

NATURAL METRIC AND ANISOTROPIC MESH FOR ANISOTROPIC ELLIPTIC EQUATION

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Scientific Computing

Outline

1 Motivation

2 Anisotropic mesh generation

- Main observation
- Metric tensor
- Anisotropic Centroidal Voronoi Tessellation
- ACVT mesh generation algorithm

3 Numerical examples

- Numerical Experiments with Known Exact Solutions
- Numerical Experiments with Unknown Exact Solutions

4 Explicit Polynomial Recovery on ACVT

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Anisotropic elliptic equation

We consider the anisotropic elliptic equation in two dimensions:

$$\begin{cases} -\nabla \cdot (A \nabla u) = f, & \text{in } \Omega, \\ u = g, & \text{on } \Gamma, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary Γ , the anisotropic diffusion $A \in \mathbb{R}^{2 \times 2}$ is symmetric positive definite matrix,

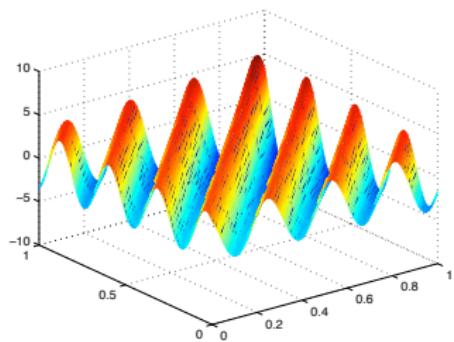
$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

The $\sqrt{\frac{\lambda_2}{\lambda_1}}$ represents the strength of the anisotropy.

Anisotropic solution

$$A = \begin{pmatrix} 1001/2000 & 999/2000 \\ 999/2000 & 1001/2000 \end{pmatrix} \quad f = 1$$

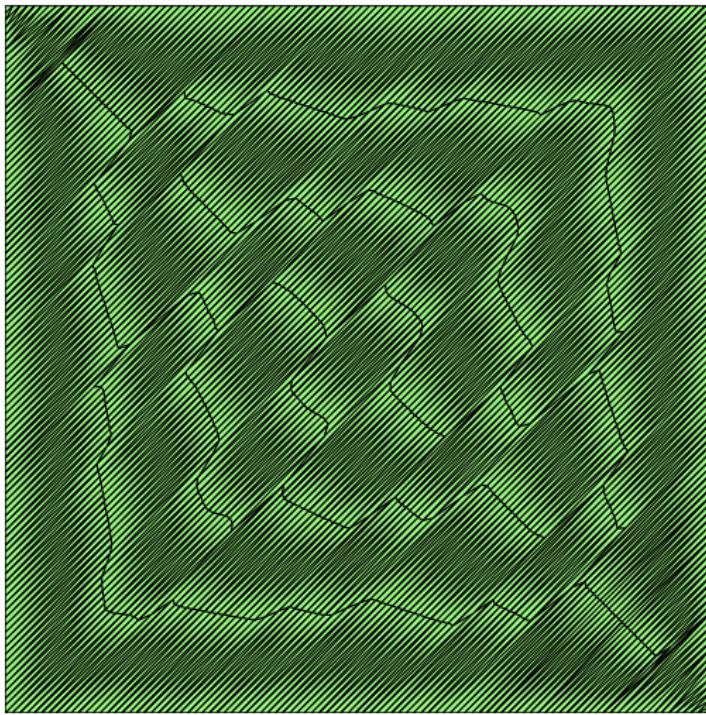
$$u = \left(e^{\frac{\sqrt{2}(x+y)}{2}} + e^{-\frac{\sqrt{2}(x+y)}{2}} \right) \left(\sin \frac{y-x}{\sqrt{0.002}} + \cos \frac{y-x}{\sqrt{0.002}} \right) - \frac{(x+y)^2}{4}$$



It is natural to require an anisotropic mesh which resolve the anisotropic solution:

- what is the "perfect" (most suitable) mesh?
- how to define and generate the anisotropic mesh?

Example of Anisotropic mesh, ACVT



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Anisotropic mesh generation

There are two main issues in anisotropic mesh generation:

- ♠ define a proper metric tensor such that the corresponding mesh's elements are aligned to the anisotropy of the solution.
- ♣ efficiently generate an anisotropic mesh with high element quality in the sense that the elements are well shaped according to the chosen metric tensor.

ANSWER:

- ♡ Metric $M = A^{-1}$, the inverse of anisotropic diffusion matrix.
- ◇ Anisotropic mesh generation: based on ACVT.

Anisotropic mesh generation

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ANSWER:

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- ◊ Anisotropic mesh generation: based on ACVT.

Main observation

By choosing $M = A^{-1}$:

- (i) The resulting discrete algebraic system is the best conditioned system in the sense of condition numbers and more efficient on linear solvers.
- (ii) The finite element solution of the anisotropic elliptic problem is more accurate on the anisotropic mesh with $M = A^{-1}$ in comparison with other selected metrics. This means that the anisotropic mesh generated with metric $M = A^{-1}$ can efficiently resolve the anisotropic behavior of the solution.
- (iii) More importantly, there are superconvergence of the linear finite element solution on the anisotropic mesh generated with metric $M = A^{-1}$.

Since the negative norm error estimates for linear element only provide second order, there is no superconvergence for the approximate solution in general. We first observed the superconvergence of the approximation at nodes for the Poisson equation on the meshes based on Centroidal Voronoi Delaunay Tessellation (CVDT), which is in turn consistent with $M = A^{-1} = I$.

Metric tensor

Given a point $p \in \Omega$, a metric tensor at p is the specification of a positive definite matrix

$$M(p) = \begin{pmatrix} a(p) & b(p) \\ b(p) & c(p) \end{pmatrix}.$$

The directional distance between two points p and q , as measured by metric $M(p)$:

$$d_{M(p)}(p, q) = \sqrt{\overrightarrow{pq}^t M(p) \overrightarrow{pq}}$$

$$M(p) = E^t U E, \text{ with } E = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad U = \begin{pmatrix} \lambda_1(p) & 0 \\ 0 & \lambda_2(p) \end{pmatrix}.$$

$\lambda_1(p)$ and $\lambda_2(p)$ are the two eigenvalues, and E are the corresponding eigenvectors of $M(p)$. $M(p)$ can be considered as it has transformed a unit circle into an ellipse.

Anisotropic Centroidal Voronoi Tessellation

Given a set of points $\{z_i\}_{i=1}^k$ in the domain Ω and a positive definite Riemannian metric M :

- Anisotropic Voronoi region (AVR):

$$V_i = \{x \in \Omega \mid d_{M(x)}(x, z_i) < d_{M(x)}(x, z_j) \forall z_i \neq z_j\}.$$

- Anisotropic energy F of V_i :

$$F(z_i) = \int_{V_i} d_{M(x)}^2(x, z_i) dx.$$

- Anisotropic Voronoi tessellation (AVT): $\{z_i, V_i\}_1^k$.
- The anisotropic center of mass (centroid) of the AVR V_i :

$$z_i^c = \left(\int_{V_i} M(x) dx \right)^{-1} \int_{V_i} (M(x), x) dx.$$

- Definition: AVT=ACVT if **generators** $z_i = z_i^c$ **centroids** for $i = 1, 2, \dots$,
- ACVDT: the dual triangulation.

ACVT: Optimal property

Define the energy functional for $\{(z_i, V_i)\}_{i=1}^k$ by

$$\mathcal{F}(\{(z_i, V_i)\}_{i=1}^k) = \sum_{i=1}^n \int_{x \in V_i} d_{M(x)}^2(x, z_i) dx$$

- ♡ A necessary condition for \mathcal{F} to be minimized is that $\{(z_i, V_i)\}_{i=1}^k$ is an ACVT of Ω .

ACVT mesh generation algorithm

Some notations:

- M the metric tensor on the domain Ω .
- h the target mesh size under the metric tensor.
- B the 1d CVT discretization of the boundary of Ω , according to h .
- E the directional edge set of B . For each edge $e \in E$, the lefthand side of e is the inside of the domain.
- T the initial constrained anisotropic Delaunay triangulation (Constrained ADT) of B .
- F the front which is an directional edge set of T .
- VP the front point set constructed from the front edges which will be inserted into T .

