

The monotone finite volume scheme for diffusion equation

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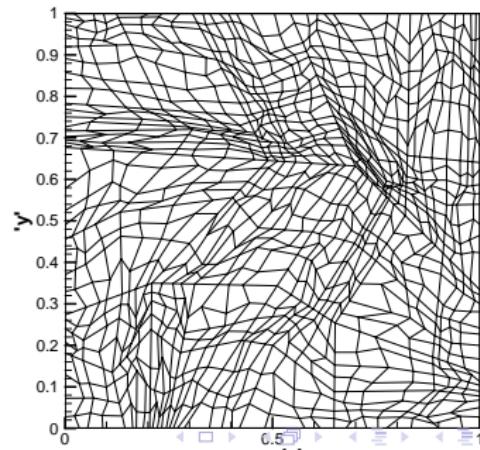
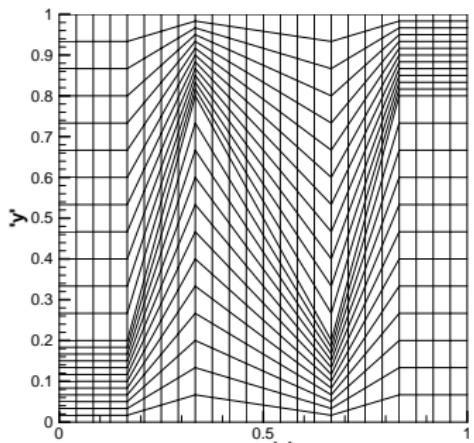
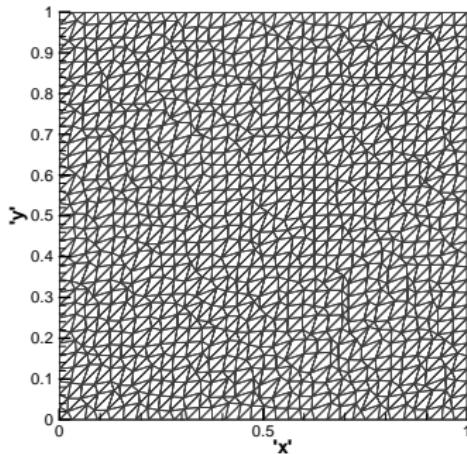
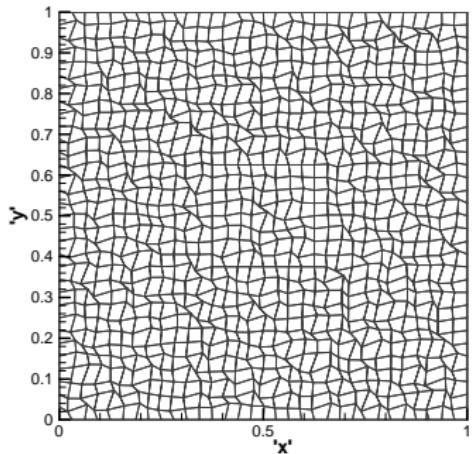
Diffusion equation

$$\begin{aligned}-\nabla \cdot (\kappa \nabla u) &= f && \text{in } \Omega, \\ u(x) &= g && \text{on } \partial\Omega,\end{aligned}$$

where

- κ is full anisotropic diffusion tensor
- f is the source

Some distorted meshes



Challenging problems

- ★ negative temperature, non-physical oscillation
- ★ violation of the second law of thermodynamics

The discrete method should preserve some important properties of PDE:

conservation, monotonicity, maximum principle ...

Some requirements for scheme

The schemes should

- ★ be locally conservative
- ★ be monotone, or satisfy maximum principle
- ★ be reliable on unstructured anisotropic meshes
- ★ allow heterogeneous full diffusion tensor
- ★ result in a sparse system with minimal number of non-zero entries
- ★ have second order accuracy

- ★ C. Le Potier (2005), C.C.Acad. Sci. Paris, Ser.
nonlinear monotone scheme, triangle meshes, Δt
small enough.

- ★ K. Lipnikov, M. Shashkov, D. Svyatskiy, Yu.
Vassilevski (2007), J. Comput. Phys.
specific definition of collocation points.

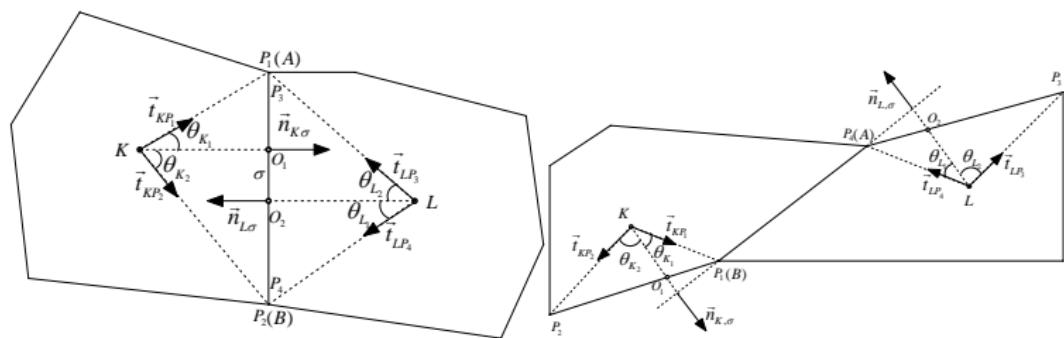
- ★ J.M. Nordbotten and I. Aavatsmark (2007),
Numer. Math.

It is impossible to construct linear nine-point methods which unconditionally satisfy the monotonicity criteria when the discretization satisfies local conservation and exact reproduction of linear solution.

- ★ R.Liska, and M.Shashkov (2008), Commun. Comput. Phys.

Two approaches have been suggested to enforce discrete extremum principle for linear finite element solutions on triangular meshes.

