Multilevel Correction Method for Symmetric Eigenvalue Problem

Hehu Xie[1] Yu Li[1]

[1]The Institute of Computational Mathematics and Scientific/Engineering Computing of Chinese Academy of Sciences

September 11, 2013

Contents

J. Xu, Iterative methods by space decomosition and subspace correction, SIAM Rev. 34(1992), pp. 581-613.

J. Xu, A new class of iterative methods for nonselfadjoint or indefinite problems, SIAM J. Numer. Anal. 29(1992), pp. 303-319.

J. Xu, Iterative methods by SPD and small subspace solvers for nonsymmetric or indefinite problems, Proceedings of the 5th International Conference on Domain Decomposition Method (1992), pp. 106-118.

J. Xu, A novel two-grid method for semilinear equations, SIAM J. Sci. Comput. 15(1994), pp. 231-237.

J. Xu, Two-level discretization techniques for linear and nonlinear PDEs, SIAM J. Numer. Anal. 33(1996), pp. 1759-1778.

W. Layton and L. Tobiska, A two-level method with backtracking for Navier-Stokes equations, SIAM J. Sci. Comput. 16(1998), pp. 2035-2054.

W. Layton, H. K. Lee and J. Peterson, Numerical solution of the stationary Navier-Stokes equations using a multilevel finite element method, SIAM J. Sci. Comput. 20(1998), pp. 1-12.

X. Dai and X. Cheng, A two-level method based on Newton iteration for the Navier-Stokes equations, J. Comput. App. Math, 220(2008), pp. 556-573.

J. Xu and A. Zhou, A two-grid discretization scheme for eigenvalue problems, Math. Comput., 70 (233) (2001), 17-25.

J. Xu and A. Zhou, Local and parallel finite element algorithm for eigenvalue problems, Acta Math. Appl. Sin. Engl. Ser., 18(2) (2002), pp. 185-200.

Qun Lin, Huhe Xie, An Observation on Aubin-Nitsche Lemma and Its Applications, Mathematics in Practice and Theory, Vol41, No.17 (2011), pp. 247-258.

Yu Li, Xiaole Han and Hehu Xie, A Multilevel Correction Method for Steklov Eigenvalue Problem by Nonconforming Finite Element Methods, finished.

Yu Li, Xiaole Han, Qun Lin and Hehu Xie, Local and Parallel Finite Element Algorithms Based On Multi-Grid Discretizations for Eigenvalue Problem, finished.

Possion Eigenvalue Problem

$$\left\{\begin{array}{rcll} -\Delta u & = & \lambda u, & \text{in } \Omega, \\ u & = & 0, & \text{on } \partial\Omega, \\ \int_\Omega |u|^2 \mathrm{d}x & = & 1. & \end{array}\right.$$

Questions

- Design $O(N)$ algorithm for eigenvalue problem
- Parallelization for multi-eigenpairs

Eigen value problem

Weak Form

Find $(\lambda, u) \in \mathcal{R} \times H_0^1(\Omega)$ such that

$$a(u, v) \; = \; \lambda b(u, v), \quad \forall v \in H_0^1(\Omega),$$

where $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x$, $b(u, v) = \int_\Omega uv \,\mathrm{d}x$.

Discretization

Find $(\lambda_h, u_h) \in \mathcal{R} \times V_{0h}(\Omega)$, $V_{0h}(\Omega) \subset H_0^1(\Omega)$ such that

$$a(u_h, v_h) \; = \; \lambda_h b(u_h, v_h), \quad \forall v_h \in V_{0h}(\Omega).$$

Notation

$$
\begin{aligned}
M(\lambda_i) &= \left\{ w \in H_0^1(\Omega) : w \text{is an eigenvector of } \lambda_i, \|w\|_b = 1 \right\}, \\
\delta_{\lambda_i}(h) &= \sup_{w \in M(\lambda_i)} \inf_{v \in V_{0h}(\Omega)} \|w - v\|_{1,\Omega}, \\
\eta_\Omega(h) &= \inf_{v_h \in V_{0h}(\Omega)} \sup_{\|w\|_{1,\Omega}=1} \|(L^*)^{-1} w - v_h)\|_{1,\Omega}.
\end{aligned}
$$