ACVT mesh generation algorithm

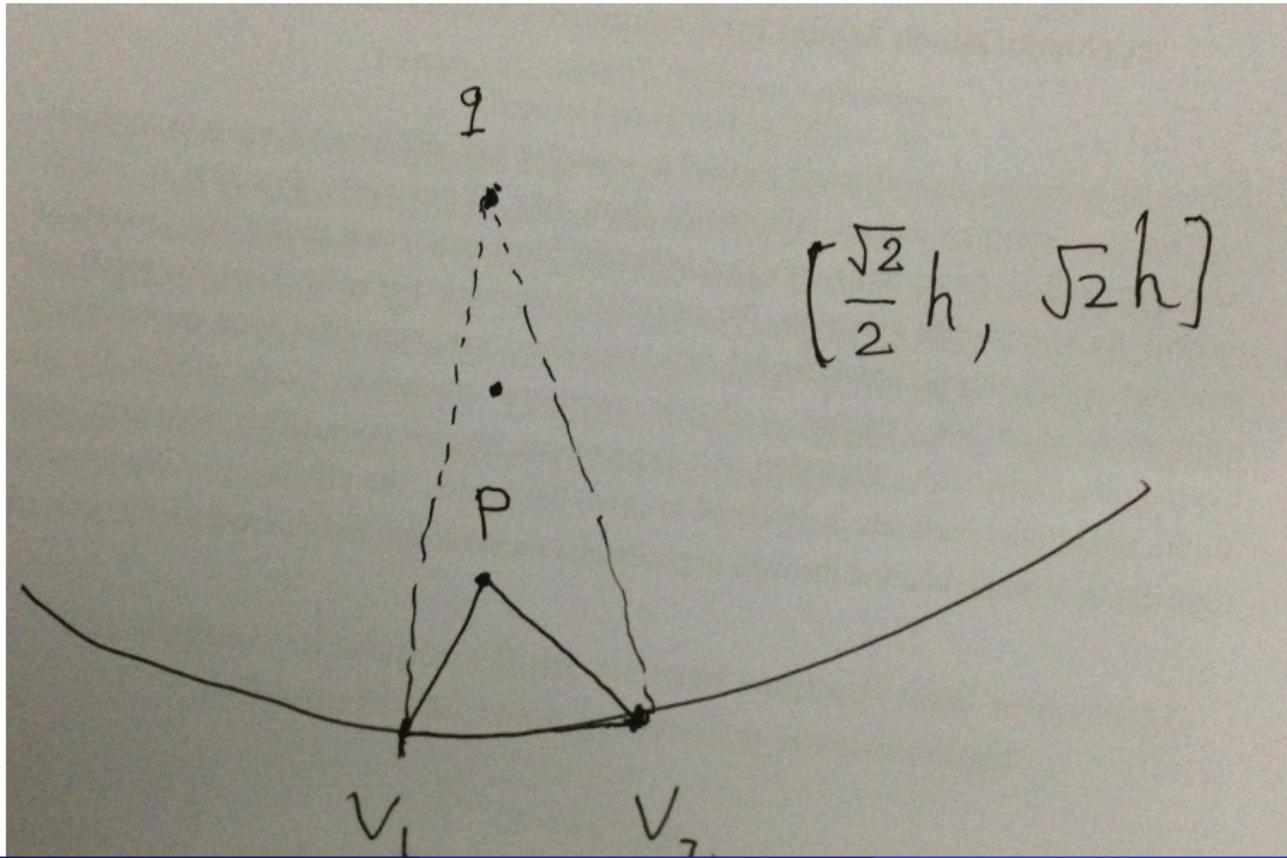
Given $M, h, \mathcal{B}, \mathcal{E}$

- ① Initialize the constrained ADT \mathcal{T} based on the 1d CVT discretization of \mathcal{B} .
- ② Initialize the front $\mathcal{F} := \mathcal{E}$, and use AFT method to generate new front points and insert them into \mathcal{T} , until no new front point insert into \mathcal{T} .
- ③ Optimize \mathcal{T} by 2D Lloyd method to generate ACVDT mesh.

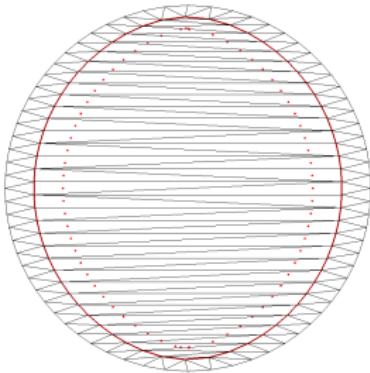
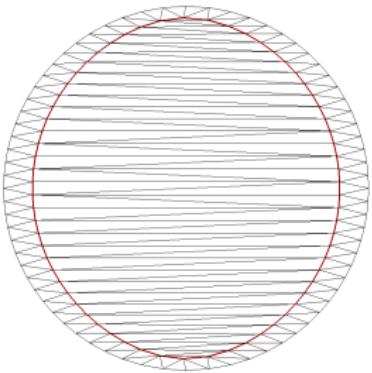
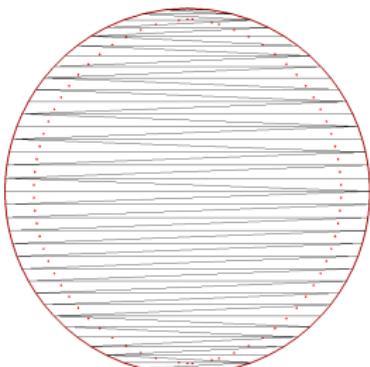
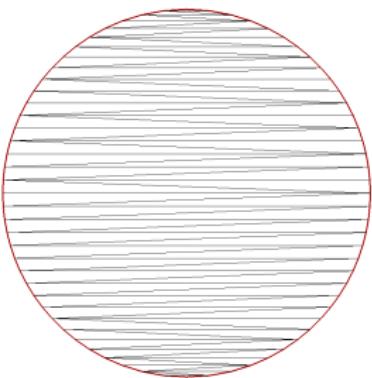
Front points generation and front update algorithm

- 1 Initialize the front $\mathcal{F} := \mathcal{E}$, and tag the vertex in \mathcal{F} as *Old* vertex.
- 2 For each directional edge $e := (v_1, v_2) \in \mathcal{F}$
 - 1 Construct new point p in the lefthand side of e , which satisfies the following conditions: $d(v_1, p)_{M(v_1)} = d(v_2, p)_{M(v_2)} = h$
 - 2 If segments (v_1, p) and (v_2, p) do not intersect with and p are not too close to other edges in \mathcal{F} , put p into \mathcal{VP} .
- 3 For each point $p \in \mathcal{VP}$, find $\tau \in \mathcal{T}$ which include p . Denote q as the closest vertex of τ to p .
If p is too close to q , replace the vertex q by $\frac{p+q}{2}$, and update \mathcal{T} be edge flipping to make sure \mathcal{T} be constrained ADT.
Else insert p into \mathcal{T} and tag vertex p as *New* vertex.
- 4 Tag the vertex in \mathcal{F} as *Old* vertex.
- 5 Update front \mathcal{F} . The edge $e := (v_1, v_2)$ in \mathcal{F} should satisfy the following conditions:
 - 1 v_1 and v_2 are *New* vertex;
 - 2 For the two elements (v, v_1, v_2) and (v', v_2, v_1) shared edge e , v is an *New* vertex and v' is an *Old* vertex.
- 6 Smoothing the front \mathcal{F} by 1d Lloyd method.

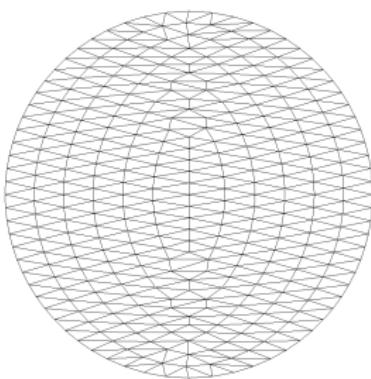
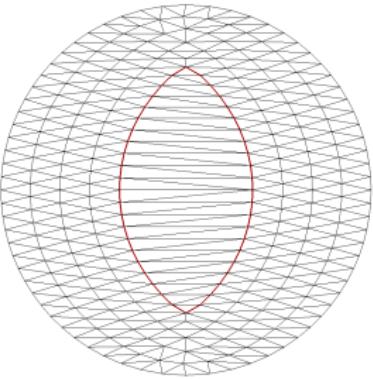
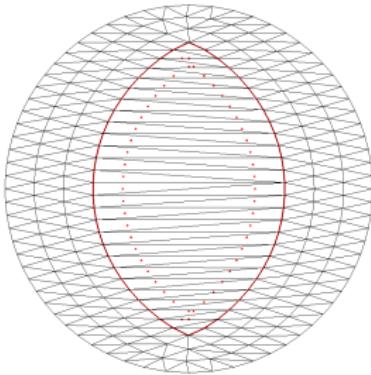
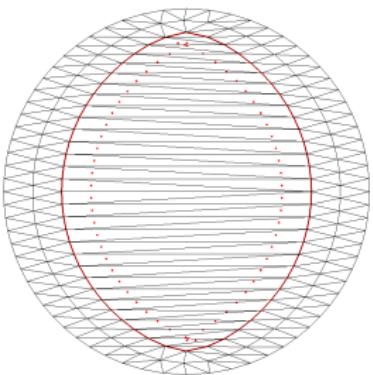
AFT example



AFT example



AFT example



Lloyd iteration method

Given a set of points $\{z_i\}_{i=1}^k$, its associated ADT and a metric tensor M on a compact domain $\Omega \in \mathbb{R}^2$,

- ① Construct the AVR $\{V_i\}_{i=1}^k$ associated with generators $\{z_i\}_{i=1}^k$.
- ② Compute the mass centers (centroids) of the Voronoi regions $\{V_i\}_{i=1}^k$.
- ③ Move the points $\{z_i\}_{i=1}^k$ to the centroid positions.
- ④ Construct an updated ADT.
- ⑤ If the new points meet some convergence criterion, terminate; otherwise, return to step 1.

Mesh quality

Given a Riemann metric M , let τ be an element in a given mesh \mathcal{T} .

- Element shape quality:

$$Q_\tau = 4\sqrt{3}S_M(\tau) / \sum_{i=1}^3 (L_\tau^i)^2$$

L_τ^i is the directional distance of the edges of τ .

- $S_M(\tau)$ (the area of τ under the metric M) can be approximated by

$$S_M(\tau) \approx \sqrt{\det(M)} S(\tau),$$

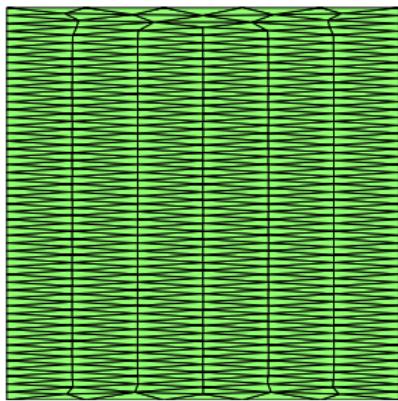
where $S(\tau)$ is the triangle area in the usual Euclidean metric.

