Computation of quasilocal effective diffusion tensors

and connections to mathematical theory of homogenization

Daniel Peterseim

Universität Augsburg

German-Russian-USA workshop, Moscow, June 13-15, 2017

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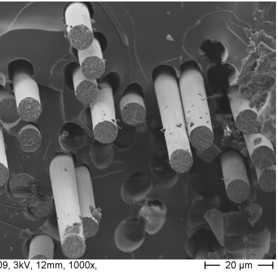
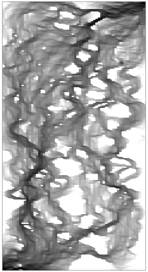
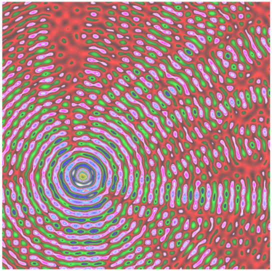
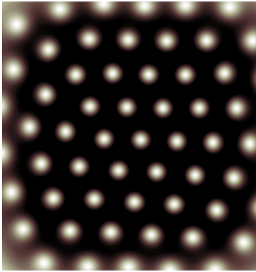
universitätbonn Institute for Numerical Simulation hausdorff CENTER FOR MATHEMATICS DFG SPP 1748

Multiscale problems









- Multi-material lightweight designs
- Transport processes in porous media
- Wave propagation in heterogeneous materials
- Vortex formation and localization effects in ultracold gases
- . . .

Characteristic features on multiple non-separable scales
$\Rightarrow$ standard numerical methods fail in under-resolved regimes

Contents

The contents of this talk is based on fruitful collaborations with

D.L. Brown (Nottingham), **D. Gallistl** (Karlsruhe),
P. Henning (Stockholm), R. Kornhuber (Berlin),
A. Målqvist (Gothenburg), R. Scheichl (Bath) and
H. Yserentant (Berlin).

Outline

Model problem

$$-\operatorname{div} A\nabla u = f \;\text{ in } D \quad \text{and} \quad u = 0 \;\text{ on } \partial D.$$

Model problem

Given data $A \in L^\infty\left(D, \mathbb{R}^{d\times d}_{\mathrm{sym}}\right)$ uniformly positive definite and $f \in L^2(D)$, find $u \in V := W^{1,2}_0(D)$ s.t.

$$\forall v \in V: \; a(u, v) := \int_D (A\nabla u) \cdot \nabla v \, \mathrm{d}x = \int_D f v \, \mathrm{d}x =: F(v).$$

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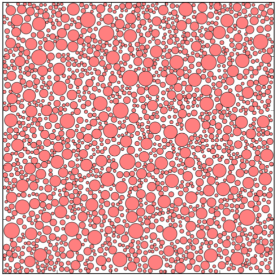
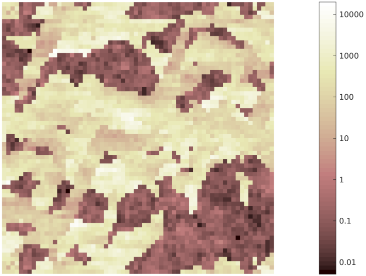
Examples (rough coefficients)



fiber reinforces composite material



porous medium (SPE10 benchmark)

Model problem

Given data $A \in L^\infty\left(D, \mathbb{R}^{d\times d}_{\mathrm{sym}}\right)$ uniformly positive definite and $f \in L^2(D)$, find $u \in V := W_0^{1,2}(D)$ s.t.

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Classical periodic homogenization:

$$\mathcal{L}_\varepsilon u_\varepsilon = f \quad \xrightarrow{\varepsilon\to 0} \quad \mathcal{L}_0 u_0 = f$$

Assuming structure such as $A(x) = A_\varepsilon = A_1(x, \frac{x}{\varepsilon})$, identify some admissible limit A_0 as $\varepsilon \to 0$ such that

$$\sup_{0\neq f\in L^2(D)} \frac{\|u_\varepsilon(f) - u_0(f)\|_{L^2(D)}}{\|f\|_{L^2(D)}} \to 0.$$

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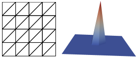

$$\rightsquigarrow \quad \mathcal{L}_{0,H} u_{0,H} = f_H$$

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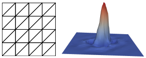
Numerical homogenization



$$\mathfrak{L}u = f \quad \rightsquigarrow \quad \mathfrak{L}_H u_H = f_H$$

Given a coarse target scale H, identify discrete effective model represented by some discrete coefficient A_H such that

$$\sup_{0\neq f\in L^2(D)} \frac{\|u(f) - u_H(f)\|_{L^2(D)}}{\|f\|_{L^2(D)}} \lesssim \sup_{0\neq f\in L^2(D)} \inf_{v_H\in V_H} \frac{\|u(f) - v_H\|_{L^2(D)}}{\|f\|_{L^2(D)}} \lesssim H$$

Some literature

⋮

- Multiscale FEM: Hou-Wu 96, ...
- Residual free bubbles: Brezzi et al. 98, ...
- Upscaling techniques: Durlofsky et al. 98, Nielsen et al. 98
- Multiscale finite volume method: Jenny et al. 03, ...
- Heterogeneous multiscale method: Engquist-E 03, Abdulle 03-...
- Equation free: Kevrekidis et al. 05
- Variational multiscale method: Målqvist 05 (Hughes 95, Hughes et al. 98), ...
- Metric based upscaling: Owhadi-Zhang 06, ...
- Flux-norm approach: Berlyand-Owhadi 10, Owhadi-Zhang 11
- GFEM based on local eigenfunctions: Babuška-Lipton 11, Efendiev-... 13-,
- AL-Basis: Sauter-Grasedyck-Greff 11
- Localized Orthogonal Decomposition: Målqvist/P. 11, Henning/P. 12, ...
- Rough polyharmonic splines: Berlyand-Owhadi-Zhang 13
- Baysian numerical homogenization: Owhadi 14
- Optimal Local Multi-scale Basis Functions: Hou-Liu 15
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Discretization scales

Macroscopic/coarse scales (H arbitrary)

- $\mathcal{T}_H$ regular coarse mesh
- $V_H \subset V$ standard P_1 FE space

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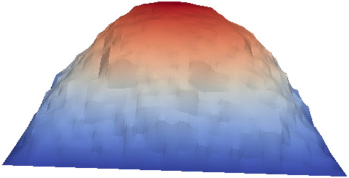
Extraction of macroscopic information by quasi-interpolation

- $I_H : V \to V_H$ quasi-local projection



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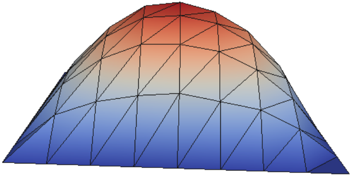

