A Fitted Finite Volume Method for Real options in Climate Change

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Contents

1 Introduction

There is broad recognition that uncertainty is an essential aspect of the economics of climate change. The real option analysis deals with the decision making problem by learning more about the uncertainty over the time and exercising the option at the most favorable time, which is important in strategic and financial analysis because traditional valuation tools such as net present value (NPV) ignore the value of flexibility. Here we will employ the real option analysis to solve optimally the problem related to rebuilding the defense higher and wider.

2 The real option model for the defense

Assuming that the sea level now is h_1 and the height of defense is h_2, we might widen the defense and increase its height in the future when the sea level reach and exceed the critical value h_k, where it is reasonable to presume that $h_1 < h_k < h_2$.

- The underlying assets

The sea level and the atmospheric temperature are two important indexes of climate change, which are the underlying assets in our real option model.

Following Bloch (2010), we can model the global mean temperature level $(Y_t)_{t\geq 0}$ be a one-dimensional Markov process under the historical measure $\mathbb{P}$:

$$dY_t = \theta(\bar{Y} - Y_t)dt + \sigma_Y d\hat{W}_Y, \tag{2.1}$$

where $\bar{Y}(t)$ is the equilibrium or mean value, σ_Y is the volatility caused by shocks and $\theta > 0$ is the rate by which these shocks dissipate and the variable reverts towards the mean.

According to Rahmstorf (2007), we consider the sea level process $(X_t)_{t\geq 0}$ to be a function of the temperature process by assuming the rate of change of the sea level to be proportional to the temperature increase under the $\mathbb{P}$-measure:

$$dX_t = \eta(Y_t - Y_0)dt + \sigma_X X_t d\hat{W}_X. \tag{2.2}$$

- The valuation equation

Denote by $F(X, Y, t)$ the real option dependent on the sea level X and the temperature Y. Then, the change in the real option can be described by the Ito's lemma:

$$\begin{aligned} dF &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{\partial F}{\partial Y}dY + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}dX^2 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2}dY^2 + \frac{\partial^2 F}{\partial X \partial Y}dXdY \\ &= \left(\frac{\partial F}{\partial t} + \eta(Y_t - Y_0)\frac{\partial F}{\partial X} + \theta(\bar{Y} - Y_t)\frac{\partial F}{\partial Y} + \frac{1}{2}\sigma_X^2\frac{\partial^2 F}{\partial X^2} + \frac{1}{2}\sigma_Y^2\frac{\partial^2 F}{\partial Y^2}\right)dt \\ &\quad +\sigma_X X\frac{\partial F}{\partial X}d\hat{W}_X + \sigma_Y\frac{\partial F}{\partial Y}d\hat{W}_Y, \end{aligned}$$

from which we can obtain the relative change by dividing on both sides by F:

$$\begin{aligned} \frac{dF}{F} &= \left((\frac{\partial F}{\partial t} + a(Y_t - Y_0)\frac{\partial F}{\partial X} + \theta(\bar{Y} - Y_t)\frac{\partial F}{\partial Y} + \frac{1}{2}\sigma_X^2 X^2\frac{\partial^2 F}{\partial X^2} + \frac{1}{2}\sigma_Y^2\frac{\partial^2 F}{\partial Y^2})/F\right)dt \\ &+ \left((\sigma_X X\frac{\partial F}{\partial X})/F\right)d\hat{W}_X + \left((\sigma_Y\frac{\partial F}{\partial Y})/F\right)d\hat{W}_Y. \end{aligned}$$

For simplicity, we define

$$k = \left(\frac{\partial F}{\partial t} + \eta(Y_t - \bar{Y}_0)\frac{\partial F}{\partial X} + \theta(\bar{Y}(t) - Y_t)\frac{\partial F}{\partial Y} + \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 F}{\partial X^2} + \frac{1}{2}\sigma_Y^2 \frac{\partial^2 F}{\partial Y^2}\right)/F,$$

$$S_1 = \left(\sigma_X X \frac{\partial F}{\partial X}\right)/F, \quad S_2 = \left(\sigma_Y \frac{\partial F}{\partial Y}\right)/F.$$