Error Estimate

$$
\begin{aligned}
\|u_i - u_{i,h}\|_{1,\Omega} &\lesssim \delta_{\lambda_i}(h), \\
\|u_i - u_{i,h}\|_{-1,\Omega} &\lesssim \eta_\Omega(h)\|u_i - u_{i,h}\|_{1,\Omega}, \\
\lambda_i \le \lambda_{i,h} &\le \lambda_i + C_i \delta^2_{\lambda_i}(h).
\end{aligned}
$$

Algorithm

Construct $V_{0H}(\Omega) \subset V_{0h}(\Omega) \subset H_0^1(\Omega)$

$$
\begin{aligned}
&\text{Step 1} \quad a(u_H, \psi) = \lambda_H b(u_H, \psi), \quad \forall \psi \in V_{0H}(\Omega)\\
&\text{Step 2} \quad a(\tilde{u}^h, \chi) = \lambda_H b(u_H, \chi), \quad \forall \chi \in V_{0h}(\Omega)\\
&\text{Step 3} \quad a(u_{H,h}, \varphi) = \lambda_{H,h} b(u_{H,h}, \varphi), \quad \forall \varphi \in V_{H,h}
\end{aligned}
$$

where $V_{H,h} = V_{0H}(\Omega) + \text{span}\{\tilde{u}^h\}$.

Error Estimate

$$\begin{aligned} \|u - u_{H,h}\|_{-1,\Omega} &\lesssim \eta_\Omega(H)\delta_\lambda(h) \\ \|u - u_{H,h}\|_{1,\Omega} &\lesssim \delta_\lambda(h) \\ |\lambda - \lambda_{H,h}| &\lesssim \delta_\lambda^2(h) \end{aligned}$$

Multilevel Correction

1. Find $(\lambda_H, u_H) \in \mathcal{R} \times V_{0H}(\Omega)$ such that $\|u_H\|_b = 1$

$$a(u_H, v_H) = \lambda_H b(u_H, v_H), \quad \forall v_H \in V_{0H}(\Omega).$$

2. Let $h_0 = H$ and construct $V_{0h_0}(\Omega) \subset V_{0h_1}(\Omega) \subset \cdots \subset V_{0h_N}(\Omega)$
Do k = 0:N-1
    - Solve boundary value problem:
    find $\widetilde{u}_{h_{k+1}} \in V_{0h_{k+1}}(\Omega)$ such that $\forall v_{h_{k+1}} \in V_{0h_{k+1}}(\Omega)$

$$a(\widetilde{u}_{h_{k+1}}, v_{h_{k+1}}) = \lambda_{h_k} b(u_{h_k}, v_{h_{k+1}}),$$

    - define new finite-dimensional space $V_{H,h_{k+1}} = V_{0H}(\Omega) + \mathrm{span}\{\widetilde{u}_{h_{k+1}}\}$,
    and solve eigenvalue problem:
    find $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{H,h_{k+1}}$ such that $\forall v_{H,h_{k+1}} \in V_{H,h_{k+1}}$

$$a(u_{h_{k+1}}, v_{H,h_{k+1}}) = \lambda_{h_{k+1}} b(u_{h_{k+1}}, v_{H,h_{k+1}}),$$

    satisfying $\|u_{h_{k+1}}\|_b = 1$.

END Do

$$
\begin{aligned}
\|u - u_{h_N}\|_1 &\lesssim \sum_{k=0}^{N} \eta_\Omega(H)^{N-k} \delta_\lambda(h_k) \sim h_N \\
|\lambda - \lambda_{h_N}| &\lesssim \Big(\sum_{k=0}^{N} \eta_\Omega(H)^{N-k} \delta_\lambda(h_k)\Big)^2 \sim h_N^2.
\end{aligned}
$$

where $h_k = 2^{N-k} h_N$.