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Extraction of macroscopic information by quasi-interpolation

- $I_H : V \rightarrow V_H$ quasi-local projection
- Stability und L^2 approximation

$$H^{-1}\|v - I_H v\|_{L^2(T)} + \|\nabla I_H v\|_{L^2(T)} \leq C_{I_H}\|\nabla v\|_{L^2(\mathrm{N}(T))}$$

- Example: $I_H := E_H \circ \Pi_H$
  $\Pi_H \ldots \mathcal{T}_H$-piecewise L^2-Projektion onto $P_1(\mathcal{T}_H)$
  $E_H \ldots$ nodal averaging operator

Discretization scales

Macroscopic/coarse scales (H arbitrary)

- $\mathcal{T}_H$ regular coarse mesh
- $V_H \subset V$ standard P_1 FE space

Resolution limit ($h < \varepsilon < H$)

- $\mathcal{T}_h$ regular refinement of $\mathcal{T}_H$
- $V_h \subset V$ standard P_1 FE space

Extraction of macroscopic information by quasi-interpolation

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Scale decomposition

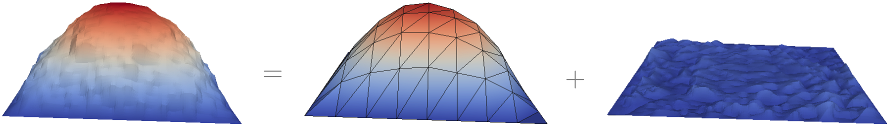
Microscopic/fine scale functions

$$W := \{v \in V \ : \ I_H(v) = 0\} = \text{kernel } I_H = \text{range}(1 - I_H)$$

Stable decomposition

$$V = V_H \oplus W$$

Example



$u \in V$ $\quad$ $I_H u \in V_H$ $\quad$ $(1 - I_H)u \in W$

a-Orthogonalization of scales

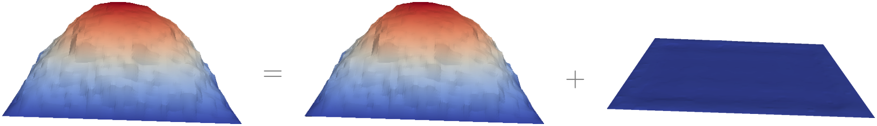
Finescale corrector $\mathcal{C}\colon V_H \to W$: Given $v_H \in V_H$, $\mathcal{C}v_H$ solves

$$a(w, \mathcal{C}v_H) = a(w, v_H) \quad \text{for all } w \in W$$

a-Orthogonal Decomposition

$$V = \underbrace{(1 - \mathcal{C})V_H}_{\text{corrected coarse space } \tilde{V}_H} \oplus W \quad \text{and} \quad a(W, (1 - \mathcal{C})V_H) = 0$$

Example



$$u \in V \quad = \quad (1 - \mathcal{C})I_H u \in \tilde{V}_H \quad + \quad \mathcal{C}u \in W$$

Ideal method

- Standard FE space V_H
- Finescale corrector $\mathcal{C} : V_H \to W$

Ideal method seeks $u_H \in V_H$ such that

$$a((1-\mathcal{C})u_H, (1-\mathcal{C})v_H) = F((1-\mathcal{C})v_H) \quad \text{for all } v_H \in V_H.$$

- Variational characterization of I_H, i.e., $u_H = I_H u$

$$a(I_H u, (1-\mathcal{C})v_H) = a(u - \underbrace{(1-I_H)u}_{\in W}, (1-\mathcal{C})v_H) = F(v)$$

- Quasi-optimality

$$\|u - u_H\| = \|(1-I_H)u\| \leq \|I_H\| \min_{v_H \in V_H} \|u - v_H\|$$

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$$\|u - u_H\| = \|(1 - I_H)u\| \leq \|I_H\| \min_{v_H \in V_H} \|u - v_H\| + H\|f\|_{L^2(D)}$$

Localization of correctors

Given $v_H \in V_H$, $\nabla v_H = \sum_{T\in\mathcal{T}_H} \sum_{j=1}^d \left(\frac{\partial}{\partial x_j} v_H|_T\right) e_j$ and

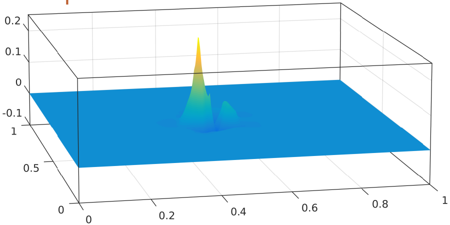
$$\mathcal{C}\ v_H = \sum_{T\in\mathcal{T}_H} \sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j} v_H|_T\right) q_{T,j}$$

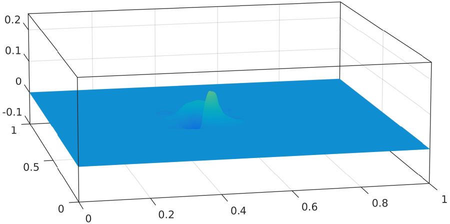
with correctors $q_{T,j} \in W$ that solve the

"Cell" problem

$$\int_D (A\nabla w)\cdot\nabla q_{T,j}\,dx = \int_T (A\nabla w)\cdot e_j\,dx \quad \text{for all } w \in W .$$

Example





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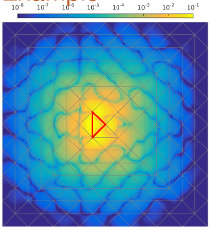

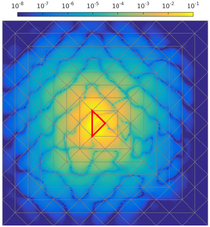

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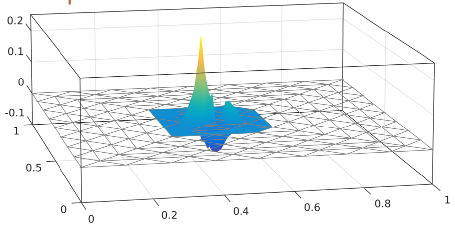
Cell problem

$$\int_{\mathrm{N}^\ell(T)} (A\nabla w)\cdot\nabla q_{T,j}^\ell \, dx = \int_T (A\nabla w)\cdot e_j \, dx \quad \text{for all } w \in W(\mathrm{N}^\ell(T)).$$

