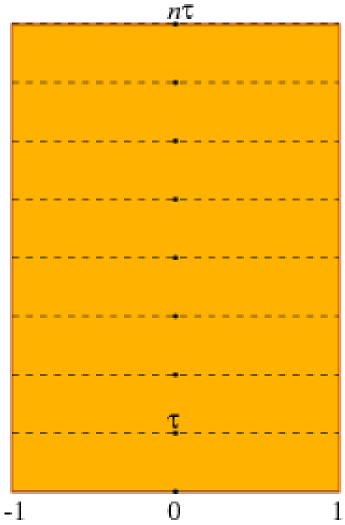
Example: Zolotarev fraction

1877: $E^{\pm} = \pm[1, 1/k]$, $0 < k < 1$, $F(x) = Sign(x)$

Solution is given in terms of elliptic functions and looks like



Example: Zolotarev fraction



$$x_\tau(u) : \Pi_\tau, -1, 0, 1 \longrightarrow \mathbb{H}, -1, 0, 1$$

$$Z_n(x_{n\tau}(u)) = x_\tau(u)$$

$$x_\tau(u) = sn(K(\tau)u|\tau)$$

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{\pi}{2}\theta_3^2(\tau)$$

$$k = (\theta_2(\tau)/\theta_3(\tau))^2$$

Ad: Z_n has lots of interesting properties!

Genesis: signal processing

Analogue	Digital
Function $x(t), t \in \mathbb{R}$	Sequence $x(n), n \in \mathbb{Z}$
$x(t) \to \boxed{RCLM..\text{-scheme}} \to y(t)$	$y(n+1) = \sum_{j=0}^{N} p_j x(n-j)$
$P_N(D)[x(t)] = Q_M(D)[y(t)]$	$+ \sum_{j=1}^{M} q_j y(n-j)$
Fourier (Laplace) transform	Z-transform (generating function)
$\hat{h}(\omega) = P_N(i\omega)/Q_M(i\omega)$	$\hat{h}(z) = P_N(z)/Q_M(z)$

In both cases the operator $x(\cdot) \to y(\cdot)$ is

- linear and real
- commutes with time shifts
- causal

Filtering

Therefore signal processing is just a convolution with h, response to the impulse $\delta-$input. In any case the spectrum of the output is that of input multiplied by rational function $\hat{h}$. For multiband filtering the magnitude response function

$$\left|\frac{\hat{y}(\omega)}{\hat{x}(\omega)}\right|^2 = |\hat{h}(\omega)|^2 = \hat{h}(\omega)\hat{h}(-\omega) = H(\omega^2)$$

should uniformly approximate 0 at the stopbands and 1 at the passbands. Two problems arise:

- Find the minimal degree filter meting given filter specification
- Given the degree, the pass- and stop- bands find filter with minimal uniform deviation.

Example Zolotarev fraction is used as a magnitude response function for the Cauer (elliptic) filters.

Approaches to optimization

- Direct numerical optimization based on Remez-type methods. Built-in instability, problems with initial approximation.

Degree	Mantissa	Source
$n \sim 10$	$1.E - 15$	Matlab toolbox
$n \sim 100$	$1.E - 2500$	S.Suetin
$n \sim 1000$	$1.E - 150000$	Yu.Matiyasevich

- Electrical engineers: decompose problem into many simple ones and use one-band (Cauer) filters. Substantial loss of degree (and therefore size, cost of production, energy consumption, cooling...)
- Ansatz method – explicit analytical formula generalizing Zolotarev fractions. [B., 2010]; [B., Goreinov, Lyamayev, 2017]

Novel approach

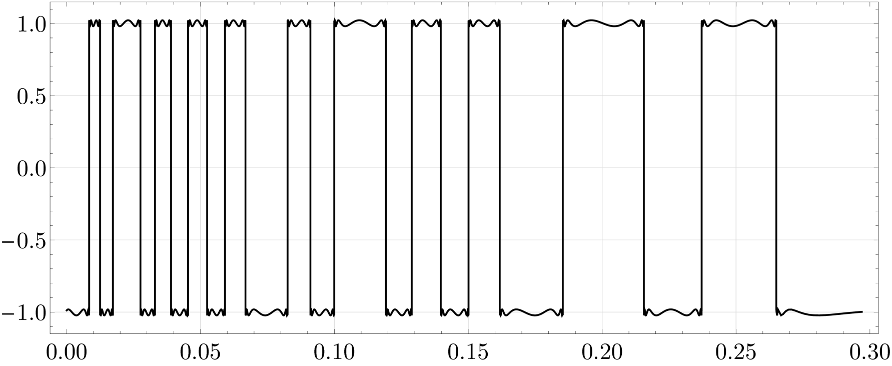
Use equiripple property of the solution: $2n+2$ alternation points is roughly equal to the number $2n-2$ of critical points of degree n rational function. This means that almost all (with $g-1$ properly counted exceptions) critical points are simple with the values in a 4-element set, e.g. $\pm\theta, \pm 1/\theta$ in the settings 1,2,3 or $\pm 1 \pm \mu$ in setting 4). This very rare property eventually gives the few-parametric representation of the solution

$$R(x) = \mathrm{sn}\left(\int_e^x d\zeta + A(e) \,\middle|\, \tau\right), \tag{1}$$

where τ is related to μ and $d\zeta$ is a holomorphic differential on a (beforehand unknown) hyperelliptic Riemann surface of genus g. The arising surface is not arbitrary: it is so called Calogero-Moser curve describing the dynamics of points on a torus interacting with potential $\wp(u)$.