Matrix Expression

$$V_{0H}(\Omega) \subset V_{0h_1}(\Omega) \subset V_{0h_2}(\Omega)$$

$$\begin{aligned} V_{0H}(\Omega) &= \operatorname{span}\{\phi_H^1, \phi_H^2, \ldots, \phi_H^{n_0}\}, \\ V_{0h_1}(\Omega) &= \operatorname{span}\{\phi_{h_1}^1, \phi_{h_1}^2, \ldots, \phi_{h_1}^{n_1}\}, \\ V_{0h_2}(\Omega) &= \operatorname{span}\{\phi_{h_2}^1, \phi_{h_2}^2, \ldots, \phi_{h_2}^{n_2}\}. \end{aligned}$$

And $\mathcal{I}_1^2 \in \mathcal{R}^{n_2 \times n_1}$, $\mathcal{I}_0^2 \in \mathcal{R}^{n_2 \times n_0}$ such that

$$\begin{aligned} (\phi_{h_1}^1, \phi_{h_1}^2, \ldots, \phi_{h_1}^{n_1}) &= (\phi_{h_2}^1, \phi_{h_2}^2, \ldots, \phi_{h_2}^{n_2})\mathcal{I}_1^2, \\ (\phi_H^1, \phi_H^2, \ldots, \phi_H^{n_0}) &= (\phi_{h_2}^1, \phi_{h_2}^2, \ldots, \phi_{h_2}^{n_2})\mathcal{I}_0^2 \end{aligned}$$

1. First, one knows an approximate of the eigenpair $(\lambda_{h_1}, u_{h_1}) \in \mathcal{R} \times V_{0h_1}(\Omega)$, where $u_{h_1} = (\phi_{h_1}^1, \phi_{h_1}^2, \ldots, \phi_{h_1}^{n_1})\mathcal{U}_{h_1}$ and $\mathcal{U}_{h_1} \in \mathcal{R}^{n_1}$.
2. Define the linear algebraic system:
Find $\mathcal{E}_{h_2} \in \mathcal{R}^{n_2}$ satisfying

$$\mathcal{A}_{h_2}\mathcal{E}_{h_2} = \lambda_{h_1}\mathcal{B}_{h_2}\mathcal{I}_1^2\mathcal{U}_{h_1} - \mathcal{A}_{h_2}\mathcal{I}_1^2\mathcal{U}_{h_1},$$

where $(\mathcal{A}_{h_2})_{ij} = a(\phi_{h_2}^j, \phi_{h_2}^i)$ and $(\mathcal{B}_{h_2})_{ij} = b(\phi_{h_2}^j, \phi_{h_2}^i)$.
Then $\widetilde{\mathcal{U}}_{h_2} = \mathcal{I}_1^2\mathcal{U}_{h_1} + \mathcal{E}_{h_2} \in \mathcal{R}^{n_2}$ and
$\widetilde{u}_{h_2} = (\phi_{h_2}^1, \phi_{h_2}^2, \ldots, \phi_{h_2}^{n_2})\widetilde{\mathcal{U}}_{h_2} \in V_{0h_2}(\Omega)$.

3. Define the auxiliary generalized eigenvalue problem:
Find $(\lambda_{h_2}, \mathcal{U}_{H,h_2}) \in \mathcal{R} \times \mathcal{R}^{n_0+1}$ such that $\mathcal{U}_{H,h_2}^T \mathcal{U}_{H,h_2} = 1$ and

$$\mathcal{A}_{H,h_2}\mathcal{U}_{h_2} = \lambda_{H,h_2}\mathcal{B}_{H,h_2}\mathcal{U}_{h_2},$$

where

$$\mathcal{A}_{H,h_2} = \begin{bmatrix} \mathcal{A}_H & \mathcal{C} \\ \mathcal{C}^T & \widetilde{\mathcal{U}}_{h_2}^T \mathcal{A}_{h_2} \widetilde{\mathcal{U}}_{h_2} \end{bmatrix}, \ \mathcal{B}_{H,h_2} = \begin{bmatrix} \mathcal{B}_H & \mathcal{D} \\ \mathcal{D}^T & \widetilde{\mathcal{U}}_{h_2}^T \mathcal{B}_{h_2} \widetilde{\mathcal{U}}_{h_2} \end{bmatrix},$$