Example

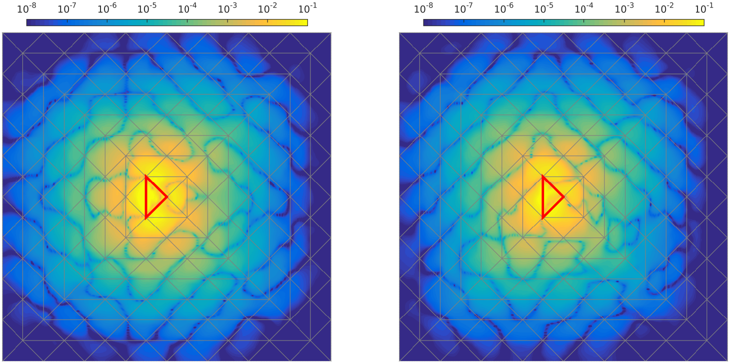


Exponential decay of correctors

Theorem

There exists $c > 0$ (that only depends on contrast) such that, for any T and $j = 1 \ldots, d$ and $\ell \in \mathbb{N}$,

$$\|\nabla q_{T,j}\|_{L^2(D \setminus B_{\ell H}(z))} \lesssim \exp(-c\ell)\,\sqrt{|T|}.$$



A. Målqvist, D. P. Localization of elliptic multiscale problems *Mathematics of Computation, 2014*

R. Kornhuber, D.P., H. Yserentant An Analysis of a class of variational multiscale methods based on subspace decomposition

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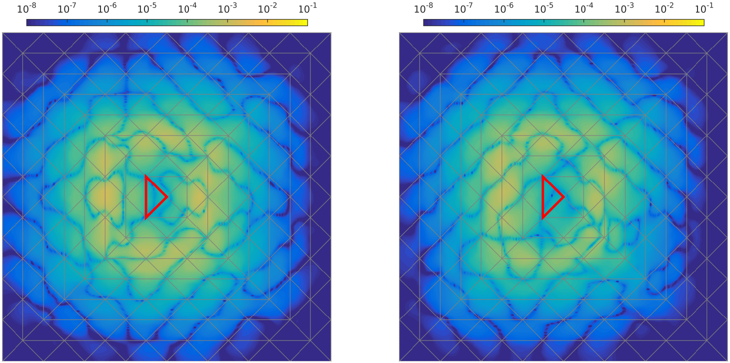

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Quasi-local homogenization method

- Standard FE space V_H
- Localized finescale corrector $\mathcal{C}_\ell : V_H \to W$

Quasi-local homogenization method (msPG) seeks $u_{H,\ell} \in V_H$ such that

$$a(u_{H,\ell}, (1 - \mathcal{C}_\ell) v_H) = F(v_H) \quad \text{for all } v_H \in V_H.$$

Quasi-optimality

Theorem

The quasi-local homogenization method satisfies

$$\sup_{0\neq f\in L^2(D)} \frac{\|u(f) - u_{H,\ell}(f)\|_{L^2(D)}}{\|f\|_{L^2(D)}} \lesssim \sup_{0\neq f\in L^2(D)} \min_{v_H\in V_H} \frac{\|u(f) - v_H\|_{L^2(D)}}{\|f\|_{L^2(D)}} + H^2 + e^{-c\ell}.$$

Hence, the oversampling condition $\ell \gtrsim |\log H|$ implies optimal convergence rates in $L^2(D)$.

The theorem holds without any assumptions on periodicity, scale separation or regularity!

D. Gallistl, D. P. Computation of local and quasi-local effective diffusion tensors in elliptic homogenization. *ArXiv, 2016*

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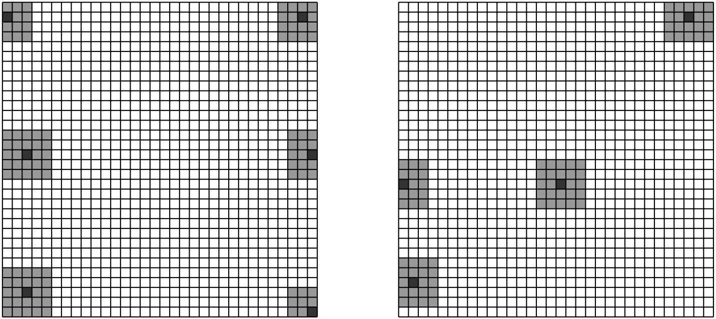

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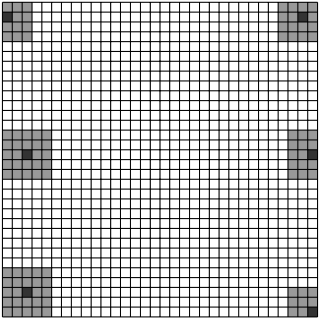

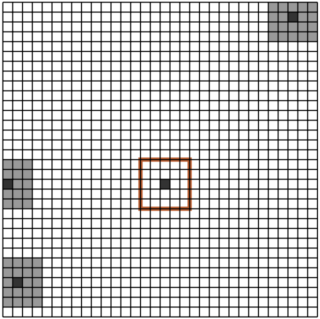

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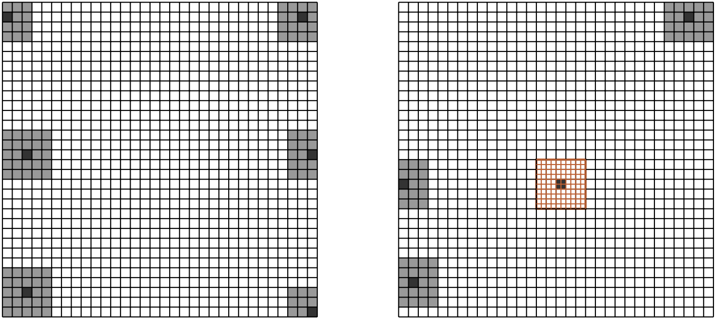

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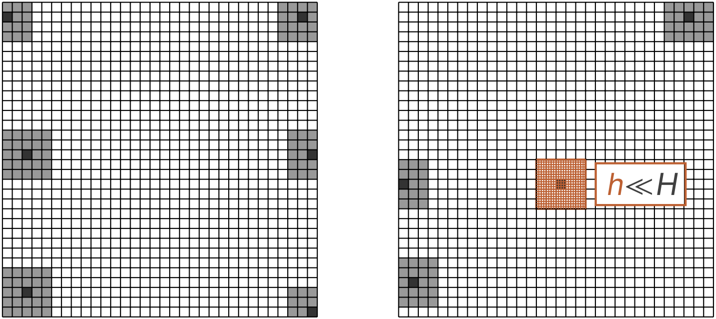
$$\sup_{0\neq f\in L^2(D)} \frac{\|u(f) - u_{H,\ell}(f)\|_{L^2(D)}}{\|f\|_{L^2(D)}} \lesssim \sup_{0\neq f\in L^2(D)} \min_{v_H\in V_H} \frac{\|u(f) - v_H\|_{L^2(D)}}{\|f\|_{L^2(D)}} + H^2 + e^{-c\ell}.$$

Hence, the oversampling condition $\ell \gtrsim |\log H|$ *implies optimal convergence rates in* $L^2(D)$.