- ★ G.Yuan and Z. Sheng (2008), J. Comput. Phys.
 - based on an adaptive approach of choosing stencil
 - no severe restrictions on meshes
 - no any special restriction for the choice of collocation point



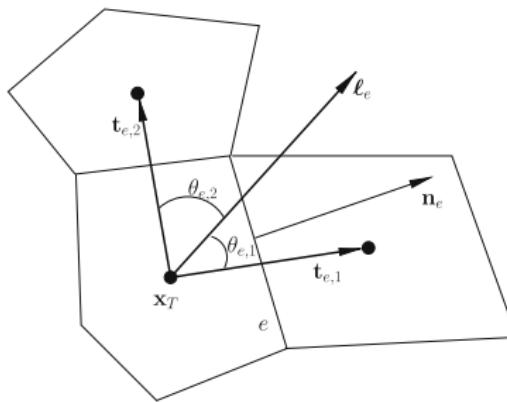
- ★ Z. Sheng, J. Yue and G. Yuan (2009), SIAM J. Sci. Comput.
present monotone finite volume scheme for nonequilibrium radiation diffusion problems,
more efficient than nine point scheme

Comparison of computational efficient

	$it_{nonlinear}^{\#}$	$it_{linear}^{\#}$
monotone scheme	5.80	56.26
nine point scheme	15.39	99.42

Recent progress(5)

- ★ K. Lipnikov, D. Svyatskiy, Y. Vassilevski(2009),
J. Comput. Phys.
An interpolation-free nonlinear monotone
scheme.

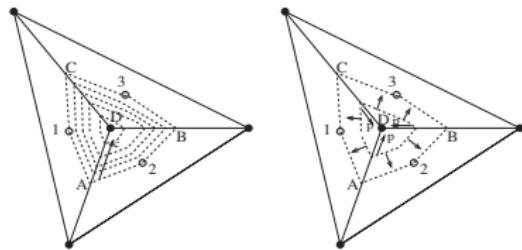


- ★ A. Danilov, Y. Vassilevski(2009), Russ. J.
Numer. Anal. Math. Modelling.
It has been extended to three dimension.

- ★ K. Lipnikov, D. Svyatskiy, Y. Vassilevski (2010), J. Comput. Phys.
A monotone finite volume method for advection-diffusion equations on unstructured polygonal meshes.

- ★ K. Nikitin, Y. Vassilevski(2010), Russ. J. Numer. Anal. Math. Modelling.
It has been extended to three dimension.

- ★ H. A. Friis, M.G. Edwards (2010, 2011), J. Comput. Phys.
MPFA finite volume scheme with full pressure support.



- ★ M.G. Edwards, H. Zheng (2011), SIAM J. Sci. Comput.
It has been extended to three dimension.

- ★ Z. Sheng, G. Yuan (2011), J. Comput. Phys.
The nonlinear finite volume scheme preserving maximum principle.
- ★ J. Droniou, C. Le Potier (2011), SIAM J. Numer. Anal.
Construction and analysis of scheme preserving maximum principle.
- ★ K. Lipnikov, D. Svyatskiy, Y. Vassilevski, (2012), Russ. J. Numer. Anal. Math. Modelling.
Minimal stencil finite volume scheme with the discrete maximum principle.

- ★ Z. Sheng, G. Yuan (2012), J. Comput. Phys.
An improved monotone finite volume scheme.
- ★ C. Cancès, M. Cathala, and C. Le Potier,
(2013), Numer. Math.
The nonlinear technique to correct a general finite
volume scheme.
- ★ ...

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The steady diffusion equation

$$-\nabla \cdot (\kappa \nabla u) = f \quad \text{in } \Omega, \quad (1)$$

$$u(x) = g \quad \text{on } \partial\Omega. \quad (2)$$

Integrate equation (1) over the cell K to obtain

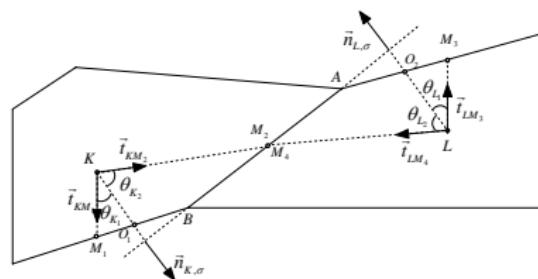
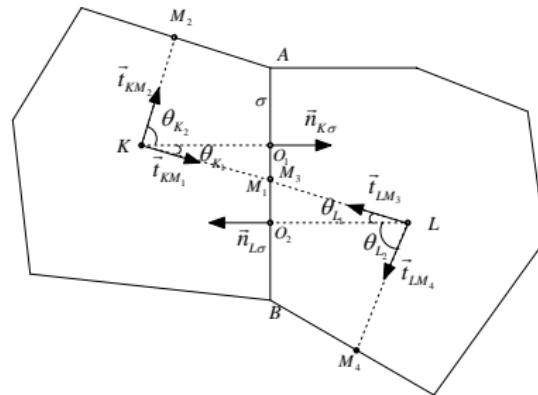
$$\sum_{\sigma \in \mathcal{E}_K} \mathcal{F}_{K,\sigma} = \int_K f(x) dx,$$

where the continuous flux on the edge σ is

$$\mathcal{F}_{K,\sigma} = - \int_{\sigma} \nabla u(x) \cdot \kappa(x)^T \vec{n}_{K\sigma} dl.$$