- $Q_\tau \in (0, 1]$, and τ is an equilateral triangle under the metric M if and only if $Q_\tau = 1$.
- Mesh average quality: N_e is the number of the elements,

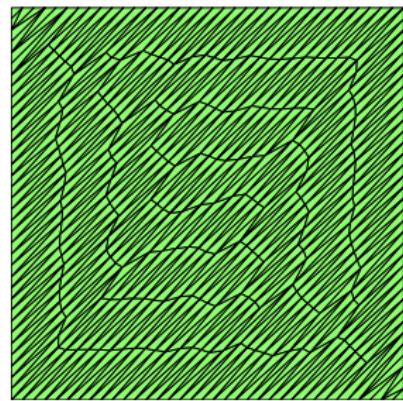
$$Q_{avg} = \frac{1}{N_e} \sum_{i=1}^N Q_\tau$$

ACVT mesh examples

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 50.5 & -49.5 \\ -49.5 & 50.5 \end{pmatrix}.$$



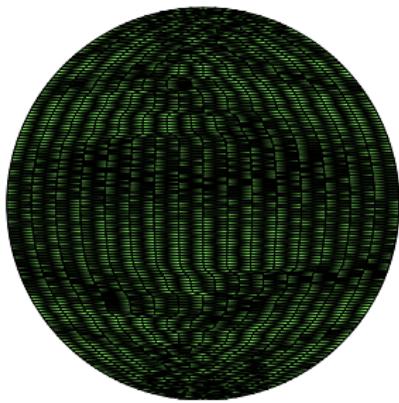
$M_1, Q_{avg} > 0.99$



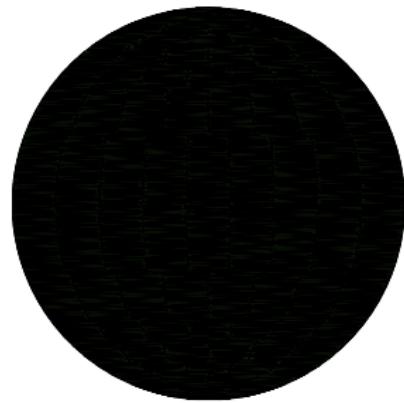
$M_2, Q_{avg} > 0.99$

ACVT mesh examples

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1000 \end{pmatrix}.$$



$M_1, Q_{avg} > 0.99$



$M_3, Q_{avg} > 0.99$

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FEM for Anisotropic Elliptic Problem

Computational setting:

- linear finite element
- anisotropic mesh generation: based on CGAL package
- finite element discretization: based on iFEM package
- linear systems solver: PCG with preconditioner AMG or ILU

Diffusion matrix A

$$A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

- C the coefficient matrix resulting from the linear finite element discretization of PDE
- $\kappa(C)$ the condition number of the coefficient matrix C
- N_e the number of mesh elements
- $\kappa(C)/N_e$ the ratio between the condition number and the number of mesh elements
- N_{its} the number of iterations in solving the linear algebraic equations by the PCG iterative method

CVT vs ACVT

Problem:

$$\begin{cases} -\nabla \cdot (A \nabla u) = f, \\ u = g, \end{cases}$$

$$\Omega = [0, 1]^2$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0.001 \end{pmatrix}$$

$$u = x(1-x)y(1-y)$$

in Ω ,
on Γ ,

- A anisotropic elliptic operator
- u isotropic solution
- isotropic mesh or anisotropic mesh?
- CVT $M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- ACVT $M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1000 \end{pmatrix}$

CVT vs ACVT

Mesh quality:

Metric	M_0	M_3
Q_{avg}	0.9934	0.9909
	0.9968	0.9993
	0.9984	0.9997
	0.9992	0.9999

CVT vs ACVT

$$\|u - u_h\|_{l^2}$$

	size	0.04	0.02	0.01	0.005
M_0	N_e	1452	5784	23102	92402
	error	6.01577e-05	1.63390e-05	3.96084e-06	9.63438e-07
	order	–	1.8861	2.0466	2.0396
		0.23	0.115	0.0575	0.02875
M_3	N_e	1376	5478	21954	87904
	error	4.42602e-05	7.93465e-07	1.16113e-07	2.18113e-08
	order	–	5.8216	2.7688	2.4106

The l^2 errors on ACVT mesh are smaller, and superconvergent.

CVT vs ACVT

$$\|u - u_h\|_{L^2}$$

	size	0.04	0.02	0.01	0.005
M_0	N_e	1452	5784	23102	92402
	error	1.01792e-04	2.63789e-05	6.52727e-06	1.61464e-06
	order	—	1.9540	2.0170	2.0154
		0.23	0.115	0.0575	0.02875
M_3	N_e	1376	5478	21954	87904
	error	1.24108e-03	3.10585e-04	7.76448e-05	1.94113e-05
	order	—	2.0054	1.9973	1.9985

The L^2 error on ACVT mesh is larger than it on CVT mesh, the reason can be seen from

$$\|u - u_h\|_{L^2} \leq \|u - u_I\|_{L^2} + \|u_I - u_h\|_{L^2}$$

On ACVT mesh, $\|u_I - u_h\|_{L^2}$ is better, but ACVT is not good for interpolation of the isotropic solution.

CVT vs ACVT

$$\|u - u_h\|_E$$

	size	0.04	0.02	0.01	0.005
M_0	N_e	1452	5784	23102	92402
	error	4.97189e-03	2.47618e-03	1.23766e-03	6.18513e-04
	order	—	1.0087	1.0016	1.0008
M_3	size	0.23	0.115	0.0575	0.02875
	N_e	1376	5478	21954	87904
	error	2.10861e-02	1.05499e-02	5.27489e-03	2.63744e-03
	order	—	1.0025	0.9986	0.9993

CVT vs ACVT

$\kappa(C)/N_e$

$m \backslash$ Metric	M_0	M_3
1	2.32e-01	9.46e-03
2	2.34e-01	8.86e-03
3	2.35e-01	8.77e-03
4	2.34e-01	8.71e-03

Anisotropic problem with an isotropic exact solution on a unit circle

Example

This example is to solve equation on the unit circle, the diffusion matrix A is chosen with $(\theta, k) = (0, 0.001)$, $f = 1$, $g = 0$ and the isotropic exact solution is given by

$$u = \frac{-1}{2(1+k)}(x^2 + y^2 - 1).$$

M_0	size	0.05	0.025	0.0125	0.0065
	N_e	3524	13698	55718	222346
	error	8.64665e-05	2.64815e-05	7.02166e-06	2.10618e-06
	order	–	1.7432	1.8922	1.7401
M_3	size	0.2526	0.1263	0.06315	0.031575
	N_e	3578	14306	57308	229926
	error	2.79643e-04	5.85851e-05	1.05297e-05	1.97471e-06
	order	–	2.2557	2.4735	2.4095

Table: $\|u - u_h\|_{l^2}$ for unit circle with isotropic solution, $(\theta, k) = (0, 0.001)$.

M_0	size	0.05	0.025	0.0125	0.0065
	N_e	3524	13698	55718	222346
	error	5.20716e-04	1.35955e-04	3.40880e-05	8.77343e-06
	order	–	1.9782	1.9720	1.9614
M_3	size	0.2526	0.1263	0.06315	0.031575
	N_e	3578	14306	57308	229926
	error	7.22981e-03	1.81635e-03	4.54050e-04	1.12990e-04
	order	–	1.9935	1.9980	2.0023

Table: $\|u - u_h\|_{L^2}$ for unit circle with isotropic solution, $(\theta, k) = (0, 0.001)$.

M_0	size	0.05	0.025	0.0125	0.0065
	N_e	3524	13698	55718	222346
	error	1.66256e-02	8.40181e-03	4.15392e-03	2.07678e-03
	order	—	1.0054	1.0041	1.0018
M_3	size	0.2526	0.1263	0.06315	0.031575
	N_e	3578	14306	57308	229926
	error	1.11840e-01	5.60952e-02	2.80522e-02	1.39949e-02
	order	—	0.9958	0.9987	1.0010

Table: $\|u - u_h\|_E$ for unit circle with isotropic solution, $(\theta, k) = (0, 0.001)$.

$$(\theta, k) = (0, 0.001)$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0.001 \end{pmatrix}$$

