Relaxing the CFL Condition for Explicit Numerical Wave Propagation

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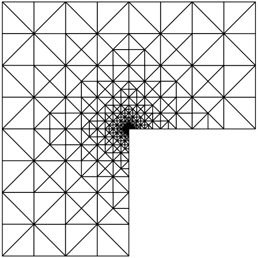
Model problem

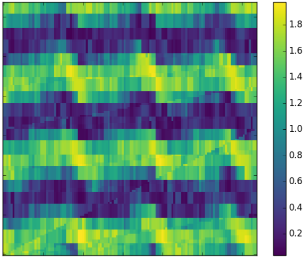
$$
\begin{aligned}
\ddot{u} - \operatorname{div} A\nabla u &= f && \text{in } (0,T)\times\Omega\\
u(0) &= u_0 && \text{in } \Omega\\
\dot{u}(0) &= v_0 && \text{in } \Omega\\
u|_{\partial\Omega} &= 0 && \text{in } (0,T)
\end{aligned}
$$

- $\Omega \subseteq \mathbb{R}^2$ bounded, Lipschitz, polygonal
- $A \in L^\infty(\Omega; \mathbb{R}^{2\times 2}_{sym})$
- $\alpha|\xi|^2 \leq A(x)\xi\cdot\xi \leq \beta|\xi|^2$ for all $\xi \in \mathbb{R}^2$, a.a. $x \in \Omega$
- $0 < \alpha \leq \beta < \infty$

Problems in the context of explicit time-stepping



(a) Adaptive mesh refinement



(b) Rough coefficients on a fine scale

$\longrightarrow$ CFL condition $\Delta t \lesssim h_{min}$ leads to small time steps

Definitions

- $V := H^1_0(\Omega)$
- $V_H \subset V$ standard P_1/Q_1-FE space based on decomposition $\mathcal{T}_H$
- $V_h \subset V$ standard P_1/Q_1-FE space based on refinement $\mathcal{T}_h$ of $\mathcal{T}_H$
- $\| \bullet \| := \| \bullet \|_{L^2(\Omega)}$

Aim: Relaxed CFL condition $\Delta t \lesssim H$
Idea: Construct coarse space $\tilde{V}_H$ which accounts for fine-scale information ($\to$ numerical homogenization)

Construction is based on $I_H : V \to V_H$ s.t.

$$\|H^{-1}(v - I_H v)\| + \|\nabla I_H v\| \leq C\|\nabla v\| \quad \text{for all } v \in V$$

and

$$\|I_H v\| \leq C\|v\| \quad \text{for all } v \in V$$

Set $W_h := \ker(I_{H|V_h})$

Define projection $\mathcal{C}v_H \in W_h$ of FE functions $v_H \in V_H$ onto W_h by

$$(\nabla \mathcal{C}v_H, A\nabla w_h)_{L^2(\Omega)} = (\nabla v_H, A\nabla w_h)_{L^2(\Omega)}$$

for all $w_h \in W_h$.

With $\tilde{V}_H := (1-\mathcal{C})V_H$, we have

$$V_h = \tilde{V}_H \oplus W_h \text{ and } (\nabla \tilde{V}_H, A\nabla W_h)_{L^2(\Omega)} = 0$$

Writing the model problem in variational form and applying the leapfrog scheme leads to:

Find $(\tilde{u}_H^n)_{n=0,..,N} \in \tilde{V}_H$, *s.t. for* $n \geq 2$

$$\begin{aligned} &\Delta t^{-2}(\tilde{u}_H^{n+1} - 2\tilde{u}_H^n + \tilde{u}_H^{n-1}, \tilde{v}_H)_{L^2(\Omega)} + (\nabla \tilde{u}_H^n, A\nabla \tilde{v}_H)_{L^2(\Omega)} \\ = \quad &(f(n\Delta t), \tilde{v}_H)_{L^2(\Omega)} \end{aligned}$$

for all $\tilde{v}_H \in \tilde{V}_H$ *and given* $\tilde{u}_H^0$ *and* $\tilde{u}_H^1$