The monotone scheme

The adaptive approach of choosing stencil



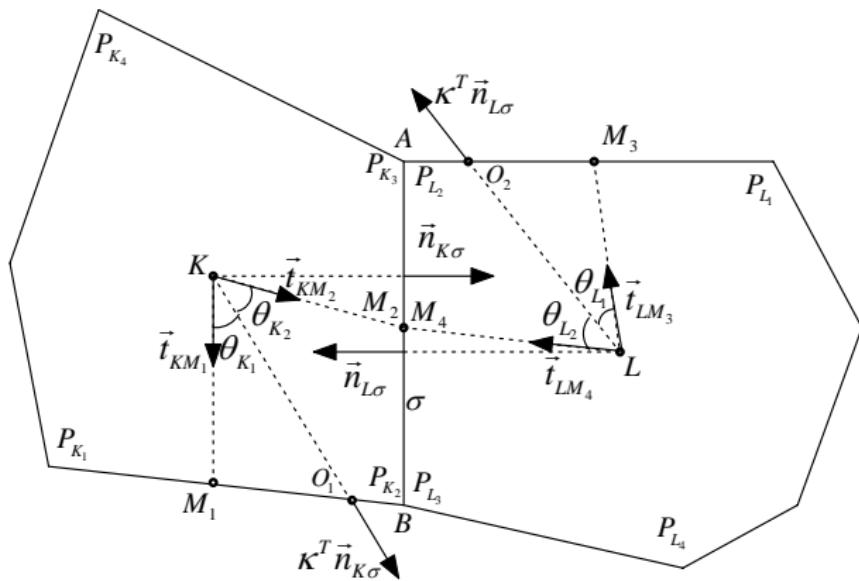
The expression of continuous flux

$$\mathcal{F}_{K,\sigma} = - \int_{\sigma} \nabla u(x) \cdot \kappa(x)^T \vec{n}_{K\sigma} dl.$$

Vector decomposition

$$\kappa(x)^T \vec{n}_{K\sigma} = ?$$

Stencil and notation



K, L, \dots cell and cell center,

A, B vertex,

$M_1, M_2, M_3, M_4, \dots$ midpoint of edge,

$\vec{n}_{K\sigma}$ and $\vec{n}_{L\sigma}$ unit outer normal vector, \vec{t}_{KM_i} and \vec{t}_{LM_i} unit tangential vectors.

The expression of flux

Vector decomposition

$$\frac{\kappa^T \vec{n}_{K\sigma}}{|\kappa^T \vec{n}_{K\sigma}|} = \frac{\sin \theta_{K_2}}{\sin \theta_K} \vec{t}_{KM_1} + \frac{\sin \theta_{K_1}}{\sin \theta_K} \vec{t}_{KM_2}.$$

$$\begin{aligned}\mathcal{F}_{K,\sigma} &= - \int_{\sigma} \nabla u(x) \cdot \kappa(x)^T \vec{n}_{K\sigma} dl \\&= - \int_{\sigma} \nabla u(x) \cdot |\kappa^T(K) \vec{n}_{K\sigma}| \left(\frac{\sin \theta_{K_2}}{\sin \theta_K} \vec{t}_{KM_1} + \frac{\sin \theta_{K_1}}{\sin \theta_K} \vec{t}_{KM_2} \right) dl \\&= - |\kappa^T(K) \vec{n}_{K\sigma}| \left(\frac{\sin \theta_{K_2}}{\sin \theta_K} \int_{\sigma} \nabla u(x) \cdot \vec{t}_{KM_1} dl \right. \\&\quad \left. + \frac{\sin \theta_{K_1}}{\sin \theta_K} \int_{\sigma} \nabla u(x) \cdot \vec{t}_{KM_2} dl \right) \\&\approx - |\kappa^T(K) \vec{n}_{K\sigma}| |\sigma| \left(\frac{\sin \theta_{K_2}}{\sin \theta_K} \frac{u_{M_1} - u_K}{|KM_1|} + \frac{\sin \theta_{K_1}}{\sin \theta_K} \frac{u_{M_2} - u_K}{|KM_2|} \right) \\&\equiv F_1\end{aligned}$$

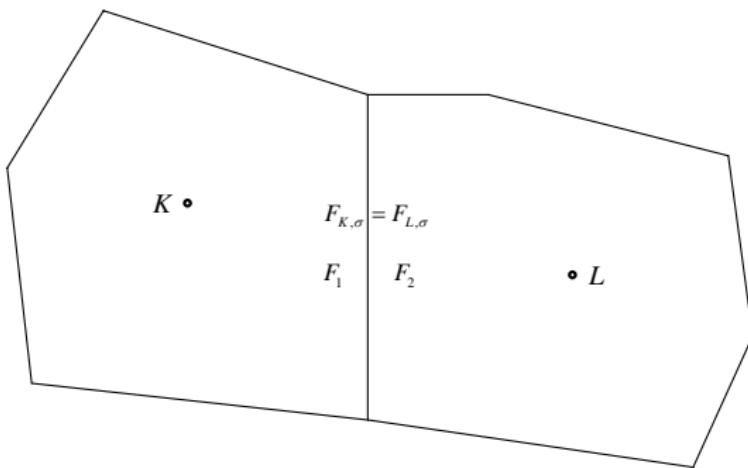
The expression of flux

The expression of flux

$$F_1 = -|\kappa^T(K)\vec{n}_{K\sigma}||\sigma| \left(\frac{\sin \theta_{K_2}}{\sin \theta_K} \frac{u_{M_1} - u_K}{|KM_1|} + \frac{\sin \theta_{K_1}}{\sin \theta_K} \frac{u_{M_2} - u_K}{|KM_2|} \right),$$

$$F_2 = -|\kappa^T(L)\vec{n}_{L\sigma}||\sigma| \left(\frac{\sin \theta_{L_2}}{\sin \theta_L} \frac{u_{M_3} - u_L}{|LM_3|} + \frac{\sin \theta_{L_1}}{\sin \theta_L} \frac{u_{M_4} - u_L}{|LM_4|} \right).$$

The expression of flux



The expression of conservative flux

$$F_{K,\sigma} = \mu_1 F_1 - \mu_2 F_2,$$

$$F_{L,\sigma} = \mu_2 F_2 - \mu_1 F_1.$$

The expression of flux

The expression of conservative flux

$$\begin{aligned} F_{K,\sigma} = & \mu_1 \frac{|\kappa^T(K) \vec{n}_{K\sigma} ||\sigma|}{\sin \theta_K} \left(\frac{\sin \theta_{K_2}}{|KM_1|} + \frac{\sin \theta_{K_1}}{|KM_2|} \right) u_K \\ & - \mu_2 \frac{|\kappa^T(L) \vec{n}_{L\sigma} ||\sigma|}{\sin \theta_L} \left(\frac{\sin \theta_{L_2}}{|LM_3|} + \frac{\sin \theta_{L_1}}{|LM_4|} \right) u_L \\ & - \mu_1 \frac{|\kappa^T(K) \vec{n}_{K\sigma} ||\sigma|}{\sin \theta_K} \left(\frac{\sin \theta_{K_2}}{|KM_1|} u_{M_1} + \frac{\sin \theta_{K_1}}{|KM_2|} u_{M_2} \right) \\ & + \mu_2 \frac{|\kappa^T(L) \vec{n}_{L\sigma} ||\sigma|}{\sin \theta_L} \left(\frac{\sin \theta_{L_2}}{|LM_3|} u_{M_3} + \frac{\sin \theta_{L_1}}{|LM_4|} u_{M_4} \right). \end{aligned}$$

