Explicit schemes for parabolic problems, new opportunities

Petr Vabishchevich

Nuclear Safety Institute of RAS, Moscow, Russia, vabishchevich@gmail.com

September 13, 2013

Content

Introduction

Illustrative example

The boundary value problem for one-dimensional parabolic equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < l, \quad 0 < t \le T.$$

$$u(0,t) = 0, \quad u(l,t) = 0,$$

$$u(x,0) = u^0(x), \quad 0 < x < l.$$

$$\frac{y_i^{n+1} - y_i^n}{\tau} - \frac{y_{i+1}^n - 2y_i^n + y_{i-1}^n}{h^2} = 0.$$

The stability:

$$\tau \leq \frac{1}{2} h^2.$$

The convergence:

$$\|z^n\| \leq M(h^2 + \tau).$$

Explicit-implicit scheme

A red-black point-wise decomposition:

$$\chi_i = \begin{cases} 0, & i - even, \\ 1, & i - odd. \end{cases}$$

ADI type scheme:

$$\frac{y_i^{n+1/2} - y_i^n}{\tau} - \chi_i \frac{y_{i+1}^{n+1/2} - 2y_i^{n+1/2} + y_{i-1}^{n+1/2}}{h^2} - (1-\chi_i) \frac{y_{i+1}^{n} - 2y_i^{n} + y_{i-1}^{n}}{h^2} = 0,$$

$$\frac{y_i^{n+1} - y_i^n}{\tau} - \chi_i \frac{y_{i+1}^{n+1/2} - 2y_i^{n+1/2} + y_{i-1}^{n+1/2}}{h^2} - (1-\chi_i) \frac{y_{i+1}^{n+1} - 2y_i^{n+1} + y_{i-1}^{n+1}}{h^2} = 0.$$

The convergence:

$$\|z^n\| \le M \left(h^2 + \left(\sigma - \frac{1}{2} \right) \tau + \tau^2 + \frac{\tau}{h} \right).$$

Alternating triangle method

$$\frac{y_i^{n+1/2} - y_i^n}{\tau} - \frac{y_{i+1}^{n+1/2} - y_i^{n+1/2}}{h^2} + \frac{y_i^n - y_{i-1}^n}{h^2} = 0,$$

$$\frac{y_i^{n+1} - y_i^n}{\tau} - \frac{y_{i+1}^{n+1/2} - y_i^{n+1/2}}{h^2} + \frac{y_i^{n+1} - y_{i-1}^{n+1}}{h^2} = 0.$$

The convergence:

$$\|z^n\| \le M \left(h^2 + \left(\sigma - \frac{1}{2} \right) \tau + \tau^2 + \frac{\tau^2}{h^2} \right).$$

The main work

Soviet period:

- V.K. Saul'yev, 1961, Asymmetric schemes
- A.A. Samarskii, 1964, Alternating triangle method
- V.P. Il'in, 1967, Explicit ADI

West period:

- D.J. Evans, A.R.B. Abdullah, 1983, Group explicit method

Chinese period:

- G.W.Yuan, L.J.Shen, Y.L.Zhou, 2001, Yu Zhuang, 2005, Zhang Baolin, Li Wenzhi, 2007, Parallel computing
- Wang Wen-qia, 2003, Convection-diffusion problems

Alternating triangle method

Operator formulation

The Cauchy problem:

$$\frac{du}{dt} + Au = f(t), \quad 0 < t \le T,$$

$$u(0) = u^0.$$

The triangular splitting:

$$A = A^* \ge 0, \quad A = A_1 + A_2, \quad A_1 = A_2^*.$$

1D problem:

$$(A_1 u)_i = -\frac{u_{i+1} - u_i}{h^2}, \quad (A_2 u)_i = \frac{u_i - u_{i-1}}{h^2}.$$

Two-level scheme

$$B\frac{y^{n+1}-y^n}{\tau}+Ay^n=\varphi^n,$$

where

$$B=(E+\sigma\tau A_1)(E+\sigma\tau A_2).$$

We have

$$B=E+\sigma\tau A+\sigma^2\tau^2 A_1 A_2,$$

$$A_1 A_2 \geq 0, \quad A_1 A_2 = (A_1 A_2)^* \quad (B=B^*).$$

A necessary and sufficient condition for stability in H_A (Samarskii, 1967):

$$B \geq \frac{\tau}{2} A.$$

For the alternating triangular method: $\sigma \geq 0.5$.
An a priori estimate:

$$\|y^{n+1}\|_A \leq \|y^n\|_A + \tau \|\varphi^n\|.$$

The convergence:

$$\|z^n\|_A \leq M \left(h^2 + \left(\sigma - \frac{1}{2} \right) \tau + \tau^2 + \tau^2 \left\| A_1 A_2 \frac{du}{dt} \right\| \right).$$

Modified alternating triangle method

Standard scheme

$$(E+\sigma\tau A)\frac{y^{n+1}-y^n}{\tau}+\sigma^2\tau^2A_1A_2\frac{y^{n+1}-y^n}{\tau}+Ay^n=\varphi^n.$$

Modification:

$$(E+\sigma\tau A)\frac{y^{n+1}-y^n}{\tau}+\sigma^2\tau^2A_1A_2\frac{y^{n+1}-y^n}{\tau}-\sigma^2\tau^2A_1A_2\frac{y^{n}-y^{n-1}}{\tau}+Ay^n=\varphi^n.$$

Another form:

$$(E+\sigma\tau A)\frac{y^{n+1}-y^n}{\tau}+\sigma^2\tau^3A_1A_2\frac{y^{n+1}-2y^n+y^{n-1}}{\tau^2}+Ay^n=\varphi^n.$$

The main result

The stability condition: $\sigma \geq 0.5$.
The convergence:

$$\|z^n\|_A \leq M \left(h^2 + \left(\sigma - \frac{1}{2} \right) \tau + \tau^2 + \tau^3 \left\| A_1 A_2 \frac{d^2 u}{dt^2} \right\| \right).$$

1D problem:

$$\|z^n\|_A \leq M \left(h^2 + \left(\sigma - \frac{1}{2} \right) \tau + \tau^2 + \frac{\tau^3}{h^2} \right).$$