Quasi-optimality

Theorem

The quasi-local homogenization method satisfies

$$\sup_{0\neq f\in L^2(D)} \frac{\|u_h(f) - u_{H,\ell,h}(f)\|_{L^2(D)}}{\|f\|_{L^2(D)}} \lesssim \sup_{0\neq f\in L^2(D)} \min_{v_H\in V_H} \frac{\|u_h(f) - v_H\|_{L^2(D)}}{\|f\|_{L^2(D)}} + H^2 + e^{-c\ell}.$$

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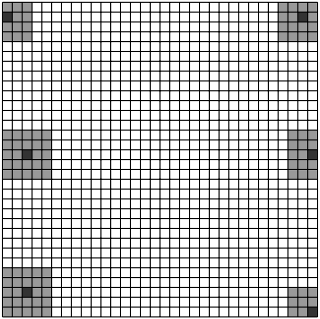

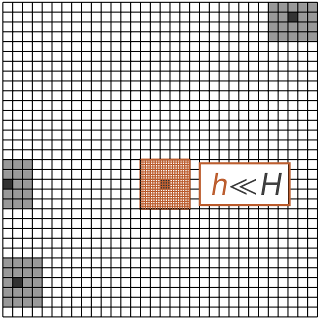

Error of corrected approximation

Correction: $u_{H,\ell} \longrightarrow (1-\mathcal{C}_\ell)u_{H,\ell}$ (or Galerkin in $(1-\mathcal{C}_\ell)V_H$)

Theorem

The quasi-local homogenization method satisfies

$$\sup_{0\neq f\in L^2(D)} \frac{\|u(f)-(1-\mathcal{C}_\ell)u_{H,\ell}(f)\|_{L^2(D)}}{\|f\|_{L^2(D)}} \lesssim H^2 + e^{-c\ell}.$$

Hence, the oversampling condition $\ell \gtrsim |\log H|$ implies optimal convergence rates in $L^2(D)$.

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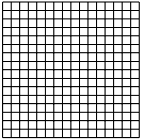
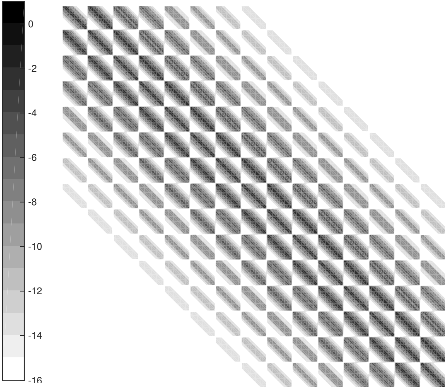
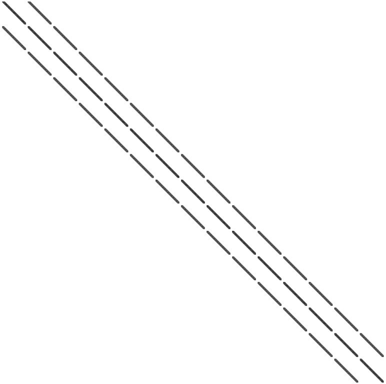
Hence, the oversampling condition $\ell \gtrsim |\log H|$ implies optimal convergence rates in $H^1(D)$.

Matrix perspective



- S_h/S_H FE stiffness matrices on fine/coarse scale
- I_H Matrix representation of quasi-interpolation





$$S_H^{eff} = (I_H \cdot S_h^{-1} \cdot I_H^T)^{-1} \quad (\ell = \infty)$$

$$S_H \approx I_H \cdot S_h \cdot I_H^T$$

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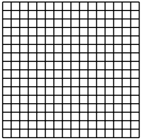

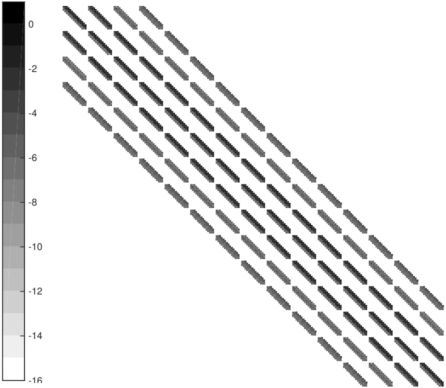

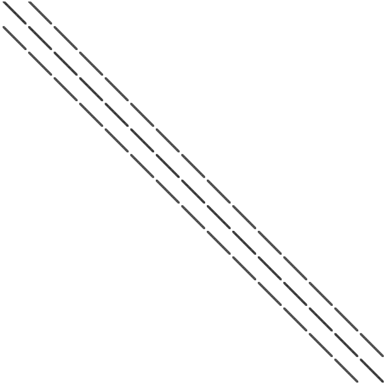

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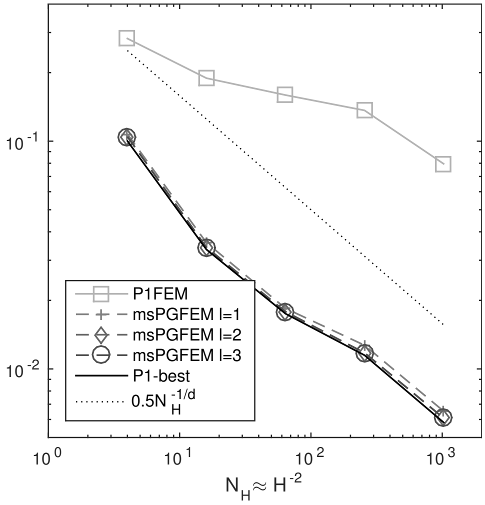
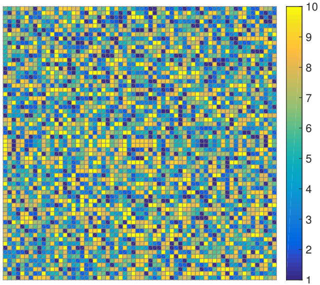
Numerical experiment