μ_1 and μ_2 satisfy the following relation:

$$\begin{cases} \mu_1 + \mu_2 = 1 \\ -a_1 \mu_1 + a_2 \mu_2 = 0 \end{cases}$$

$$a_1 = \frac{|\kappa^T(K) \vec{n}_{K\sigma} ||\sigma|}{\sin \theta_K} \left(\frac{\sin \theta_{K_2}}{|KM_1|} u_{M_1} + \frac{\sin \theta_{K_1}}{|KM_2|} u_{M_2} \right), a_2 = \frac{|\kappa^T(L) \vec{n}_{L\sigma} ||\sigma|}{\sin \theta_L} \left(\frac{\sin \theta_{L_2}}{|LM_3|} u_{M_3} + \frac{\sin \theta_{L_1}}{|LM_4|} u_{M_4} \right).$$

Coefficient

$$\begin{cases} \mu_1 + \mu_2 = 1 \\ -a_1\mu_1 + a_2\mu_2 = 0 \end{cases}$$

- ★ $a_1 + a_2 \neq 0$

$$\mu_1 = \frac{a_2}{a_1+a_2},$$

$$\mu_2 = \frac{a_1}{a_1+a_2}.$$

- ★ $a_1 + a_2 = 0$

$$\mu_1 = \mu_2 = \frac{1}{2},$$

The expression of conservative flux

$$F_{K,\sigma} = A_{K,\sigma} u_K - A_{L,\sigma} u_L,$$

where

$$A_{K,\sigma} = \mu_1 \frac{|\kappa^T(K) \vec{n}_{K\sigma}| |\sigma|}{\sin \theta_K} \left(\frac{\sin \theta_{K_2}}{|KM_1|} + \frac{\sin \theta_{K_1}}{|KM_2|} \right),$$

$$A_{L,\sigma} = \mu_2 \frac{|\kappa^T(L) \vec{n}_{L\sigma}| |\sigma|}{\sin \theta_L} \left(\frac{\sin \theta_{L_2}}{|LM_3|} + \frac{\sin \theta_{L_1}}{|LM_4|} \right).$$

The expression of conservative flux

$$F_{K,\sigma} = A_{K,\sigma} u_K - A_{L,\sigma} u_L,$$

where

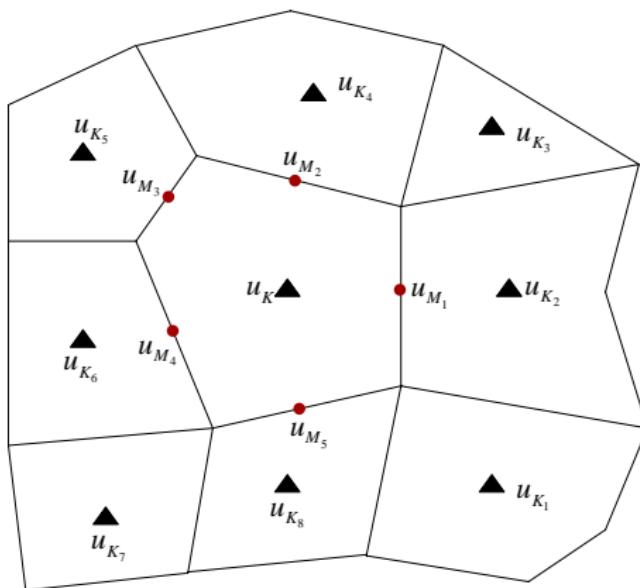
$$A_{K,\sigma} = \mu_1 \frac{|\kappa^T(K) \vec{n}_{K\sigma}| |\sigma|}{\sin \theta_K} \left(\frac{\sin \theta_{K_2}}{|KM_1|} + \frac{\sin \theta_{K_1}}{|KM_2|} \right),$$

$$A_{L,\sigma} = \mu_2 \frac{|\kappa^T(L) \vec{n}_{L\sigma}| |\sigma|}{\sin \theta_L} \left(\frac{\sin \theta_{L_2}}{|LM_3|} + \frac{\sin \theta_{L_1}}{|LM_4|} \right).$$

$$A_{K,\sigma} > 0, \quad A_{L,\sigma} > 0.$$

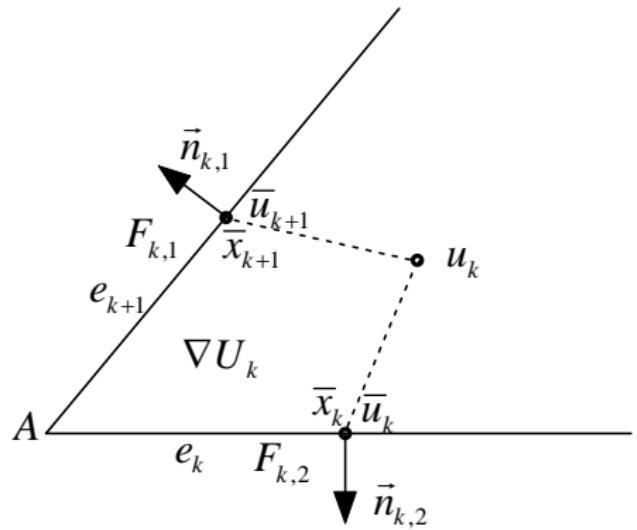
The monotone scheme

Question: How to eliminate cell edge unknowns?