Remarks

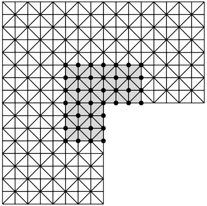
- Mass and stiffness matrices are based on the basis functions $\tilde{\Lambda}_z = (1 - \mathcal{C})\Lambda_z$, for $z \in \mathcal{N}$
- $\mathcal{C}\Lambda_z$ have global support but decay exponentially fast[1]

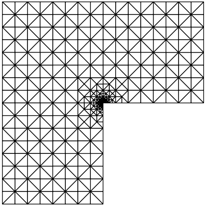
$\Longrightarrow$ Localize the correctors and solve local cell problems

[1] Målqvist and Peterseim 2014.

Setting

- $A = \mathbb{1}$
- $\Omega = (-1,1)^2 \setminus ([0,1] \times [-1,0])$
- $T = 0.5,\ \Delta t = 0.3H$
- Initial conditions and f s.t.
  $u(t,x) = \sin(\pi t)\,(r(x))^{2/3}\,\sin(2\theta(x)/3)$





[2] Peterseim and Schedensack 2017.

Theoretical result

Assumptions:

- $\Delta t \lesssim H$
- exact solution $u \in C^4$ w.r.t. time
- right-hand side $f \in C^2$ w.r.t. time
- semi-discrete solution $z_H \in C^4$ w.r.t time

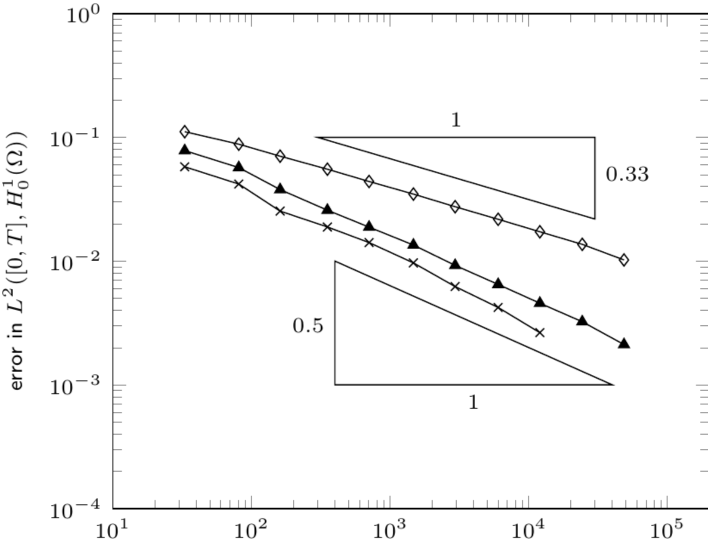
Then

$$\|\frac{\tilde{u}_H^{n+1} - \tilde{u}_H^n}{\Delta t}\| + \|\nabla \tilde{u}_H^{n+1}\|$$
$$\leq C\left(\|\frac{\tilde{u}_H^1 - \tilde{u}_H^0}{\Delta t}\| + \|\nabla \tilde{u}_H^0\| + \|\nabla \tilde{u}_H^1\| + \sum_{k=2}^{n} \Delta t \|f(k\Delta t)\|\right)$$

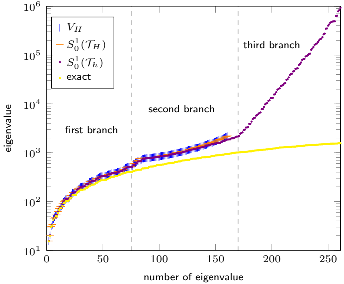
and

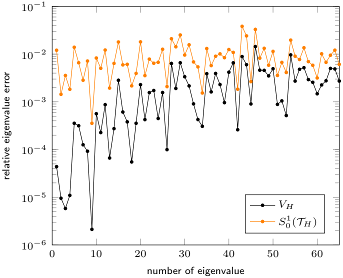
$$\|u - \tilde{u}_H\|_{L^2([0,T],H_0^1(\Omega))} \lesssim H + (\Delta t)^2$$