Brennan and Schwartz (1979) proposed a method to form a portfolio P by investing amounts of x_1, x_2 and x_3 in three bonds of maturity τ_1, τ_2 and τ_3, respectively. The rate of return on this portfolio is

$$\frac{dP}{P} = [x_1 k + x_2 k + x_3 k]dt + [x_1 S_1 + x_2 S_1 + x_3 S_1]d\hat{W}_X + [x_1 S_2 + x_2 S_2 + x_3 S_2]d\hat{W}_Y.$$

The rate of return on the portfolio will be non-stochastic if the portfolio proportions are chosen so that the coefficients of $d\hat{W}_X$ and $d\hat{W}_Y$ are zero. That is

$$\begin{cases} x_1 S_1 + x_2 S_1 + x_3 S_1 = 0, \\ x_1 S_2 + x_2 S_2 + x_3 S_2 = 0. \end{cases} \quad (2.3)$$

To prevent arbitrage the return on this portfolio must be risk-less over short time intervals, this is to say, the rate of the return is equal to r, the instantaneous risk-free interest rate. As a consequence, the portfolio risk premium is zero:

$$x_1(k(\tau_1) - r) + x_2(k(\tau_2) - r) + x_3(k(\tau_3) - r) = 0. \tag{2.4}$$

The no arbitrage condition Eq.(2.4) and the two zero risk conditions Eq.(2.3) will have a solution only if

$$k - r = \lambda_X S_1 + \lambda_Y S_2, \tag{2.5}$$

where λ_X and λ_Y are the market price per unit of sea level and temperature, respectively, from which we can obtain

$$\left(\frac{\partial F}{\partial t} + \eta(Y_t - \bar{Y}_0)\frac{\partial F}{\partial X} + \theta(\bar{Y}(t) - Y_t)\frac{\partial F}{\partial Y} + \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 F}{\partial X^2} + \frac{1}{2}\sigma_Y^2 \frac{\partial^2 F}{\partial Y^2}\right)/F - r$$
$$= \lambda_X \left(\sigma_X X \frac{\partial F}{\partial X}\right)/F + \lambda_Y \left(\sigma_Y \frac{\partial F}{\partial Y}\right)/F.$$

By simplifying and moving all terms to the left hand side, we finally get the valuation equation as follows:

$$\frac{\partial F}{\partial t} + \left(\eta(Y_t - \bar{Y_0}) - \lambda_X \sigma_X X\right) \frac{\partial F}{\partial X} + \left(\theta(\bar{Y}(t) - Y_t) - \lambda_Y \sigma_Y\right) \frac{\partial F}{\partial Y} + \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 F}{\partial X^2} + \frac{1}{2}\sigma_Y^2 \frac{\partial^2 F}{\partial Y^2} - rF = 0. \tag{2.6}$$

- Final and boundary conditions

At expiration, the value of the real option will be

$$F(X, Y, T) = \max(V(X_T) - K, 0) \tag{2.7}$$

where $V(X)$ denotes the function of avoided damages caused by sea level rise by increasing the defense in height and it is an increasing function about X, and K denotes the cost of increasing the defense to a specific height.

To solve the above equation, we need the following two classic value-matching and smooth-pasting boundary conditions:

$$F(X^\star(t), Y, t) = V(X^\star(t)) - K, \tag{2.8}$$

$$\frac{\partial F(X^\star(t), Y, t)}{\partial X} = V'(X^\star(t)), \tag{2.9}$$

where $X^\star(t)$ is the critical value on which point the investment is triggered. If $X(t) > X^\star(t)$, we should commit the investment immediately. If $X(t) < X^\star(t)$, it is advisable to delay the decision to investment.

The lowest sea level results in a call option being worthless. Therefore, a practical Dirichlet boundary condition is

$$F(X_{min}, Y, t) = 0, \tag{2.10}$$

and the boundary condition at $X = X_{max}$ is simply taken to be the extension of the final condition at the point, i.e.,

$$F(X_{max}, Y, t) = F(X_{max}, Y, T) = V(X_{max}) - K, \tag{2.11}$$

and

$$F(X, Y_{min}, t) = g_1(X, t), \quad F(X, Y_{max}, t) = g_2(X, t), \tag{2.12}$$

where g_1 and g_2 will be determined by solving the one-dimensional equation obtained by taking the third and the fifth terms of (2.6) to equal to zero for two particular values $Y = Y_{min}$ and $Y = Y_{max}$.