$$u(M_i) = \sum_{j=1}^n \omega_j u(K_j) + O(h^2).$$

The expression of cell edge unknowns



$$\nabla U_k \cdot (\bar{x}_{k+1} - x_k) = \bar{u}_{k+1} - u_k,$$

$$\nabla U_k \cdot (\bar{x}_k - x_k) = \bar{u}_k - u_k,$$

⇒

$$\nabla U_k = \frac{1}{T_k} [R(\bar{x}_k - x_k)(\bar{u}_k - u_k) - R(\bar{x}_{k+1} - x_k)(\bar{u}_{k+1} - u_k)].$$

The expression of flux on sub-edge

$$\begin{aligned}F_{k,1} &= \omega_{k,1,1}(\bar{u}_{k+1} - u_k) + \omega_{k,1,2}(\bar{u}_k - u_k), \\F_{k,2} &= \omega_{k,2,1}(\bar{u}_{k+1} - u_k) + \omega_{k,2,2}(\bar{u}_k - u_k).\end{aligned}$$

where

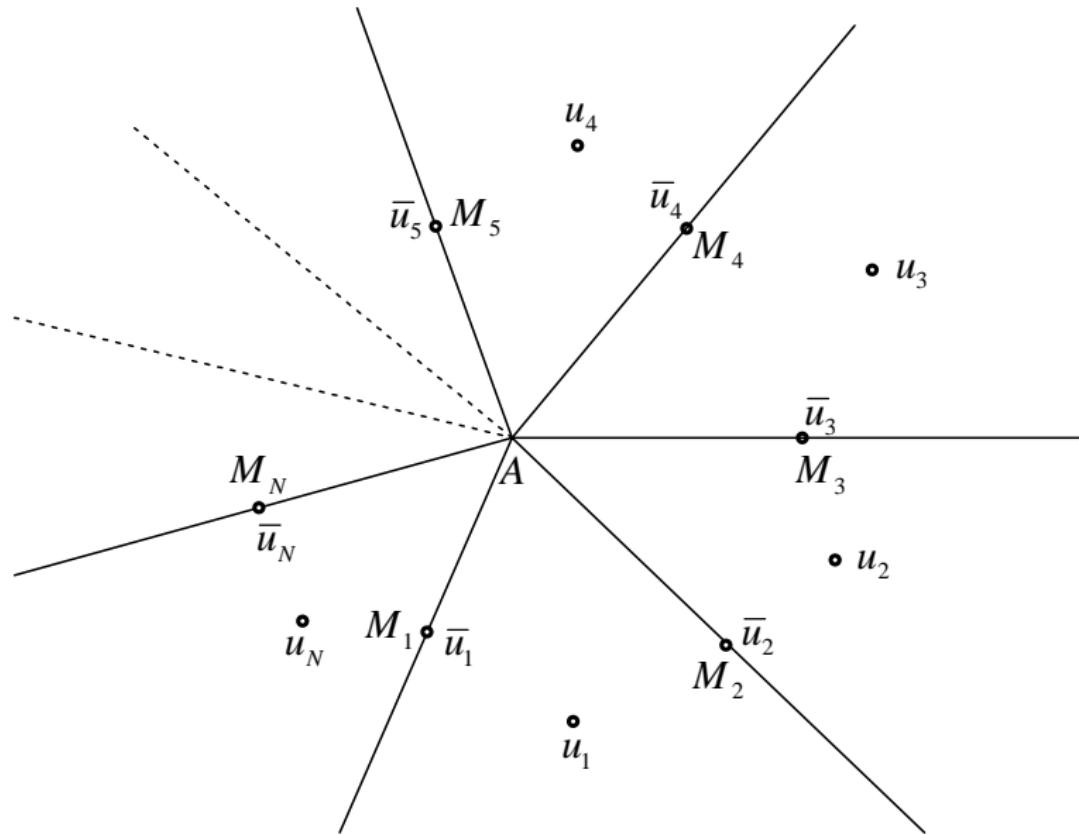
$$\omega_{k,1,1} = -\frac{|\boldsymbol{e}_{k+1}|}{T_k} \boldsymbol{R}(\bar{\boldsymbol{x}}_k - \boldsymbol{x}_k) \cdot \boldsymbol{\kappa}_k^T \vec{\boldsymbol{n}}_{k,1},$$

$$\omega_{k,1,2} = \frac{|\boldsymbol{e}_{k+1}|}{T_k} \boldsymbol{R}(\bar{\boldsymbol{x}}_{k+1} - \boldsymbol{x}_k) \cdot \boldsymbol{\kappa}_k^T \vec{\boldsymbol{n}}_{k,1},$$

$$\omega_{k,2,1} = -\frac{|\boldsymbol{e}_k|}{T_k} \boldsymbol{R}(\bar{\boldsymbol{x}}_k - \boldsymbol{x}_k) \cdot \boldsymbol{\kappa}_k^T \vec{\boldsymbol{n}}_{k,2},$$

$$\omega_{k,2,2} = \frac{|\boldsymbol{e}_k|}{T_k} \boldsymbol{R}(\bar{\boldsymbol{x}}_{k+1} - \boldsymbol{x}_k) \cdot \boldsymbol{\kappa}_k^T \vec{\boldsymbol{n}}_{k,2}.$$

The expression of cell edge unknowns



The expression of cell edge unknowns

The continuity of normal flux on the sub-edge

$$F_{k,1} + F_{k+1,2} = 0$$

⇒

$$\begin{aligned} & \omega_{k,1,2}\bar{u}_k + (\omega_{k,1,1} + \omega_{k+1,2,2})\bar{u}_{k+1} + \omega_{k+1,2,1}\bar{u}_{k+2} \\ = & (\omega_{k,1,1} + \omega_{k,1,2})u_k + (\omega_{k+1,2,1} + \omega_{k+1,2,2})u_{k+1} \end{aligned}$$

⇒

$$\mathbf{A}\bar{\mathbf{u}} = \mathbf{B}\mathbf{u}$$

$$\bar{u}_k = \sum_{j=1}^N \omega_{k,j} u_j$$

Scheme

$$\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} = f_K m(K), \quad \forall K \in \mathcal{P}_{in},$$
$$u_{M_i} = g_{M_i}, \quad \forall M_i \in \mathcal{P}_{out}.$$