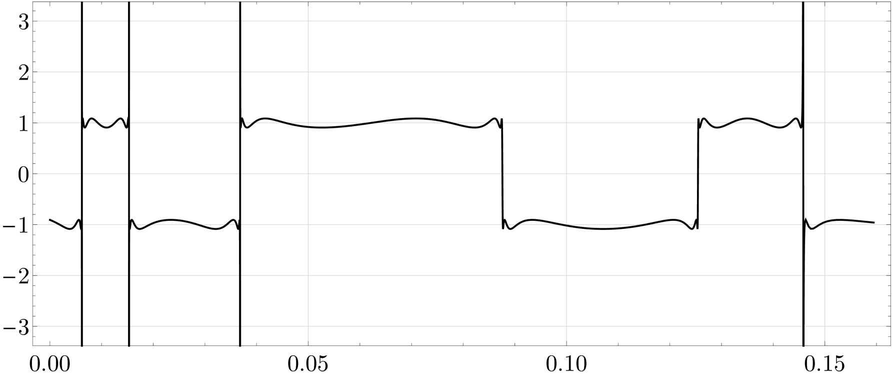
Example of filter design

Optimal magnitude response with 23 bands (computed by Sergei Lyamaev)



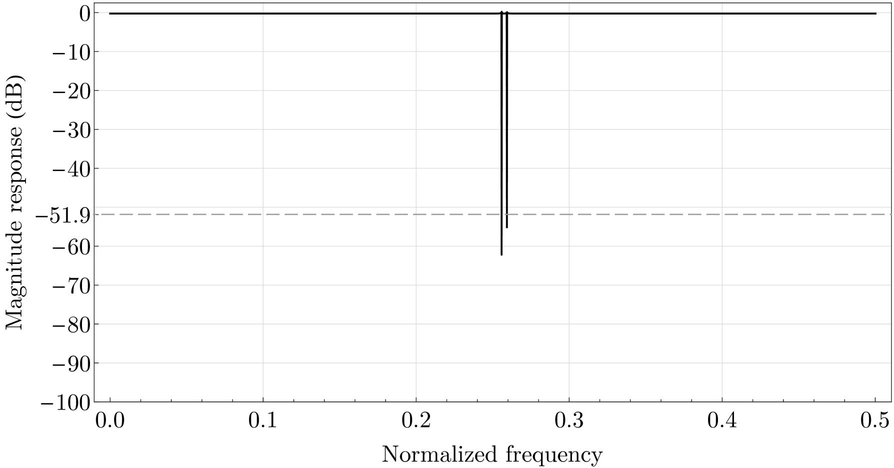
Example of filter design 2

Optimal magnitude response with 7 bands, class admits poles in transient bands (computed by Sergei Lyamaev)



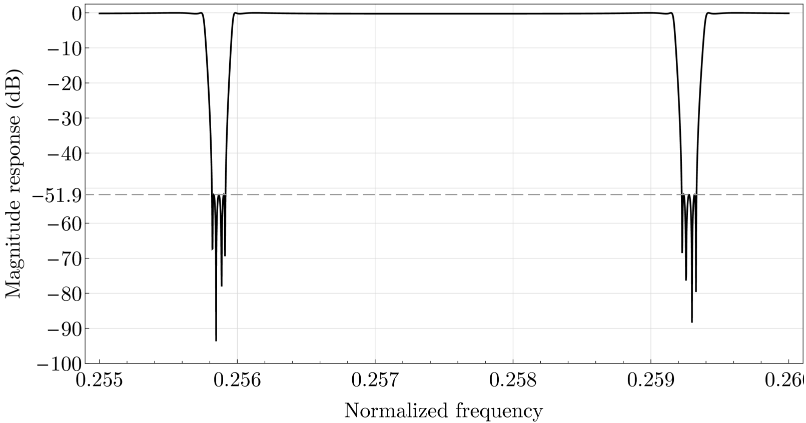
Comparison to other approach

Magnitude response of a double notch filter of degree $n = 16$ (S.Lyamaev).



Comparison to other approach, cont'd

Magnification of the previous picture at two cut off frequencies



Same specification composite filter has degree $n = 62$

Cπacuδo ζa bHumaHue!

Vielen Dank!

Thanks for the patience!