$D = [0, 1]^2$, $f \equiv 1$, A as depicted





- H variable
- ℓ variable
- $h = 2^{-9}$

D. P. Variational Multiscale Stabilization and the Exponential Decay of Finescale Correctors. *LNCSE*, 2016.

Numerical experiment: high contrast

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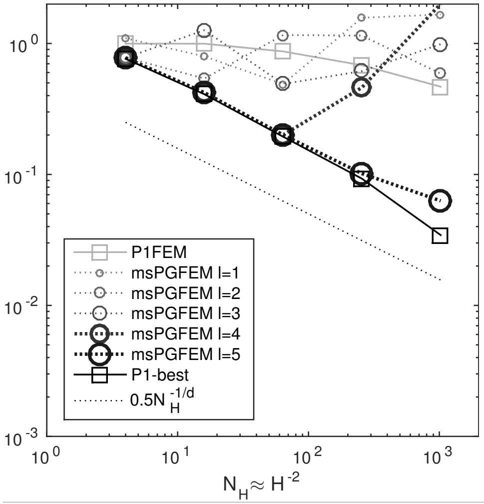

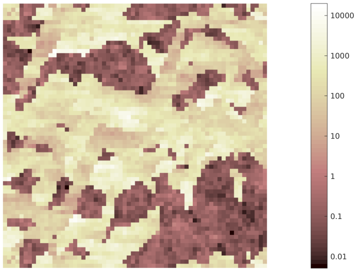

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D. P., R. Scheichl. Rigorous Numerical Upscaling at High Contrast. *CMAM*, 2016.

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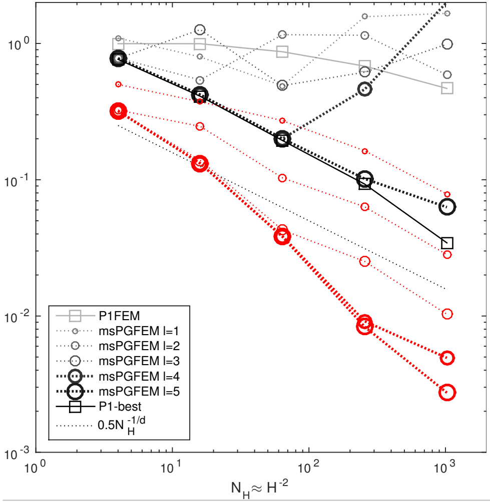

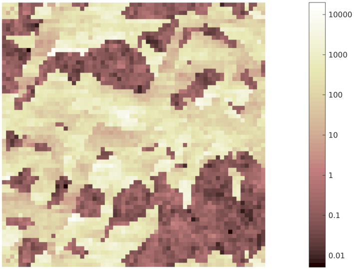

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Outline

Quasi-local method revisited

- Standard FE space V_H
- Localized finescale corrector $\mathcal{C}_\ell : V_H \to W$,

$$\mathcal{C}_\ell v_H = \sum_{T\in\mathcal{T}_H} \sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j} v_H|_T \right) q_{T,j}^{\ell}$$

with correctors $(\mathrm{N}^\ell(T))$ that solve the cell problem

$$\int_{\mathrm{N}^\ell(T)} (A\nabla w)\cdot \nabla q_{T,j}^{\ell}\, dx = \int_T (A\nabla w)\cdot e_j\, dx \quad \text{for all } w \in W(\mathrm{N}^\ell(T)).$$

Quasi-local homogenization method (msPG) seeks $u_{H,\ell} \in V_H$ such that

$$a(u_{H,\ell}, (1-\mathcal{C}_\ell)v_H) = F(v_H) \quad \text{for all } v_H \in V_H.$$

Reinterpretation: integral operator

How can we interpret the form

$$a(v_H, (1 - \mathcal{C}^\ell) z_H)$$

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$$\mathfrak{a}^\ell(v_H, z_H) := \int_D \int_D (\mathcal{A}_H^\ell(x, y) \nabla v_H(x)) \cdot \nabla z_H(y)\, dy\, dx$$

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with quasi-local effective coefficient/kernel $\mathcal{A}_H^\ell \in P_0(\mathcal{T}_H \times \mathcal{T}_H, \mathbb{R}^{d\times d})$:

$$\left[\mathcal{A}_H^\ell|_{T,K}\right]_{jk} := \delta_{T,K} \frac{1}{|T|} \int_T A_{jk}\, dx - \frac{1}{|T|} \int_K (Ae_k) \cdot \nabla q_{T,j}^\ell\, dx.$$

(skip construction and jump to slide 19)

Construction of the kernel

$$\int_D \nabla u_H \cdot (A \nabla \mathcal{C}^\ell v_H)\, dx = \sum_{T\in\mathcal{T}_H} \sum_{j=1}^{d} (\partial_j v_H|_T) \int_D \nabla u_H \cdot (A \nabla q_{T,j}^\ell)\, dx$$

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$$= \sum_{K, T \in \mathcal{T}_H} \int_K \nabla u_H \cdot \left(\sum_{j=1}^{d} \fint_K (A \nabla q^\ell_{T,j})\, dx\, (\partial_j v_H|_T) \right) dx$$

Construction of the kernel

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\int_D \nabla u_H \cdot (A \nabla \mathcal{C}^\ell v_H)\, dx &= \sum_{T \in \mathcal{T}_H} \sum_{j=1}^{d} (\partial_j v_H|_T) \int_D \nabla u_H \cdot (A \nabla q^\ell_{T,j})\, dx \\
&= \sum_{K,T \in \mathcal{T}_H} \int_K \nabla u_H \cdot \left(\sum_{j=1}^{d} \fint_K (A \nabla q^\ell_{T,j})\, dx \, (\partial_j v_H|_T) \right) dx \\
&= \sum_{K,T \in \mathcal{T}_H} \int_K \int_T \nabla u_H|_K \cdot (\mathcal{K}^\ell_{T,K} \nabla v_H|_T)\, dy\, dx
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$$

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$$\int_D \nabla u_H \cdot (A \nabla \mathcal{C}^\ell v_H)\, dx = \sum_{T \in \mathcal{T}_H} \sum_{j=1}^{d} (\partial_j v_H|_T) \int_D \nabla u_H \cdot (A \nabla q_{T,j}^\ell)\, dx$$

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$$= \sum_{K,T \in \mathcal{T}_H} \int_K \int_T \nabla u_H|_K \cdot (\mathcal{K}_{T,K}^\ell \nabla v_H|_T)\, dy\, dx$$

for the matrix

$$(\mathcal{K}_{T,K}^\ell)_{j,k} := \frac{1}{|T|\,|K|} e_k \cdot \int_K A \nabla q_{T,j}^\ell\, dx.$$

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Define the piecewise constant matrix field over $\mathcal{T}_H \times \mathcal{T}_H$, for $T, K \in \mathcal{T}_H$ by

$$\mathcal{A}^\ell_H|_{T,K} := \delta_{T,K} \fint_T A\,dx - \mathcal{K}^\ell_{T,K}.$$

Compression

We have

$$\begin{aligned} \mathfrak{a}^\ell(v_H, w_H) &= a(v_H^\ell, (1 - \mathcal{C}^\ell) w_H) \\ &= \int_D \int_D \nabla v_H(x) \cdot (\mathcal{A}_H^\ell(x,y) \nabla w_H(y))\, dy\, dx. \end{aligned}$$

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Effective coefficient A_H^{ℓ} is piecewise constant on the scale H.