The discrete system

$$A(U)U = F.$$

Solution procedure:

Choose a small value $\varepsilon_{non} > 0$ and initial vector U^0 , and repeat for $k = 1, 2, \dots$,

1. Solve $A(U^{k-1})U^k = F$
2. Stop if $\|A(U^k)U^k - F\| \leq \varepsilon_{non}\|A(U^0)U^0 - F\|$.

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3 Numerical experiments

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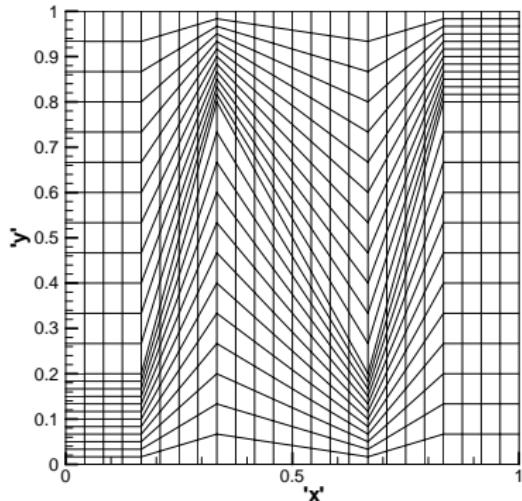
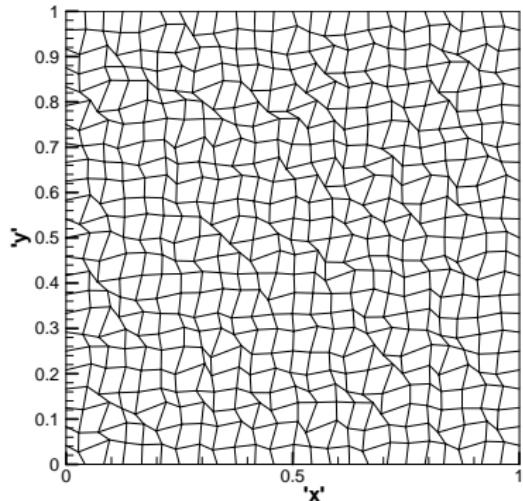
Example 1: The accuracy of scheme

Take $\kappa = RDR^T$,

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad D = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix},$$

where $\theta = \frac{5\pi}{12}$, $k_1 = 1 + 2x^2 + y^2$, $k_2 = 1 + x^2 + 2y^2$. The exact solution is $u(x, y) = \sin(\pi x) \sin(\pi y)$.

Computational meshes



The accuracy on random quadrilateral meshes

New monotone scheme

number of cell	144	576	2304	9216	36864
ε_2^u rate	3.96e-3 –	9.78e-4 2.02	2.74e-4 1.84	7.45e-5 1.88	2.27e-5 1.71
ε_2^F rate	9.33e-2 –	4.51e-2 1.05	2.33e-2 0.95	1.11e-2 1.07	5.49e-3 1.02

Existing monotone scheme (G. Yuan and Z. Sheng, JCP,2008)

number of cell	144	576	2304	9216	36864
ε_2^u rate	4.38e-3 –	1.00e-3 2.13	2.81e-4 1.83	6.76e-5 2.06	1.75e-5 1.95
ε_2^F rate	1.06e-1 –	5.42e-2 0.97	2.34e-2 1.21	1.09e-2 1.10	5.25e-3 1.05

The accuracy on Kershaw meshes

New monotone scheme

Number of cell	144	576	2304	9216	36864
ε_2^U rate	1.96e-2 —	5.91e-3 1.73	1.24e-3 2.25	3.47e-4 1.84	9.43e-5 1.88
ε_2^F rate	6.70e-1 —	2.66e-1 1.33	1.04e-1 1.35	3.85e-2 1.43	1.39e-2 1.47

Existing monotone scheme (G. Yuan and Z. Sheng, JCP,2008)

Number of cell	144	576	2304	9216	36864
ε_2^U rate	3.42e-2 —	2.14e-2 0.68	9.63e-3 1.15	3.22e-3 1.58	9.02e-4 1.84
ε_2^F rate	1.39 —	8.78e-1 0.66	4.23e-1 1.05	1.69e-1 1.32	6.33e-2 1.42

Example 2: The problem with point source

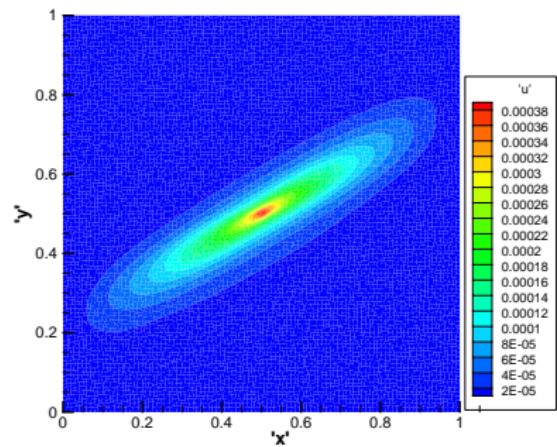
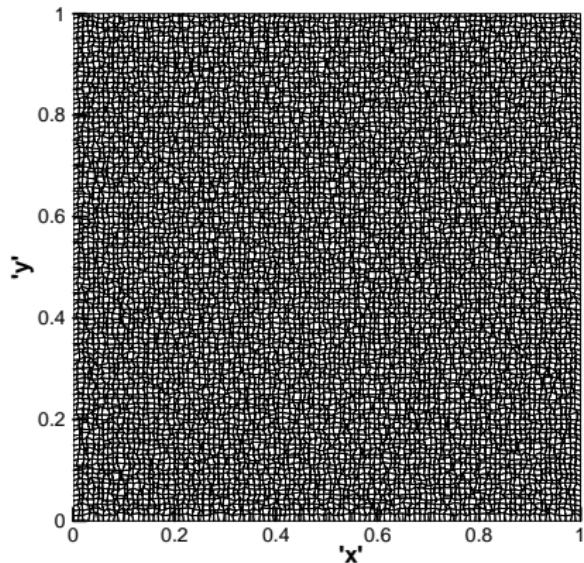
Take

$$\kappa = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

where $k_1 = 10000$, $k_2 = 1$, $\theta = \pi/6$.