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1000 \end{pmatrix}.$$

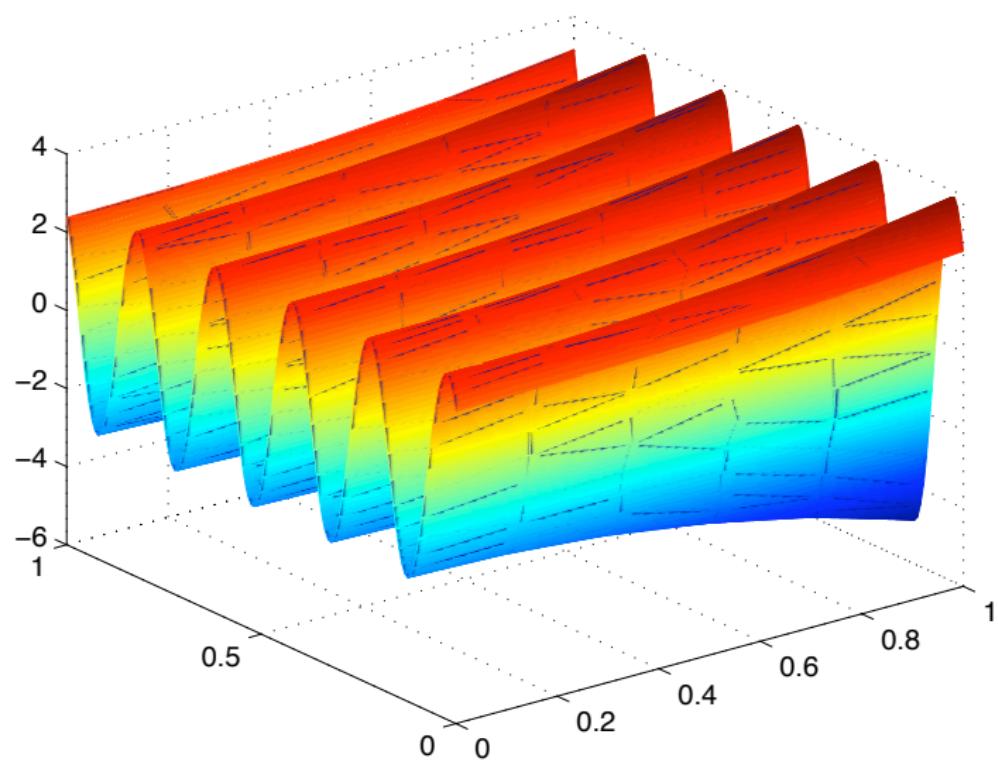
In this case,

$$M_3 = A^{-1}, \quad f = 1, \quad \Omega = [0, 1]^2$$

and

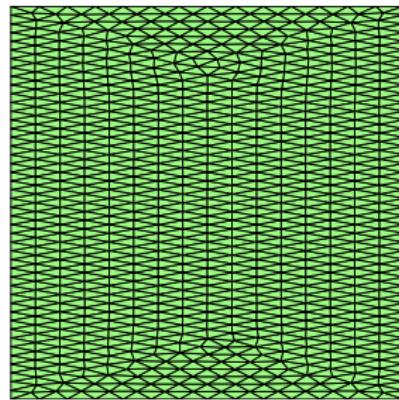
$$u = (e^x + e^{-x}) \left(\sin \frac{y}{\sqrt{k}} + \cos \frac{y}{\sqrt{k}} \right) - \frac{x^2}{2}$$

$(\theta, k) = (0, 0.001)$, solution, square domain

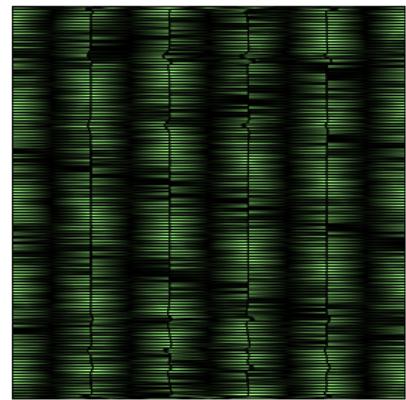


$(\theta, k) = (0, 0.001)$, square domain

Mesh



M_1



M_3

$(\theta, k) = (0, 0.001)$, square domain

Mesh quality

Metric	M_0	M_1	M_2	M_3
Q_{avg}	0.9934	0.9957	0.9913	0.9909
	0.9968	0.9978	0.9984	0.9993
	0.9984	0.9991	0.9993	0.9997
	0.9992	0.9996	0.9995	0.9999

$(\theta, k) = (0, 0.001), \|u - u_h\|_{L^2}, M_3 = A^{-1}$, square domain

M_0	size	0.04	0.02	0.01	0.005
	N_e	1452	5784	23102	92402
	error	1.24226e+00	5.64212e-01	1.86950e-01	5.14036e-02
	order	—	1.14	1.60	1.86
M_1	size	0.07	0.035	0.0175	0.00875
	N_e	1450	5928	23758	95236
	error	2.23942e-01	6.09431e-02	1.54656e-02	3.83280e-03
	order	—	1.85	1.98	2.01
M_2	size	0.13	0.065	0.0325	0.01625
	N_e	1402	5482	22052	87308
	error	3.13945e-02	7.83850e-03	1.94319e-03	4.80036e-04
	order	—	2.04	2.00	2.03
M_3	size	0.23	0.115	0.0575	0.02875
	N_e	1376	5478	21954	87904
	error	6.28650e-03	1.52549e-03	3.81340e-04	9.52004e-05
	order	—	2.05	2.00	2.00

$(\theta, k) = (0, 0.001)$, $\|u - u_h\|_E$, $M_3 = A^{-1}$, square domain

	size	0.04	0.02	0.01	0.005
M_0	N_e	1452	5784	23102	92402
	error	6.89162e+00	4.35671e+00	2.39769e+00	1.23279e+00
	order	—	0.66	0.86	0.96
M_1	size	0.07	0.035	0.0175	0.00875
	N_e	1450	5928	23758	95236
	error	2.14297e+00	1.13365e+00	5.71897e-01	2.84529e-01
	order	—	0.90	0.99	1.01
M_2	size	0.13	0.065	0.0325	0.01625
	N_e	1402	5482	22052	87308
	error	5.31623e-01	2.73221e-01	1.36143e-01	6.73174e-02
	order	—	0.98	1.00	1.02
M_3	size	0.23	0.115	0.0575	0.02875
	N_e	1376	5478	21954	87904
	error	2.60323e-01	1.30054e-01	6.49727e-02	3.24634e-02
	order	—	1.00	1.00	1.00

$(\theta, k) = (0, 0.001)$, $\|u - u_h\|_{l^2}$, $M_3 = A^{-1}$, square domain

M_0	size	0.04	0.02	0.01	0.005
	N_e	1452	5784	23102	92402
	error	1.13164e+00	5.24367e-01	1.75920e-01	4.86389e-02
	order	—	1.11	1.58	1.85
M_1	size	0.07	0.035	0.0175	0.00875
	N_e	1450	5928	23758	95236
	error	1.60794e-01	4.48355e-02	1.14503e-02	2.84343e-03
	order	—	1.81	1.97	2.01
M_2	size	0.13	0.065	0.0325	0.01625
	N_e	1402	5482	22052	87308
	error	9.82092e-03	2.14944e-03	5.36949e-04	1.30232e-04
	order	—	2.23	1.99	2.06
M_3	size	0.23	0.115	0.0575	0.02875
	N_e	1376	5478	21954	87904
	error	5.72305e-04	6.24420e-05	1.08959e-05	1.90314e-06
	order	—	3.21	2.52	2.52

$(\theta, k) = (0, 0.001)$, $M_3 = A^{-1}$, square domain

$\kappa(C)/N_e$

$m \backslash$ Metric	M_0	M_1	M_2	M_3
1	2.32e-01	7.64e-02	2.34e-02	9.46e-03
2	2.34e-01	7.48e-02	2.43e-02	8.86e-03
3	2.35e-01	7.47e-02	2.41e-02	8.77e-03
4	2.34e-01	7.49e-02	2.34e-02	8.71e-03

$(\theta, k) = (0, 0.001)$, $M_3 = A^{-1}$, square domain

N_{its}

	PCG_{AMG}				PCG_{ILU}			
Metric	M_0	M_1	M_2	M_3	M_0	M_1	M_2	M_3
N_{its}	23	17	11	8	70	37	15	16
	31	23	14	11	151	70	25	26
	48	31	17	13	355	158	45	48
	64	38	19	14	920	308	90	92

$$(\theta, k) = (0, 0.001)$$

Observations:

$M = A^{-1}$ is the best metric.

- (i) The resulting discrete algebraic system is the best conditioned system in the sense of condition numbers and more efficient on linear solvers.
- (ii) The finite element solution of the anisotropic elliptic problem is more accurate on the anisotropic mesh with $M = A^{-1}$ in comparison with other selected metrics. This means that the anisotropic mesh generated with metric $M = A^{-1}$ can efficiently resolve the anisotropic behavior of the solution.
- (iii) More importantly, there are superconvergence of the linear finite element solution on the anisotropic mesh generated with metric $M = A^{-1}$.

$$(\theta, k) = (0, 0.01)$$

To see more clearly, we take the matrix A with parameter $(\theta, k) = (0, 0.01)$. In this case

$$M_2 = A^{-1}, \quad f = 1, \quad \Omega = [0, 1]^2$$

and

$$u = (e^x + e^{-x}) \left(\sin \frac{y}{\sqrt{k}} + \cos \frac{y}{\sqrt{k}} \right) - \frac{x^2}{2}$$

$(\theta, k) = (0, 0.01), \|u - u_h\|_{l^2}, M_2 = A^{-1}, \text{ square domain}$

M_0	size	0.04	0.02	0.01	0.005
	N_e	1452	5784	23102	92402
	error	4.06931e-02	1.07760e-02	2.67071e-03	6.69122e-04
	order	–	1.92	2.01	2.00
M_1	size	0.07	0.035	0.0175	0.00875
	N_e	1450	5928	23758	95236
	error	2.81733e-03	7.09365e-04	1.73101e-04	4.22076e-05
	order	–	1.96	2.03	2.03
M_2	size	0.13	0.065	0.0325	0.01625
	N_e	1402	5482	22052	87308
	error	3.38351e-04	4.04959e-05	6.11325e-06	1.18976e-06
	order	–	3.11	2.72	2.38
M_3	size	0.23	0.115	0.0575	0.02875
	N_e	1376	5478	21954	87904
	error	1.21216e-03	3.52108e-04	6.91314e-05	1.66069e-05
	order	–	1.79	2.35	2.06

$(\theta, k) = (0, 0.01)$, $M_2 = A^{-1}$, square domain

$\kappa(C)/N_e$

$m \backslash$ Metric	M_0	M_1	M_2	M_3
1	2.30e-01	7.54e-02	2.95e-02	6.66e-02
2	2.32e-01	7.41e-02	2.75e-02	7.54e-02
3	2.33e-01	7.41e-02	2.80e-02	7.21e-02
4	2.32e-01	7.44e-02	2.86e-02	7.40e-02