$(\mathcal{A}_H)_{ij} = a(\phi_H^j, \phi_H^i)$, $(\mathcal{B}_H)_{ij} = b(\phi_H^j, \phi_H^i)$, $(\mathcal{C})_i = a(\widetilde{u}_{h_2}, \phi_H^i)$ and $(\mathcal{D})_i = b(\widetilde{u}_{h_2}, \phi_H^i)$,

4. Set $u_{h_2} = (\phi_H^1, \phi_H^2, \ldots, \phi_H^{n_0}, \widetilde{u}_{h_2})\mathcal{U}_{H,h_2} \in V_{0h_2}(\Omega)$ and $\lambda_{h_2} = \lambda_{H,h_2}$.

minimum eigenvalue, 7 level, maximum ndofs = 1050625

MaxLevel	NDofs	Time	Error	Order
1	289	0.01	0.047645	
2	1089	0.04	0.011896	2.00
3	4225	0.18	0.002973	2.00
4	16641	0.71	0.000743	1.99
5	66049	2.81	0.000186	1.99
6	263169	11.63	0.000046	2.01
7	1050625	48.38	0.000012	1.93

- Achieve a linear algorithm for solving the eigenvalue problem

Numerical Tests II

128 eigenpairs, 5 level, Ti – run time in i-th level,
maximum ndofs = 1050625, minimum ndofs = 4225

NProcs	T1	T2	T3	T4	T5	TotalTime
1	1.33	18.68	29.43	80.75	250.09	380.28
2	1.34	11.48	16.80	48.56	202.19	280.37
4	1.32	7.16	11.31	34.33	135.71	135.71
8	1.33	4.91	7.43	20.13	75.51	109.31
16	1.33	3.84	5.55	13.46	47.57	71.75
32	1.35	3.44	4.69	10.21	35.89	55.58
64	1.33	3.23	4.16	8.56	27.12	44.40
128	1.38	3.03	4.04	7.73	23.41	39.58

- Achieve a parallel algorithm for solving the eigenvalue problem

Weak Form

Find $(\lambda, u) \in \mathcal{R} \times H^1(\Omega)$ such that

$$a(u, v) \;=\; \lambda b(u, v), \quad \forall v \in H^1(\Omega),$$

where $a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\ \mathrm{d}\Omega,\ b(u,v) = \int_{\partial\Omega} uv\,\mathrm{d}s.$

Discretization

Find $(\lambda_h, u_h) \in \mathcal{R} \times V_h(\Omega)$, such that

$$a_h(u_h, v_h) \;=\; \lambda_h b(u_h, v_h), \quad \forall v_h \in V_h(\Omega),$$

where $a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \int_K (\nabla u_h \cdot \nabla v_h + u_h v_h)\ \mathrm{d}K.$

Notation

CR element is define on the triangle partition and

$$V_h = \{v \in L^2(\Omega) : v|_K \in \text{span}\{1, x, y\}, \int_l v|_{K_1}\, \mathrm{d}s = \int_l v|_{K_2}\, \mathrm{d}s, \text{ when } K_1 \cap K_2 = l \in \mathcal{E}_h^i\},$$

where $K, K_1, K_2 \in \mathcal{T}_h$ and $\mathcal{E}_h^i$ denotes the internal edge set.