$$f(x, y) = \begin{cases} 101 \times 101 & \text{if } (x, y) \in [50/101, 51/101]^2, \\ 0 & \text{otherwise.} \end{cases}$$

Computational meshes and numerical results



Example 3: Positivity of numerical solutions

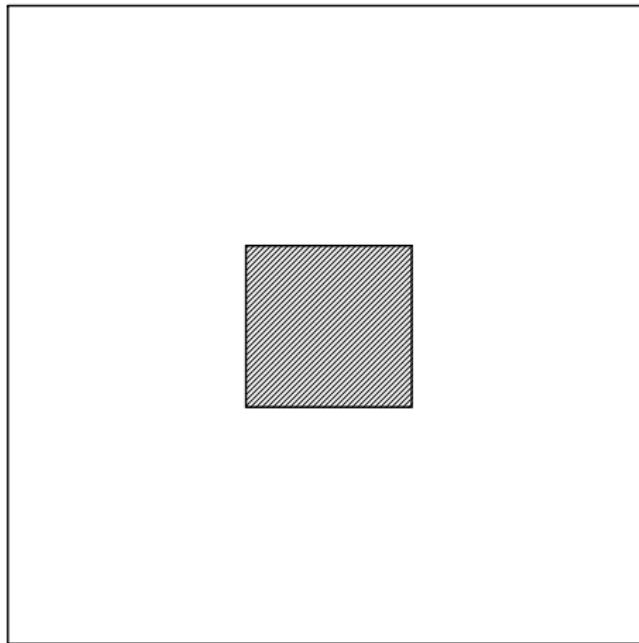
Take

$$\kappa = \begin{pmatrix} y^2 + \varepsilon x^2 + \varepsilon & -(1 - \varepsilon)xy \\ -(1 - \varepsilon)xy & \varepsilon y^2 + x^2 + \varepsilon \end{pmatrix}, \quad \varepsilon = 5 \cdot 10^{-3},$$

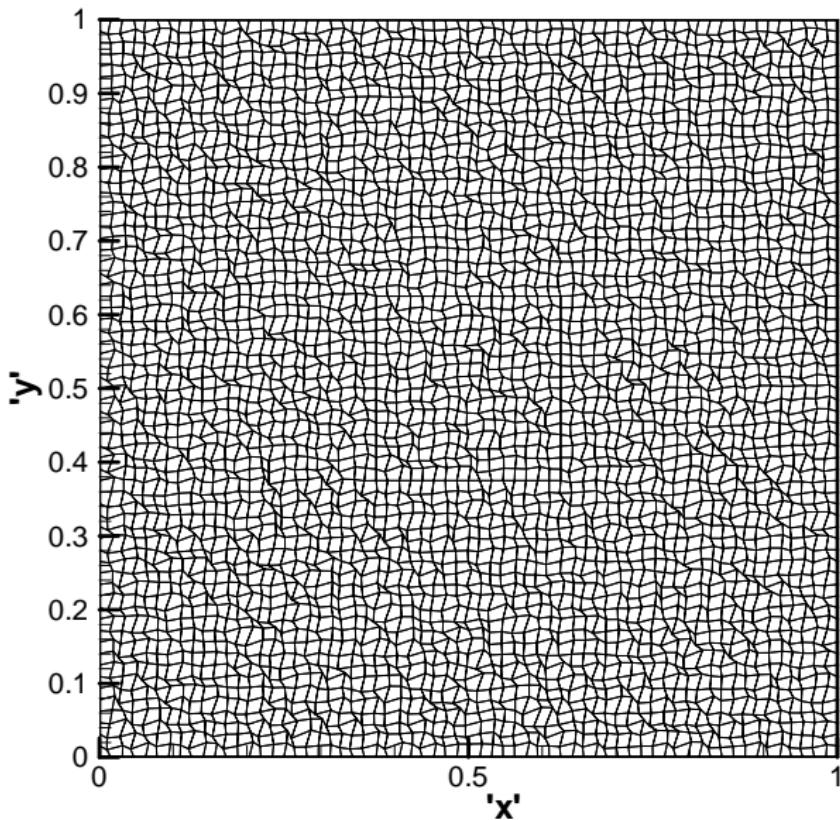
$$f(x, y) = \begin{cases} 1 & \text{if } (x, y) \in [3/8, 5/8]^2, \\ 0 & \text{otherwise.} \end{cases}$$

We impose the homogeneous Dirichlet boundary condition on $\partial\Omega$.