$(\theta, k) = (0, 0.01)$, $M_2 = A^{-1}$, square domain

N_{its}

	PCG_{AMG}				PCG_{ILU}			
Metric	M_0	M_1	M_2	M_3	M_0	M_1	M_2	M_3
N_{its}	22	14	10	12	63	25	26	37
	26	16	12	17	125	43	41	77
	32	18	14	20	254	89	82	144
	40	20	15	22	531	166	159	282

$(\theta, k) = (\pi/4, 0.001)$, square domain

$$A = \begin{pmatrix} 1001/2000 & 999/2000 \\ 999/2000 & 1001/2000 \end{pmatrix}.$$

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} \frac{11}{2} & -\frac{9}{2} \\ -\frac{9}{2} & \frac{11}{2} \end{pmatrix}$$

$$M_2 = \begin{pmatrix} \frac{101}{2} & -\frac{99}{2} \\ -\frac{99}{2} & \frac{101}{2} \end{pmatrix}, \quad M_3 = \begin{pmatrix} \frac{1001}{4} & -\frac{999}{2} \\ -\frac{999}{2} & \frac{1001}{2} \end{pmatrix}$$

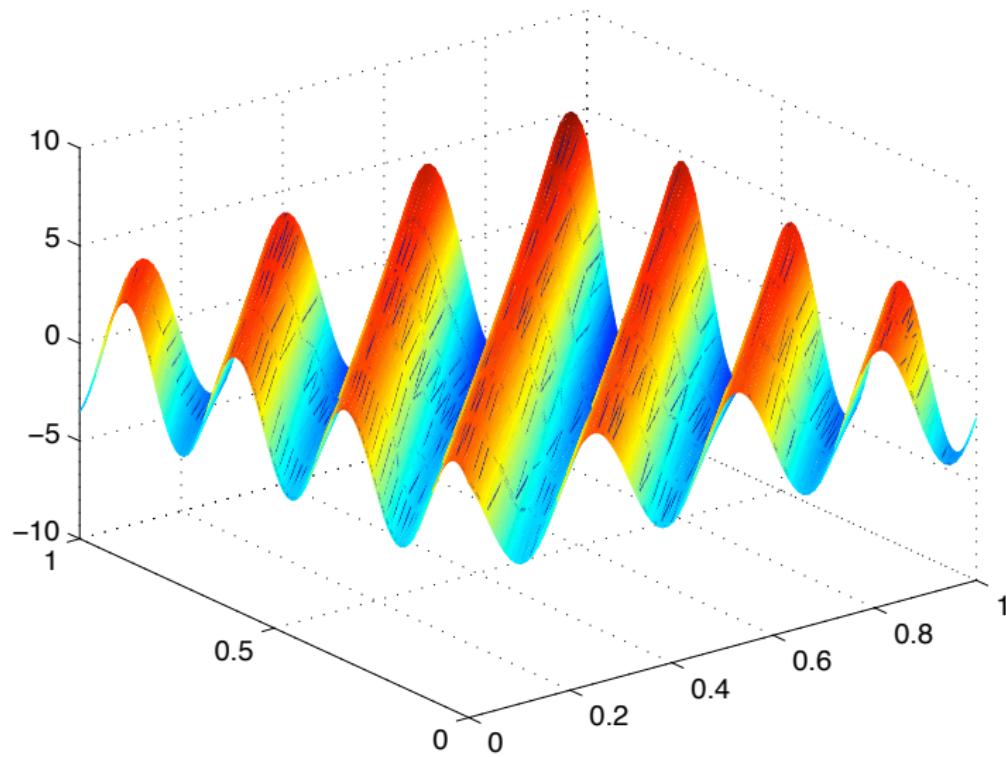
In this case,

$$M_3 = A^{-1}, \quad f = 1, \quad \Omega = [0, 1]^2$$

and

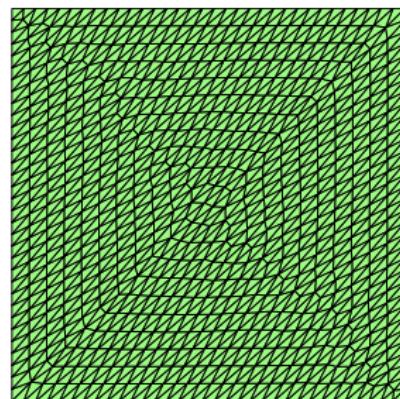
$$u = \left(e^{\frac{\sqrt{2}(x+y)}{2}} + e^{-\frac{\sqrt{2}(x+y)}{2}} \right) \left(\sin \frac{y-x}{\sqrt{2k}} + \cos \frac{y-x}{\sqrt{2k}} \right) - \frac{(x+y)^2}{4}$$

$(\theta, k) = (\pi/4, 0.001)$, solution, square domain

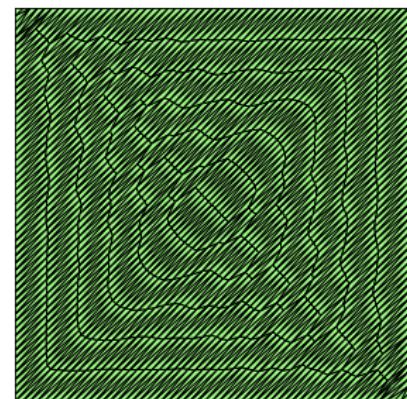


$$(\theta, k) = (\pi/4, 0.001), \text{ square domain}$$

Mesh



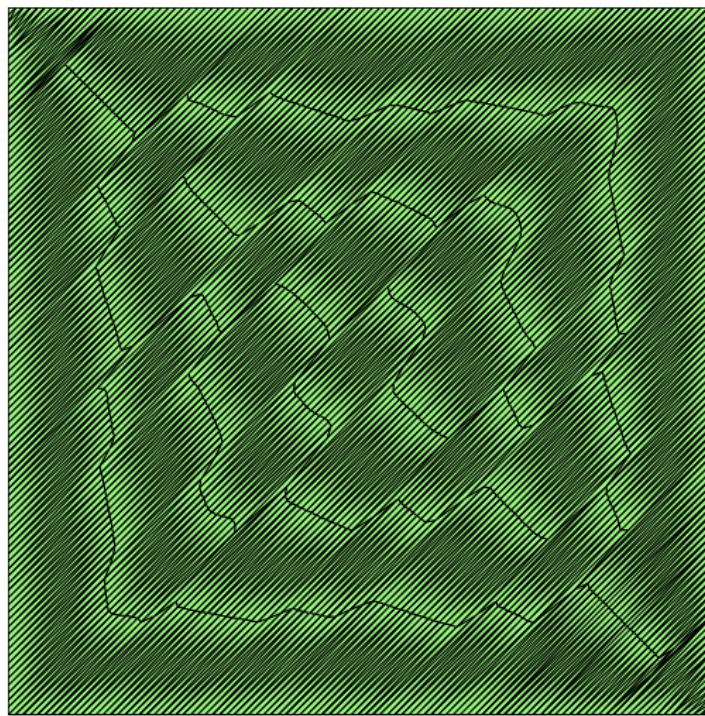
M_1



M_2

$(\theta, k) = (\pi/4, 0.001)$, square domain

Mesh M_3



$(\theta, k) = (\pi/4, 0.001)$, square domain

Mesh quality

Metric	M_0	M_1	M_2	M_3
Q_{avg}	0.9934	0.9915	0.9859	0.9696
	0.9968	0.9955	0.9942	0.9900
	0.9984	0.9973	0.9971	0.9965
	0.9992	0.9982	0.9982	0.9986

$(\theta, k) = (\pi/4, 0.001)$, $\|u - u_h\|_{L^2}$, $M_3 = A^{-1}$, square domain

M_0	size	0.04	0.02	0.01	0.005
	N_e	1452	5784	23102	92402
	error	1.50454e+00	7.00693e-01	2.30585e-01	6.29556e-02
	order	—	1.11	1.61	1.87
M_1	size	0.07	0.035	0.0175	0.00875
	N_e	1494	6034	24162	96792
	error	1.91610e-01	4.64033e-02	1.21458e-02	3.06158e-03
	order	—	2.03	1.93	1.99
M_2	size	0.13	0.065	0.0325	0.01625
	N_e	1376	5484	21932	87940
	error	3.55638e-02	8.67515e-03	2.16046e-03	5.34917e-04
	order	—	2.04	2.01	2.01
M_3	size	0.23	0.115	0.0575	0.02875
	N_e	1416	5536	22082	88316
	error	7.44689e-03	1.76134e-03	4.35045e-04	1.07760e-04
	order	—	2.11	2.02	2.01

$(\theta, k) = (\pi/4, 0.001)$, $\|u - u_h\|_E$, $M_3 = A^{-1}$, square domain

	size	0.04	0.02	0.01	0.005
M_0	N_e	1452	5784	23102	92402
	error	7.95465e+00	4.85577e+00	2.64403e+00	1.35758e+00
	order	—	0.71	0.88	0.96
M_1	size	0.07	0.035	0.0175	0.00875
	N_e	1494	6034	24162	96792
	error	1.81941e+00	8.86902e-01	4.41710e-01	2.20437e-01
	order	—	1.03	1.00	1.00
M_2	size	0.13	0.065	0.0325	0.01625
	N_e	1376	5484	21932	87940
	error	6.06499e-01	3.00695e-01	1.50522e-01	7.50588e-02
	order	—	1.01	1.00	1.00
M_3	size	0.23	0.115	0.0575	0.02875
	N_e	1416	5536	22082	88316
	error	3.08995e-01	1.53805e-01	7.66443e-02	3.82702e-02
	order	—	1.02	1.01	1.00

$(\theta, k) = (\pi/4, 0.001)$, $\|u - u_h\|_{l^2}$, $M_3 = A^{-1}$, square domain