Local effective coefficient and method

- Quasi-local effective coefficient $\mathcal{A}_H^\ell \in P_0(\mathcal{T}_H, \mathbb{R}^{d\times d}) \times P_0(\mathcal{T}_H, \mathbb{R}^{d\times d})$
- Correctors $q_{T,j}^\ell$

Local effective coefficient $A_H^\ell \in P_0(\mathcal{T}_H, \mathbb{R}^{d\times d})$:

$$\left[A_H^\ell|_T\right]_{jk} = \sum_K |K| \left[\mathcal{A}_H^\ell|_{T,K}\right]_{jk} := \frac{1}{|T|} \int_{\mathrm{N}^\ell(T)} (Ae_k) \cdot (\chi_T e_j - \nabla q_{T,j}^\ell)\, dx$$

Local homogenization method seeks $\tilde{u}_{H,\ell} \in V_H$ such that

$$\int_D \nabla \tilde{u}_H^\ell \cdot (A_H^\ell \nabla v_H)\, dx = F(v_H) \quad \text{for all } v_H \in V_H.$$

D. Gallistl, D. P. Computation of local and quasi-local effective diffusion tensors in elliptic homogenization. *ArXiv, 2016*

Well-posedness / regularized coefficient

Spectral bounds

$$\alpha_H|\xi|^2 \leq \xi \cdot (A_H^\ell(x)\xi) \leq \beta_H|\xi|^2$$

must be verified a posteriori (but hold true with $\alpha_H \approx \alpha$ and $\beta_H \approx \beta$ for slightly perturbed compression).

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Regularized coefficient

There exists a Lipschitz continuous coefficient $A_H^{\text{reg}} \in W^{1,\infty}(D;\mathbb{R}^{d\times d})$ with the following properties:

1. $\int_T A_H^{\text{reg}}\,dx = \int_T A_H^\ell\,dx$ for all $T \in \mathcal{T}_H$.
2. The eigenvalues of $\operatorname{sym}(A_H^{\text{reg}})$ lie in $[\alpha_H/2, 2\beta_H]$.
3. $\|\nabla A_H^{\text{reg}}\|_{L^\infty(D)} \leq C \underbrace{H^{-1}\|[A_H^\ell]\|_{L^\infty(\mathcal{F}_H)}\left(1+\alpha_H^{-1}\|[A_H^\ell]\|_{L^\infty(\mathcal{F}_H)}\right)}_{=:\eta(A_H^\ell)\text{ model-error estimator}}$

Error estimate

Let u^{reg} denote the solution to the PDE with regularized coefficient $A_H^{\mathrm{reg}} \in W^{1,\infty}(D;\mathbb{R}^{d\times d})$.

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Proposition

Technical assumptions: *Let $d = 2$ and D convex. Let $1 \leq p \leq 2$ such that for all interior angles ω of D there holds $2\omega/(p\pi) \notin \mathbb{Z}$. Furthermore, let $q \in [2,\infty)$ with $1/p + 1/q = 1$.*

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Result: *If the solution u^{reg} has smoothness $W^{1+s,q}(D)$ for $0 < s \le 1$ and $f \in L^q(D)$ and $\ell \approx |\log H|$, then*

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$$\|u - \tilde{u}_H^\ell\|_{L^2(D)} \lesssim \left(H + H^{s-2(p-1)/p} |\log H|^2 \left(1 + \eta(A_H^\ell)\right)^{2s}\right) \|f\|_{L^q(D)}.$$

D. Gallistl, D. P. Computation of local and quasi-local effective diffusion tensors in elliptic homogenization. *ArXiv, 2016*

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A posteriori homogenization criterion

$$H^{-1}\left\|[\![A_H^\ell]\!]\right\|_{L^\infty(\mathcal{F}_H)} \lesssim 1 \quad \text{and} \quad \alpha_H^{-1} H \lesssim 1$$

implies almost optimal error estimates (without regularity of A!).

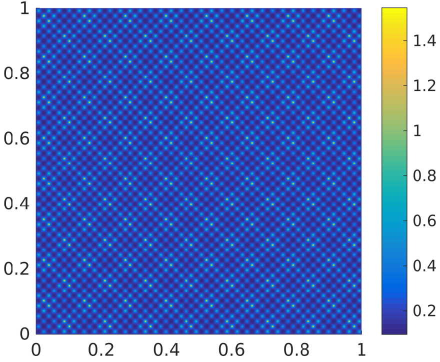
D. Gallistl, D. P. Computation of local and quasi-local effective diffusion tensors in elliptic homogenization. *ArXiv, 2016*

Experiment 1: Setup

Domain $D = (0,1)^2$. Scalar coefficient

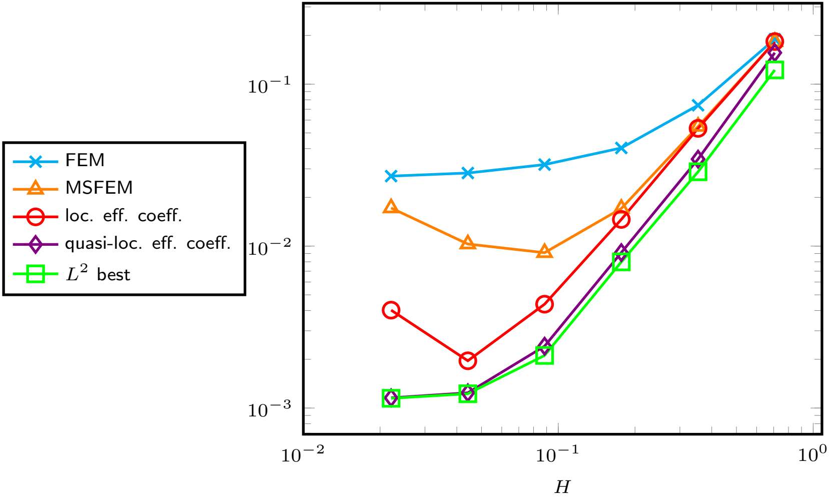
$$A(x_1, x_2) = \left(\frac{11}{2} + \sin\left(\frac{2\pi}{\varepsilon_1}x_1\right)\sin\left(\frac{2\pi}{\varepsilon_1}x_2\right) + 4\,\sin\left(\frac{2\pi}{\varepsilon_2}x_1\right)\sin\left(\frac{2\pi}{\varepsilon_2}x_2\right)\right)^{-1}$$

with $\varepsilon_1 = 2^{-3}$ and $\varepsilon_2 = 2^{-5}$.