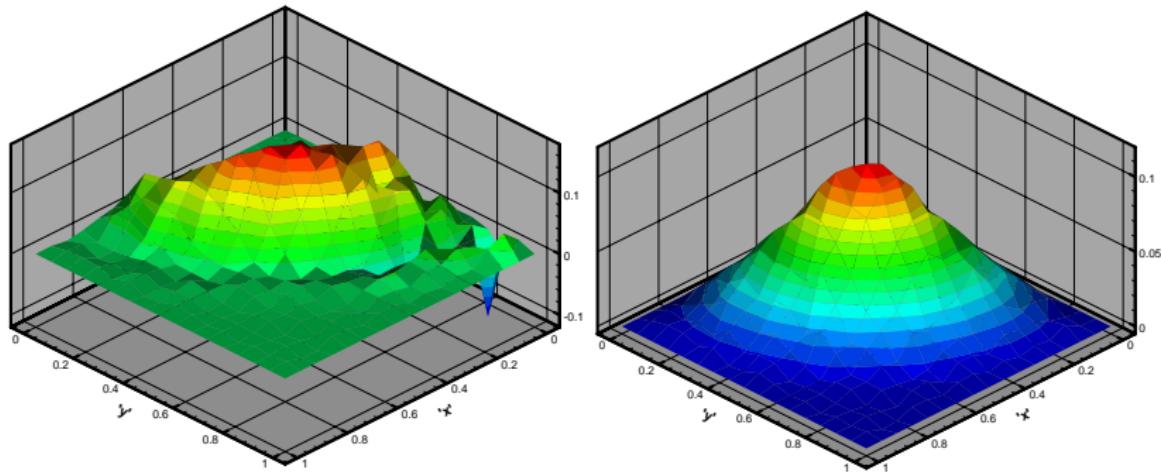
Computational domain



Random quadrilateral meshes



Numerical results



Left MPFA,

right: new monotone scheme

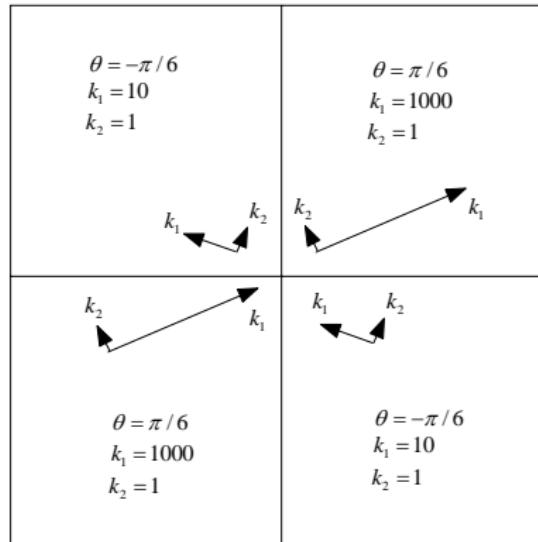
$$(u_{min}^{in} = -0.11, u_{max}^{in} = 0.17), \quad (u_{min}^{in} = 2.39e-13, u_{max}^{in} = 0.1252)$$

Exapmple 4: Heterogeneous diffusion tensor

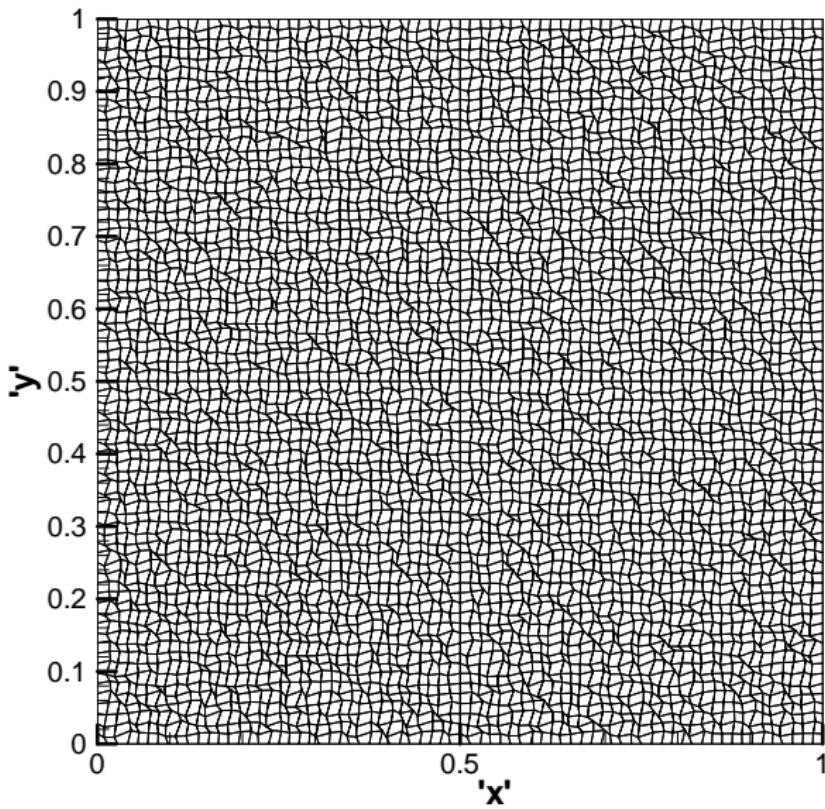
Take

$$\kappa = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

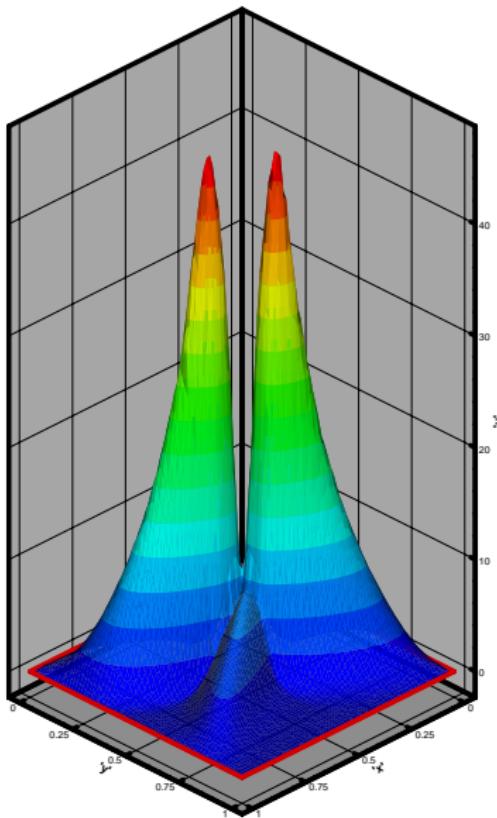
$$f(x, y) = \begin{cases} 10000 & \text{if } (x, y) \in [7/18, 11/18]^2, \\ 0 & \text{otherwise.} \end{cases}$$



Computational meshes



Numerical results



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New monotone scheme

- Conservation
- Polygonal meshes
- Second order accuracy
- Reduce to the classical five point scheme

New monotone scheme

- Conservation
- Polygonal meshes
- Second order accuracy
- Reduce to the classical five point scheme

Perspective

- New expression of flux
- Method eliminating the auxiliary unknowns
- ...

Thank you for your attention!