M_0	size	0.04	0.02	0.01	0.005
	N_e	1452	5784	23102	92402
	error	1.37343e+00	6.52123e-01	2.16813e-01	5.94888e-02
	order	—	1.08	1.59	1.87
M_1	size	0.07	0.035	0.0175	0.00875
	N_e	1494	6034	24162	96792
	error	1.31436e-01	3.15497e-02	8.69826e-03	2.24027e-03
	order	—	2.04	1.86	1.96
M_2	size	0.13	0.065	0.0325	0.01625
	N_e	1376	5484	21932	87940
	error	1.23008e-02	2.87873e-03	7.06976e-04	1.71203e-04
	order	—	2.10	2.03	2.04
M_3	size	0.23	0.115	0.0575	0.02875
	N_e	1416	5536	22082	88316
	error	8.56192e-04	1.82930e-04	3.34538e-05	5.96947e-06
	order	—	2.26	2.46	2.49

$(\theta, k) = (\pi/4, 0.001)$, $M_3 = A^{-1}$, square domain

$\kappa(C)/N_e$

$m \backslash$ Metric	M_0	M_1	M_2	M_3
1	3.58e-01	1.20e-01	4.20e-02	1.71e-02
2	3.75e-01	1.25e-01	4.23e-02	1.73e-02
3	3.83e-01	1.32e-01	4.33e-02	1.69e-02
4	3.88e-01	1.43e-01	4.40e-02	1.68e-02

$(\theta, k) = (\pi/4, 0.001)$, $M_3 = A^{-1}$, square domain

N_{its}

	PCG_{AMG}				PCG_{ILU}			
Metric	M_0	M_1	M_2	M_3	M_0	M_1	M_2	M_3
N_{its}	22	19	12	9	44	18	11	17
	33	25	14	11	97	25	19	33
	47	32	17	13	205	48	34	63
	66	37	19	14	425	94	64	121

$(\theta, k) = (0, 0.001)$, circle domain

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0.001 \end{pmatrix}$$

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1000 \end{pmatrix}.$$

In this case,

$$M_3 = A^{-1}, \quad f = 1, \quad \Omega \text{ is unit circle}$$

and

$$u = (e^x + e^{-x}) \left(\sin \frac{y}{\sqrt{k}} + \cos \frac{y}{\sqrt{k}} \right) - \frac{x^2}{2}$$

$(\theta, k) = (0, 0.001)$, circle domain

Mesh quality

Metric	M_0	M_1	M_2	M_3
Quality	0.9886	0.9931	0.9949	0.9945
	0.9914	0.9958	0.9980	0.9979
	0.9938	0.9969	0.9990	0.9991
	0.9953	0.9975	0.9994	0.9996

$(\theta, k) = (0, 0.001), \|u - u_h\|_{L^2}, M_3 = A^{-1}, \text{circle domain}$

M_0	size	0.05	0.025	0.0125	0.0065
	N_e	3524	13698	55718	222346
	error	2.98404e+00	1.95595e+00	8.89379e-01	2.76967e-01
	order	—	0.6223	1.1234	1.6860
M_1	size	0.0812	0.0406	0.0203	0.01015
	N_e	3572	14584	58906	236460
	error	8.51026e-01	2.50488e-01	6.38610e-02	1.58484e-02
	order	—	1.7387	1.9580	2.0055
M_2	size	0.142	0.071	0.0355	0.01775
	N_e	3580	14290	57584	231068
	error	8.86807e-02	2.11551e-02	5.19618e-03	1.29248e-03
	order	—	2.0707	2.0147	2.0027
M_3	size	0.2526	0.1263	0.06315	0.031575
	N_e	3578	14306	57308	229926
	error	1.26711e-02	3.11945e-03	7.81728e-04	1.94400e-04
	order	—	2.0228	1.9944	2.0033

$(\theta, k) = (0, 0.001), \|u - u_h\|_E, M_3 = A^{-1}, \text{circle domain}$

M_0	size	0.05	0.025	0.0125	0.0065
	N_e	3524	13698	55718	222346
	error	1.01024e+01	7.02786e+00	4.30451e+00	2.32502e+00
	order	—	0.5346	0.6988	0.8901
M_1	size	0.0812	0.0406	0.0203	0.01015
	N_e	3572	14584	58906	236460
	error	3.73627e+00	1.96697e+00	9.84761e-01	4.91030e-01
	order	—	0.9121	0.9912	1.0014
M_2	size	0.142	0.071	0.0355	0.01775
	N_e	3580	14290	57584	231068
	error	1.01671e+00	4.97500e-01	2.46311e-01	1.22854e-01
	order	—	1.0327	1.0088	1.0012
M_3	size	0.2526	0.1263	0.06315	0.031575
	N_e	3578	14306	57308	229926
	error	4.77742e-01	2.38184e-01	1.18984e-01	5.93813e-02
	order	—	1.0045	1.0003	1.0005

$(\theta, k) = (0, 0.001), \|u - u_h\|_{l^2}, M_3 = A^{-1}, \text{circle domain}$

M_0	size	0.05	0.025	0.0125	0.0065
	N_e	3524	13698	55718	222346
	error	1.60173e+00	1.06425e+00	4.88632e-01	1.54202e-01
	order	—	0.6022	1.1096	1.6668
M_1	size	0.0812	0.0406	0.0203	0.01015
	N_e	3572	14584	58906	236460
	error	4.12560e-01	1.23010e-01	3.14733e-02	7.81447e-03
	order	—	1.7204	1.9529	2.0048
M_2	size	0.142	0.071	0.0355	0.01775
	N_e	3580	14290	57584	231068
	error	2.47613e-02	5.78802e-03	1.42351e-03	3.54669e-04
	order	—	2.1001	2.0129	2.0003
M_3	size	0.2526	0.1263	0.06315	0.031575
	N_e	3578	14306	57308	229926
	error	6.22456e-04	1.08024e-04	1.94127e-05	3.48427e-06
	order	—	2.5274	2.4737	2.4727

$(\theta, k) = (0, 0.001)$, $M_3 = A^{-1}$, circle domain

$\kappa(C)/N_e$

$m \backslash$ Metric	M_0	M_1	M_2	M_3
1	3.01e-01	9.09e-02	3.02e-02	1.11e-02
2	3.13e-01	9.62e-02	2.97e-02	1.11e-02
3	3.22e-01	1.02e-01	2.95e-02	1.10e-02
4	3.39e-01	9.46e-02	2.96e-02	1.16e-02

$(\theta, k) = (0, 0.001)$, square domain

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0.001 \end{pmatrix}$$

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$$

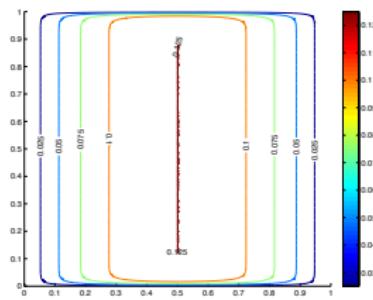
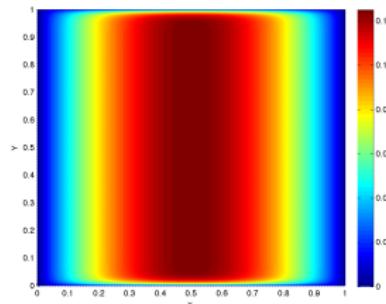
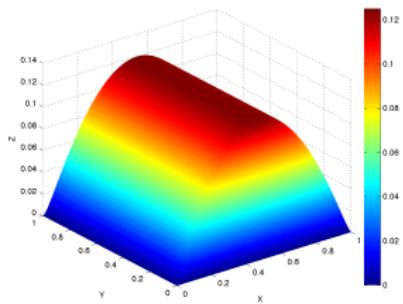
$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1000 \end{pmatrix}.$$

In this case,

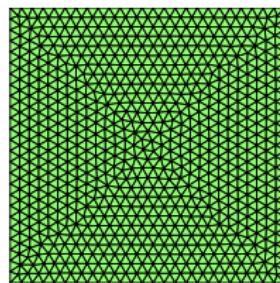
$$M_3 = A^{-1}, \quad f = 1, \quad g = 0, \quad \Omega = [0, 1]^2$$

The exact solution is unknown, we use the finite element solution on the uniform refined mesh (3 times on the 4th level mesh of M_3) as the reference solution. The anisotropic feature of the solution of this example is completely caused by the elliptic operator.