Error plot


10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^1
10^2
10^3
10^4
10^5
error in $L^2([0,T], H_0^1(\Omega))$
1
0.33
0.5
1

Alternative view on numerical homogenization





Setting[3]

- $A = a(x)\mathbb{1}$ with (piecewise constant) scalar function a
- $\Omega = (0, 1)^2$
- $T = 1$, $\Delta t = 0.25H$
- Initial conditions are zero and $f \equiv 1$

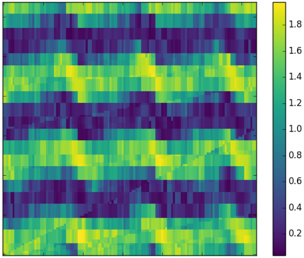
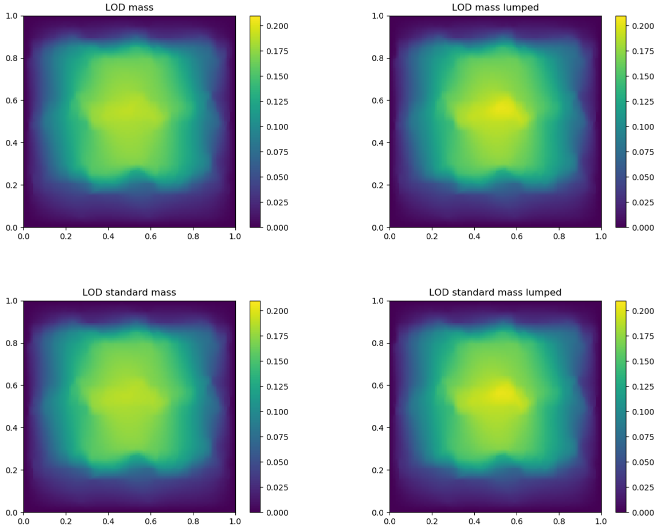


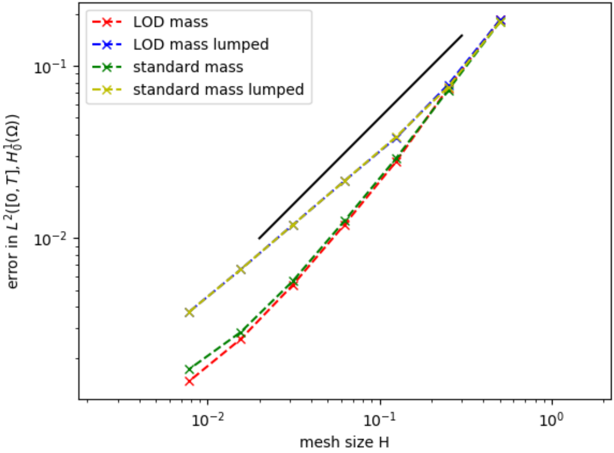
Figure: $a(x)$

[3] Abdulle and Henning 2017.

Plots ($H = 2^{-4}$, $h = 2^{-7}$, $t = 1$)



Error plot


LOD mass
LOD mass lumped
standard mass
standard mass lumped
10^{-1}
10^{-2}
error in $L^2([0,T], H_0^1(\Omega))$
10^{-2}
10^{-1}
10^{0}
mesh size H

Remarks

- symmetrized Petrov-Galerkin stiffness matrix shows similar behavior
- Coarse time steps are possible despite fine scale in space
- CFL conditions are sharp
- Open question: error estimates for rough coefficients

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- Open question: error estimates for rough coefficients

Thank you very much!

Abdulle, Assyr and Patrick Henning (2017). "Localized orthogonal decomposition method for the wave equation with a continuum of scales". In: *Mathematics of Computation* 86.304, pp. 549–587.

Målqvist, Axel and Daniel Peterseim (2014). "Localization of elliptic multiscale problems". In: *Math. Comp.* 83.290, pp. 2583–2603. ISSN: 0025-5718.

Peterseim, D. and M. Schedensack (2017). "Relaxing the CFL condition for the wave equation on adaptive meshes". In: *J. Sci. Comput.* Online First. arXiv: 1601.04812.