1. Find $(\lambda_H, u_H) \in \mathcal{R} \times V_H(\Omega)$ such that $\|u_H\|_b = 1$

$$a_H(u_H, v_H) = \lambda_H b(u_H, v_H), \quad \forall v_H \in V_H(\Omega).$$

2. Let $h_0 = H$ and construct $V_{h_1}(\Omega), \cdots, V_{h_N}(\Omega)$ on the sequence of nested meshes $\mathcal{T}_{h_1}, \cdots, \mathcal{T}_{h_N}$.
   Do k = 0:N-1
   - Solve boundary value problem:
     find $\widetilde{u}_{h_{k+1}} \in V_{h_{k+1}}(\Omega)$ such that $\forall v_{h_{k+1}} \in V_{h_{k+1}}(\Omega)$

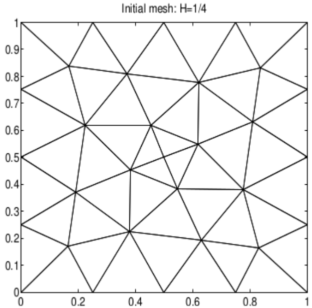
$$a_{h_{k+1}}(\widetilde{u}_{h_{k+1}}, v_{h_{k+1}}) = \lambda_{h_k} b(u_{h_k}, v_{h_{k+1}}),$$

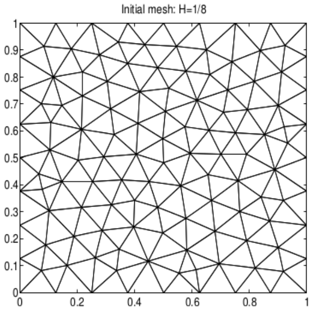
- define new finite-dimensional space $V_{H,h_{k+1}} = W_H(\Omega) + \mathrm{span}\{\widetilde{u}_{h_{k+1}}\}$, and solve eigenvalue problem:
  find $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{H,h_{k+1}}$ such that $\forall v_{H,h_{k+1}} \in V_{H,h_{k+1}}$

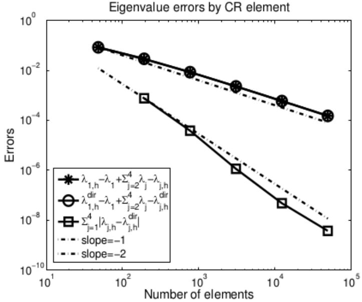
$$a_{h_{k+1}}(u_{h_{k+1}}, v_{H,h_{k+1}}) = \lambda_{h_{k+1}} b(u_{h_{k+1}}, v_{H,h_{k+1}}),$$

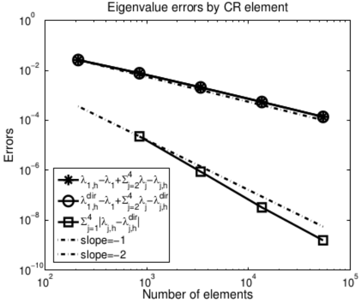
  satisfying $\|u_{h_{k+1}}\|_b = 1$.
  And $W_H(\Omega)$ is conforming finite element space on the coarsest mesh $\mathcal{T}_H$.

END Do


Initial mesh: H=1/4


Initial mesh: H=1/8


Eigenvalue errors by CR element
Errors
Number of elements
$\lambda_{1,h}-\lambda_1+\Sigma_{j=2}^4\lambda_j-\lambda_{j,h}$
$\lambda_{1,h}^{dir}-\lambda_1+\Sigma_{j=2}^4\lambda_j-\lambda_{j,h}^{dir}$
$\Sigma_{j=1}^4|\lambda_{j,h}-\lambda_{j,h}^{dir}|$
slope=−1
slope=−2


Eigenvalue errors by CR element
Errors
Number of elements
$\lambda_{1,h}-\lambda_1+\Sigma_{j=2}^4\lambda_j-\lambda_{j,h}$
$\lambda_{1,h}^{dir}-\lambda_1+\Sigma_{j=2}^4\lambda_j-\lambda_{j,h}^{dir}$
$\Sigma_{j=1}^4|\lambda_{j,h}-\lambda_{j,h}^{dir}|$
slope=−1
slope=−2

Thank You