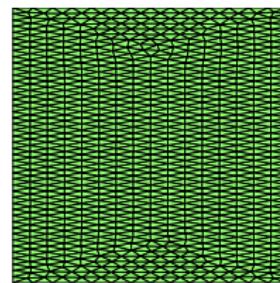
$(\theta, k) = (0, 0.001)$, square domain, Mesh
refined approximate solution



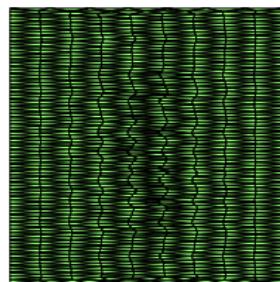
$(\theta, k) = (0, 0.001)$, square domain, Mesh



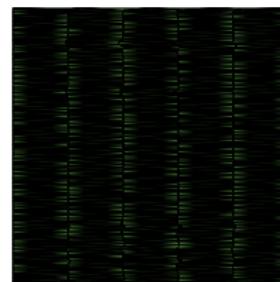
M_0



M_1



M_2



M_3

$(\theta, k) = (0, 0.001)$, $\|u - u_h\|_{l^2}$, $M_3 = A^{-1}$, square domain

M_0	size	0.04	0.02	0.01	0.005
	N_e	1452	5784	23102	92402
	error	8.7416e-04	5.74586e-04	1.87759e-04	4.92633e-05
	order	—	0.61	1.62	1.93
M_1	size	0.07	0.035	0.0175	0.00875
	N_e	1450	5928	23758	95236
	error	5.55929e-04	2.23067e-04	6.09500e-05	1.53640e-05
	order	—	1.30	1.87	1.99
M_2	size	0.13	0.065	0.0325	0.01625
	N_e	1402	5482	22052	87308
	error	3.60281e-04	6.14229e-05	1.92982e-05	4.87516e-06
	order	—	2.60	1.66	2.00
M_3	size	0.23	0.115	0.0575	0.02875
	N_e	1376	5478	21954	87904
	error	1.98958e-04	3.29049e-05	5.59777e-06	1.02042e-06
	order	—	2.61	2.56	2.45

$(\theta, k) = (0, 0.001)$, $M_3 = A^{-1}$, square domain

N_{its}

	PCG_{AMG}				PCG_{ILU}			
Metric	M_0	M_1	M_2	M_3	M_0	M_1	M_2	M_3
N_{its}	23	17	12	8	80	38	15	16
	32	24	14	11	178	78	26	27
	50	33	18	13	442	188	50	51
	68	43	21	15	1000	374	101	103

Outline

1 Motivation

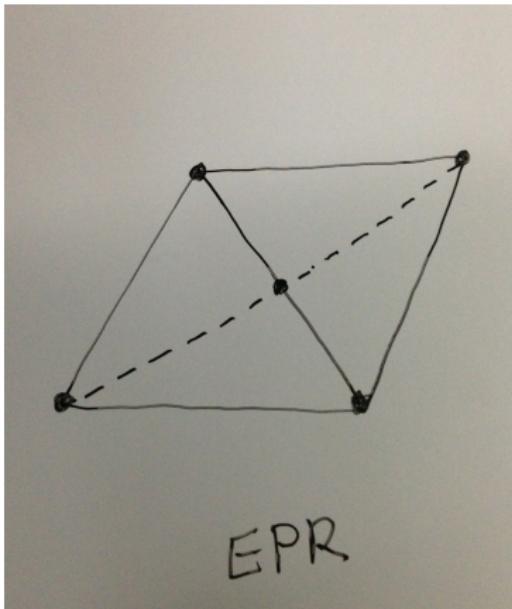
2 Anisotropic mesh generation

- Main observation
- Metric tensor
- Anisotropic Centroidal Voronoi Tessellation
- ACVT mesh generation algorithm

3 Numerical examples

- Numerical Experiments with Known Exact Solutions
- Numerical Experiments with Unknown Exact Solutions

4 Explicit Polynomial Recovery on ACVT



- Yunqing Huang, Wei Yang and Nianyu Yi, A posteriori error estimation based on the explicit polynomial recovery, Natural Science Journal of Xiangtan University, Vol 33, No. 4, 2011, pp. 1-12.
- Yunqing Huang, Wei Yang and Nianyu Yi, Superconvergence analysis for the explicit polynomial recovery method, Journal of Computational and Applied Mathematics. Accepted

Anisotropic problem with an anisotropic solution based on EPR, Rectangle domain

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1/1000 \end{pmatrix}.$$

$$u = (e^x + e^{-x}) \left(\sin \frac{y}{\sqrt{k}} + \cos \frac{y}{\sqrt{k}} \right) - \frac{x^2}{2}$$

We only test the performance of the following four metrics,

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1000 \end{pmatrix} \quad (1)$$

Note that $M_3 = A^{-1}$. Numerical results are shown in the following Tables.

M_0	size	0.04	0.02	0.01	0.005
	N_e	1452	5784	23102	92402
	$\ u - u_h\ _{L^2}$	1.2423e+00	5.6421e-01	1.8695e-01	5.1404e-02
	order	—	1.14	1.60	1.86
M_1	size	0.07	0.035	0.0175	0.00875
	N_e	1450	5928	23758	95236
	$\ u - u_h\ _{L^2}$	2.2394e-01	6.0943e-02	1.5466e-02	3.8328e-03
	order	—	1.85	1.98	2.01
M_2	size	0.13	0.065	0.0325	0.01625
	N_e	1402	5482	22052	87308
	$\ u - u_h\ _{L^2}$	3.1394e-02	7.8385e-03	1.9432e-03	4.8004e-04
	order	—	2.04	2.00	2.03
M_3	size	0.23	0.115	0.0575	0.02875
	N_e	1376	5478	21954	87904
	$\ u - u_h\ _{L^2}$	6.2865e-03	1.5255e-03	3.8134e-04	9.5200e-05
	order	—	2.05	2.00	2.00
	size	0.23	0.115	0.0575	0.02875
	N_e	1376	5478	21954	87904
	$\ u - R_h u_h\ _{L^2}$	1.9365e-03	4.1647e-04	1.0619e-04	2.6705e-05
	order	—	2.22	1.97	1.99

Table: $\|u - u_h\|_{L^2}$ for a given solution with four metrics, $(\theta, k) = (0, 0.001)$.

M_0	size	0.04	0.02	0.01	0.005
	N_e	1452	5784	23102	92402
	$\ u - u_h\ _E$	6.8914e+00	4.3567e+00	2.3977e+00	1.2328e+00
	order	—	0.66	0.86	0.96
M_1	size	0.07	0.035	0.0175	0.00875
	N_e	1450	5928	23758	95236
	$\ u - u_h\ _E$	2.1429e+00	1.1336e+00	5.7190e-01	2.8453e-01
	order	—	0.90	0.99	1.01
M_2	size	0.13	0.065	0.0325	0.01625
	N_e	1402	5482	22052	87308
	$\ u - u_h\ _E$	5.3159e-01	2.7322e-01	1.3614e-01	6.7317e-02
	order	—	0.98	1.00	1.02
M_3	size	0.23	0.115	0.0575	0.02875
	N_e	1376	5478	21954	87904
	$\ u - u_h\ _E$	2.6039e-01	1.3006e-01	6.4974e-02	3.2464e-02
	order	—	1.00	1.00	1.00
M_3	$\ u - R_h u_h\ _E$	3.1789e-02	9.0248e-03	4.2915e-03	2.1145e-03
	order	—	1.82	1.07	1.02

Table: $\|u - u_h\|_{H_E}$ for a given solution with four metrics, $(\theta, k) = (0, 0.001)$.

From above, we find the L^2 error $\|u - u_h\|_{L^2}$ of M_3 is smaller than others and the recovery L^2 error $\|u - R_h u_h\|_{L^2}$ of M_3 is also smaller than others. we find the H_E error $\|u - u_h\|_{H_E}$ of M_3 is smaller than others and the recovery H_E error $\|u - R_h u_h\|_{H_E}$ of M_3 is also smaller than others. we find EPR improves the solution both in L^2 norm and energy norm on the mesh of M_3 , more precisely,

$$\|u - R_h u_h\|_{L^2} \approx \frac{1}{3.5} \|u - u_h\|_{L^2}, \quad \|u - R_h u_h\|_{H_E} \approx \frac{1}{15} \|u - u_h\|_{H_E}$$

That means $\|R_h u_h - u_h\|_{H_E}$ could be good a posteriori error estimator

Anisotropic problem with an anisotropic solution based on EPR, Circle domain

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1/1000 \end{pmatrix}.$$

$$u = (e^x + e^{-x}) \left(\sin \frac{y}{\sqrt{k}} + \cos \frac{y}{\sqrt{k}} \right) - \frac{x^2}{2}$$

M_0	size	0.05	0.025	0.0125	0.0065
	N_e	3524	13698	55718	222346
	$\ u - u_h\ _{L^2}$	2.9840e+00	1.9559e+00	8.8938e-01	2.7697e-01
	order	—	0.62	1.12	1.69
M_1	size	0.0812	0.0406	0.0203	0.01015
	N_e	3572	14584	58906	236460
	$\ u - u_h\ _{L^2}$	8.5103e-01	2.5049e-01	6.3861e-02	1.5848e-02
	order	—	1.74	1.96	2.01
M_2	size	0.142	0.071	0.0355	0.01775
	N_e	3580	14290	57584	231068
	$\ u - u_h\ _{L^2}$	8.8681e-02	2.1155e-02	5.1962e-03	1.2925e-03
	order	—	2.07	2.01	2.00
M_3	size	0.2526	0.1263	0.06315	0.031575
	N_e	3578	14306	57308	229926
	$\ u - u_h\ _{L^2}$	1.2671e-02	3.1194e-03	7.8173e-04	1.9440e-04
	order	—	2.02	1.99	2.00
	size	0.38261e-03	9.1037e-04	2.3186e-04	5.7480e-05
	N_e	—	2.07	1.97	2.01
	$\ u - u_h\ _{L^2}$	—	—	—	—
	order	—	—	—	—

Table: $\|u - u_h\|_{L^2}$ for unit circle with anisotropic solution, $(\theta, k) = (0, 0.001)$.

M_0	size	0.05	0.025	0.0125	0.0065
	N_e	3524	13698	55718	222346
	$\ u - u_h\ _E$	1.0102e+01	7.0279e+00	4.3045e+00	2.3250e+00
	order	—	0.53	0.70	0.89
M_1	size	0.0812	0.0406	0.0203	0.01015
	N_e	3572	14584	58906	236460
	$\ u - u_h\ _E$	3.7363e+00	1.9670e+00	9.8476e-01	4.9103e-01
	order	—	0.91	0.99	1.00
M_2	size	0.142	0.071	0.0355	0.01775
	N_e	3580	14290	57584	231068
	$\ u - u_h\ _E$	1.0167e+00	4.9750e-01	2.4631e-01	1.2285e-01
	order	—	1.03	1.01	1.00
M_3	size	0.2526	0.1263	0.06315	0.031575
	N_e	3578	14306	57308	229926
	$\ u - u_h\ _E$	4.7774e-01	2.3818e-01	1.1898e-01	5.9381e-02
	order	—	1.00	1.00	1.00
	$\ u - R_h u_h\ _E$	4.7907e-02	1.8759e-02	8.7569e-03	4.1916e-03
	order	—	1.35	1.10	1.06

Table: $\|u - u_h\|_{H_E}$ for unit circle with anisotropic solution, $(\theta, k) = (0, 0.001)$.