Worst-case error under mesh-refinement

Error quantity $\sup_{f\in L^2(D)\setminus\{0\}} \frac{\|u(f)-\tilde{u}_H^{(2)}(f)\|_{L^2(D)}}{\|f\|_{L^2(D)}}$.



Estimator and spectral bounds

- Spectral bounds of A range within $[0.096, 1.55]$.
- Results:

H	$\eta(A_H^\ell)$	α_H	β_H
$\sqrt{2}\times 2^{-1}$	3.2108e-02	1.9223e-01	2.0786e-01
$\sqrt{2}\times 2^{-2}$	1.1267e-02	1.9568e-01	1.9954e-01
$\sqrt{2}\times 2^{-3}$	1.4765e-02	1.9579e-01	1.9986e-01
$\sqrt{2}\times 2^{-4}$	5.3952e-01	1.8323e-01	2.1992e-01
$\sqrt{2}\times 2^{-5}$	1.7199e+00	1.6909e-01	2.3257e-01
$\sqrt{2}\times 2^{-6}$	1.5538e+01	1.4070e-01	3.0277e-01

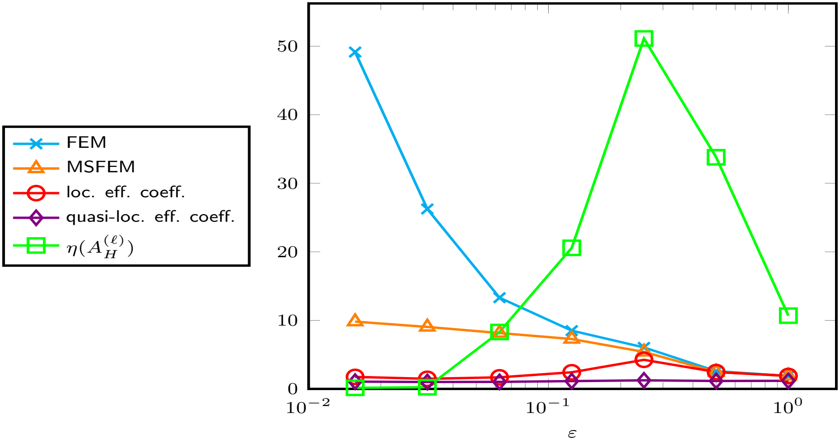
- We observe “resonance effects” close to the fine scale.

Experiment 2: Resonance effects

Consider a fixed mesh with mesh-size $H = \sqrt{2} \times 2^{-4}$ and

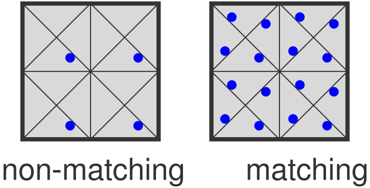
$$A(x_1, x_2) = \left(5 + 4\,\sin\left(\frac{2\pi}{\varepsilon}x_1\right)\sin\left(\frac{2\pi}{\varepsilon}x_2\right)\right)^{-1}$$

for a sequence of parameters $\varepsilon = 2^0, 2^{-1}, \ldots, 2^{-6}$.



Outline

Periodic homogenization



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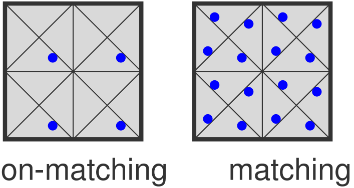

The periodic case $V = H^1_\#(D)/\mathbb{R}$

- Effective coefficient A_H is constant and independent of H.
- A_H coincides with the classical homogenization limit where the corrector $\hat{q}_k \in H^1_\#(D)/\mathbb{R}$ solves

$$\operatorname{div} A(\nabla \hat{q}_k - e_k) = 0 \text{ in } D \text{ with periodic b.c.}$$

Quantified homogenization limit

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Proposition

Any $1 \leq p \leq 2$ *und* $2 \leq q \leq \infty$ *with* $1/p + 1/q = 1$ *satisfy*

$$||u_\varepsilon - u_0||_{L^2(D)} \lesssim \varepsilon^{1-d(p-1)/p} \log|\varepsilon|^{(d+q)/q+d} ||f||_{L^q(D)}.$$

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$$\|u_\varepsilon - u_0\|_{L^2(D)} \lesssim \varepsilon^{1-d(p-1)/p} \log|\varepsilon|^{(d+q)/q+d} \|f\|_{L^q(D)}.$$

Corollary

If $f \in L^\infty(D)$ and $u_\varepsilon \in W^{2,\infty}(D)$, then one has a linear rate (up to log-factors).

Outline

Model problem

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space
- D physical domain
- Diffusion coefficient $\mathbf{A} = \mathbf{A}_\varepsilon$ is a random variable $\mathbf{A} \in L^2(\Omega; \mathcal{M}(D, \alpha, \beta))$

Seek $\mathbf{u} \in L^2(\Omega; V)$ such that

$$-\operatorname{div}(\mathbf{A}\nabla\mathbf{u}) = f \quad \text{in } D \quad \mathbb{P}\text{-a.s.}$$

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Goal: Approximation by a deterministic model.

Numerical method

- For any atom $\omega \in \Omega$, the quasi-local and local coefficients $\mathcal{A}_H(\omega)$ and $\mathbf{A}_H(\omega)$ are defined as in the deterministic case.

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Quasi-local method seeks $u_H \in V_H$ such that, for all $v_H \in V_H$,

$$\int_D \int_D \nabla u_H(x) \cdot (\bar{\mathcal{A}}_H(x,y) \nabla v_H(y))\, dy\, dx = (f, v_H)_{L^2(D)}.$$

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Local method seeks $\tilde{u}_H \in V_H$ such that

$$\int_D \nabla \tilde{u}_H \cdot (\bar{A}_H \nabla v_H)\, dx = (f, v_H)_{L^2(D)} \quad \text{for all } v_H \in V_H.$$

Model error estimator

For any $T \in \mathcal{T}_H$, denote

$$X(T) := \max_{\substack{K \in \mathcal{T}_H \\ K \cap \mathrm{N}^{\ell}(T) \neq \emptyset}} |T| \left| \mathcal{A}_H|_{T,K} - \bar{\mathcal{A}}_H|_{T,K} \right|.$$