M_0	size	0.05	0.025	0.0125	0.0065
	N_e	3524	13698	55718	222346
	$\ \nabla(u - u_h)\ _{l_2}$	9.3889e+01	6.2071e+01	2.8757e+01	9.5235e+00
	order	—	0.61	1.10	1.60
	$\ \nabla(u - R_h u_h)\ _{l_2}$	9.4088e+01	6.2101e+01	2.8641e+01	9.3381e+00
	order	—	0.61	1.10	1.62
M_1	size	0.0812	0.0406	0.0203	0.01015
	N_e	3572	14584	58906	236460
	$\ \nabla(u - u_h)\ _{l_2}$	2.7812e+01	8.9504e+00	2.9450e+00	1.1741e+00
	order	—	1.61	1.59	1.32
	$\ \nabla(u - R_h u_h)\ _{l_2}$	2.8133e+01	8.9215e+00	2.8195e+00	1.0732e+00
	order	—	1.63	1.65	1.39
M_2	size	0.142	0.071	0.0355	0.01775
	N_e	3580	14290	57584	231068
	$\ \nabla(u - u_h)\ _{l_2}$	4.3383e+00	1.8337e+00	8.6298e-01	4.1895e-01
	order	—	1.24	1.08	1.04
	$\ \nabla(u - R_h u_h)\ _{l_2}$	3.6695e+00	1.2673e+00	5.3758e-01	2.4852e-01
	order	—	1.54	1.23	1.11
M_3	size	0.2526	0.1263	0.06315	0.031575
	N_e	3578	14306	57308	229926
	$\ \nabla(u - u_h)\ _{l_2}$	4.6011e+00	2.2883e+00	1.1388e+00	5.6657e-01
	order	—	1.01	1.01	1.01
	$\ \nabla(u - R_h u_h)\ _{l_2}$	4.9676e-01	1.2533e-01	4.0088e-02	1.4136e-02
	order	—	1.99	1.64	1.50

Table: $\|\nabla(u - u_h)\|_{l_2}$ for unit circle with anisotropic solution, $(\theta, k) = (0, 0.001)$.

Anisotropic problem with an isotropic solution based on EPR, Circle domain

This example is to solve equation on the unit circle, the diffusion matrix A is chosen with $(\theta, k) = (0, 0.001)$, $f = 1$, $g = 0$ and the exact solution is given by

$$u = \frac{-1}{2(1+k)}(x^2 + y^2 - 1).$$

which is ISOTROPIC.

Special: anisotropic operator yields an isotropic solution

M_0	size	0.05	0.025	0.0125	0.0065
	N_e	3524	13698	55718	222346
	$\ \nabla(u - u_h)\ _{l_2}$	3.8288e-03	1.9402e-03	8.5460e-04	3.9779e-04
	order	—	1.00	1.17	1.11
	$\ \nabla(u - R_h u_h)\ _{l_2}$	8.6241e-03	4.2462e-03	2.0452e-03	1.0069e-03
	order	—	1.04	1.04	1.02
M_1	size	0.0812	0.0406	0.0203	0.01015
	N_e	3572	14584	58906	236460
	$\ \nabla(u - u_h)\ _{l_2}$	3.6703e-02	1.7211e-02	8.3173e-03	4.1486e-03
	order	—	1.08	1.04	1.00
	$\ \nabla(u - R_h u_h)\ _{l_2}$	2.7054e-02	1.3914e-02	6.7513e-03	3.3626e-03
	order	—	0.95	1.04	1.00
M_2	size	0.142	0.071	0.0355	0.01775
	N_e	3580	14290	57584	231068
	$\ \nabla(u - u_h)\ _{l_2}$	1.0817e-01	5.2871e-02	2.5637e-02	1.2477e-02
	order	—	1.03	1.04	1.04
	$\ \nabla(u - R_h u_h)\ _{l_2}$	6.1058e-02	2.7426e-02	1.2921e-02	6.1844e-03
	order	—	1.12	1.08	1.06
M_3	size	0.2526	0.1263	0.06315	0.031575
	N_e	3578	14306	57308	229926
	$\ \nabla(u - u_h)\ _{l_2}$	2.9254e-01	1.3786e-01	6.1030e-02	2.8343e-02
	order	—	1.09	1.17	1.10
	$\ \nabla(u - R_h u_h)\ _{l_2}$	8.4369e-02	2.8276e-02	8.9686e-03	3.2998e-03
	order	—	1.58	1.65	1.44

Table: $\|\nabla(u - u_h)\|_{l_2}$ for unit circle with isotropic solution, $(\theta, k) = (0, 0.001)$.

M_0	size	0.05	0.025	0.0125	0.0065
	N_e	3524	13698	55718	222346
	$\ u - u_h\ _{L^2}$	5.2072e-04	1.3596e-04	3.4088e-05	8.7734e-06
	order	—	1.98	1.97	1.96
	$\ u - R_h u_h\ _{L^2}$	3.5455e-04	9.4739e-05	2.4085e-05	6.3522e-06
	order	—	1.94	1.95	1.93
M_1	size	0.0812	0.0406	0.0203	0.01015
	N_e	3572	14584	58906	236460
	$\ u - u_h\ _{L^2}$	7.9218e-04	2.0201e-04	5.0041e-05	1.2839e-05
	order	—	1.94	2.00	1.96
	$\ u - R_h u_h\ _{L^2}$	4.7335e-04	1.2897e-04	3.2032e-05	8.5082e-06
	order	—	1.85	2.00	1.91
M_2	size	0.142	0.071	0.0355	0.01775
	N_e	3580	14290	57584	231068
	$\ u - u_h\ _{L^2}$	2.2441e-03	5.7311e-04	1.4323e-04	3.5802e-05
	order	—	1.97	1.99	2.00
	$\ u - R_h u_h\ _{L^2}$	1.3105e-03	3.4054e-04	8.5887e-05	2.1605e-05
	order	—	1.95	1.98	1.99
M_3	size	0.2526	0.1263	0.06315	0.031575
	N_e	3578	14306	57308	229926
	$\ u - u_h\ _{L^2}$	7.2298e-03	1.8164e-03	4.5405e-04	1.1299e-04
	order	—	1.99	2.00	2.00
	$\ u - R_h u_h\ _{L^2}$	3.5436e-03	9.0808e-04	2.2881e-04	5.7101e-05
	order	—	1.96	1.99	2.00

Table: $\|u - u_h\|_{L^2}$ for unit circle with isotropic solution, $(\theta, k) = (0, 0.001)$.

M_0	size	0.05	0.025	0.0125	0.0065
	N_e	3524	13698	55718	222346
	$\ u - u_h\ _E$	1.6626e-02	8.4018e-03	4.1539e-03	2.0768e-03
	order	—	1.01	1.00	1.00
M_1	size	0.0812	0.0406	0.0203	0.01015
	N_e	3572	14584	58906	236460
	$\ u - u_h\ _E$	3.3067e-02	1.6552e-02	8.2649e-03	4.1405e-03
	order	—	0.98	1.00	0.99
M_2	size	0.142	0.071	0.0355	0.01775
	N_e	3580	14290	57584	231068
	$\ u - u_h\ _E$	6.1614e-02	3.1157e-02	1.5582e-02	7.7925e-03
	order	—	0.99	0.99	1.00
M_3	size	0.2526	0.1263	0.06315	0.031575
	N_e	3578	14306	57308	229926
	$\ u - u_h\ _E$	1.1184e-01	5.6095e-02	2.8052e-02	1.3995e-02
	order	—	1.00	1.00	1.00
	size	0.3470e-02	1.6558e-02	8.2103e-03	4.0720e-03
	N_e	—	1.02	1.01	1.01
	$\ u - u_h\ _E$	—	—	—	—
	order	—	—	—	—

Table: $\|u - u_h\|_{H_E}$ for unit circle with isotropic solution, $(\theta, k) = (0, 0.001)$.

we find EPR improves the solution both in L^2 norm and energy norm on the mesh of M_3 , more precisely,

$$\|u - R_h u_h\|_{L^2} \approx \frac{1}{2} \|u - u_h\|_{L^2},$$

$$\|u - R_h u_h\|_{H_E} \approx \frac{1}{3} \|u - u_h\|_{H_E},$$

$$\|\nabla(u - R_h u_h)\|_{l_2} \approx \frac{1}{9} \|\nabla(u - u_h)\|_{l_2},$$

That means $\|R_h u_h - u_h\|_*$ could be good a posteriori error estimator of $\|u - u_h\|_*$

Conclusions

- $M = A^{-1}$ is the perfect metric for the linear finite element solution for anisotropic elliptic problem with ACVDT mesh.
- There is superconvergence of the finite element solutions at the nodes on ACVT meshes
- In the perfect matched anisotropic meshes, EPR technique improve the accuracy both in L^2 and H_E norms,. Superconvergence of the gradients preserves at the geometrical centers of the elements.
- Recovery type a posteriori error estimator can be provided by EPR in the anisotropic meshes.
- Anisotropic meshes have advantages even in the case of anisotropic problems with isotropic smooth solutions.

Thank you for your attention!