The model error estimator γ is defined by

$$\gamma := \max_{T \in \mathcal{T}_H} \left(\sqrt{\mathbb{E}[X(T)^2]} \right).$$

Error analysis for the quasi-local method

Proposition

The quasi-local method satisfies

$$\sqrt{\mathbb{E}[\|\mathbf{u} - u_H\|^2_{L^2(D)}]} \lesssim (H^2 + \mathbb{E}[\mathrm{wcba}(\mathbf{A}, \mathcal{T}_H)] + |\log H|^d \gamma)\|f\|_{L^2(D)}$$

and

$$\|\mathbb{E}[\mathbf{u}] - u_H\|_{L^2(D)} \lesssim (H^2 + \mathbb{E}[\mathrm{wcba}(\mathbf{A}, \mathcal{T}_H)] + |\log H|^{2d} \gamma^2)\|f\|_{L^2(D)},$$

where

$$\mathrm{wcba}(\mathbf{A}(\omega), \mathcal{T}_H) := \sup_{g \in L^2(\Omega)\setminus\{0\}} \inf_{v_H \in V_H} \frac{\|u(\mathbf{A}(\omega), g) - v_H\|_{L^2(\Omega)}}{\|g\|_{L^2(\Omega)}}.$$

Error analysis for the fully local method

Proposition

The local method satisfies for $d = 2$, D convex, and $u^{reg} \in W^{1+s,q}(D)$ the error estimate

$$\sqrt{\mathbb{E}[\|\mathbf{u} - \tilde{u}_H\|^2_{L^2(D)}]}$$

$$\lesssim |\log H|^2 \Big(H + \gamma + H^{s-2q} \left(1 + \eta(\bar{A}_H^{(\ell)})\right)^{1+s} \Big) \|f\|_{L^q(D)}.$$

Numerical experiment

Setup: $D = (0,1)^2$, $f \equiv 1$, $\mathbf{A}$ i.i.d. w.r.t. mesh $\mathcal{T}_\varepsilon$ with mesh-size ε and uniformly distributed in the interval $[\alpha, \beta] = [1, 10]$. (Reference mesh $h = 2^{-9}\sqrt{2}$.)

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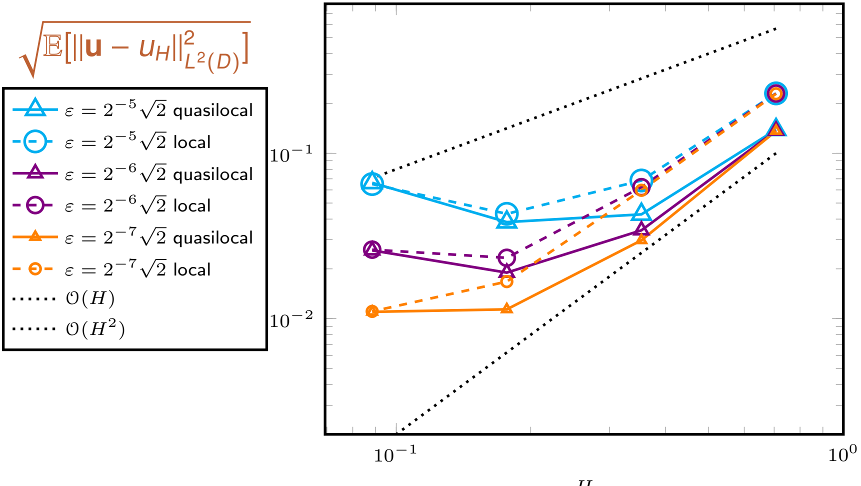

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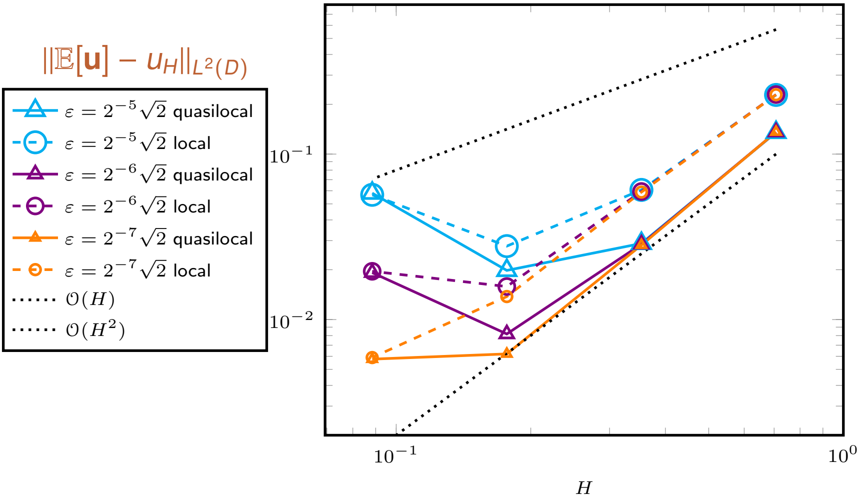
$\|\mathbb{E}[\mathbf{u}] - u_H\|_{L^2(D)}$



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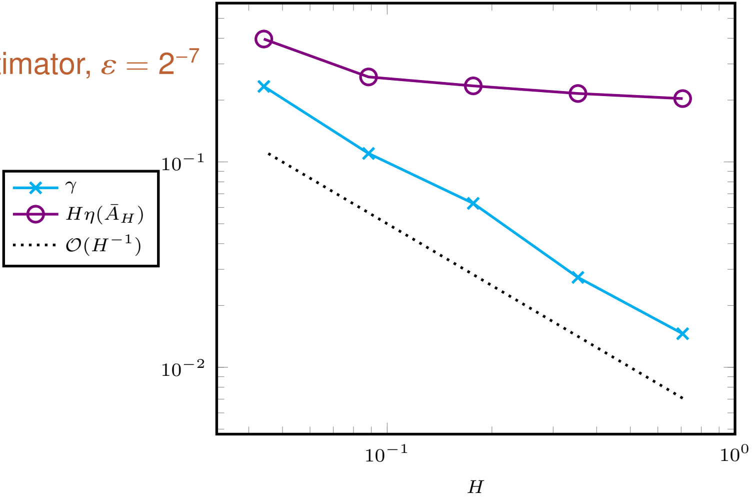
Estimator, $\varepsilon = 2^{-7}$



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Conclusion

- Numerical homogenization of general L^∞ coefficients by quasi-local discrete model method
- Under moderate assumptions on the discretization parameters, the method is stable and quasi-optimal
- Under further assumptions, truly local model possible with a posteriori model error estimator
- In the periodic case, the numerical approach coincides with classical homogenization
- Error analysis in the stochastic setting with a priori and a posteriori terms

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Thank you for your attention!

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