We restate our partial differential equation of real option as follows:

$$\frac{\partial F}{\partial t} + \left(a(Y_t - \bar{Y}_0) - \lambda_X \sigma_X\right) \frac{\partial F}{\partial X} + \left(\theta(\bar{Y}(t) - Y_t) - \lambda_Y \sigma_Y\right) \frac{\partial F}{\partial Y}$$
$$+ \frac{1}{2}\sigma_X^2 \frac{\partial^2 F}{\partial X^2} + \frac{1}{2}\sigma_Y^2 \frac{\partial^2 F}{\partial Y^2} - rF = 0, (2.6)$$

with the boundary conditions

$$F(X_{min}, Y, t) = 0, \;\; F(X_{max}, Y, t) = V(X_{max}) - K,$$

$$F(X, Y_{min}, t) = g_1(X, t), \quad F(X, Y_{max}, t) = g_2(X, t),$$

$$F(X^\star(t), Y, t) = V(X^\star(t)) - K, \;\; \frac{\partial F(X^\star(t), Y, t)}{\partial X} = V'(X^\star(t)),$$

and the final condition

$$F(X, Y, T) = \max(V(X_T) - K, 0).$$

3 A power penalty method

We formulate the above free boundary problem as a linear complementarity problem, and solve it by a power penalty method.

- Reformation of the problem

Let $F(X,Y,t)$ represent the value of the real option with expiry date T, if we define

$$LF = -\frac{\partial F}{\partial t} - \frac{1}{2}\left(\sigma_X^2 X^2 \frac{\partial^2 F}{\partial X^2} + \sigma_Y^2 \frac{\partial^2 F}{\partial Y^2}\right) - \left(h(X,Y,t)\frac{\partial F}{\partial X} + g(Y,t)\frac{\partial F}{\partial Y}\right) + rF, \tag{3.1}$$

where $h(X,Y,t) = \eta(Y_t - \bar{Y}_0) - \lambda_X \sigma_X X$ and $g(Y,t) = \theta(\bar{Y}_1 - Y_t) - \lambda_Y \sigma_Y$, the real option F satisfies the following partial differential complementarity problem:

$$\begin{cases} LF \geq 0, \\ F - F^\star \geq 0, \\ LF \cdot (F - F^\star) = 0, \end{cases} \tag{3.2}$$

for $(X, Y, t) \in \Omega \times [0, T)$ with the boundary conditions

$$\begin{cases} F(X_{min}, Y, t) = 0, F(X_{max}, Y, t) = V(X_{max}) - K, \\ F(X, Y_{min}, t) = g_1(X, t), F(X, Y_{max}, t) = g_2(X, t), \end{cases} \tag{3.3}$$

and terminal condition

$$F(X, Y, T) = F^\star(X, Y), \tag{3.4}$$

where

$$F^\star(X, Y) = \max(V(X) - K, 0)$$

is the payoff function.

For the convenience of theoretical analysis, we rewrite (3.1) as the following conservative form

$$LF = -\frac{\partial F}{\partial t} - \nabla \cdot (A\nabla F + \underline{b}F) + \bar{c}F \tag{3.5}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sigma_X^2 & 0 \\ 0 & \frac{1}{2}\sigma_Y^2 \end{pmatrix},$$
$$\underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a(Y_t - \bar{Y}_0) - \lambda_X \sigma_X X - \sigma_X^2 X \\ \theta(\bar{Y}_1 - Y_t) - \lambda_Y \sigma_Y \end{pmatrix}, \tag{3.6}$$
$$\bar{c} = r - \lambda_X \sigma_X - \sigma_X^2 - \theta.$$

Let $F_0(X, Y)$ be a twice differentiable function satisfying the boundary conditions in (3.3). We introduce a new function

$$u(X, Y, t) = e^{\beta t}(F_0 - F), \tag{3.7}$$

where $\beta = \frac{1}{2}(\sigma_X^2 + \sigma_Y^2)$. Taking LF_0 away from both sides of the first inequality of (3.2) and transforming F in (3.2) into the new function u, we have

$$\begin{cases} \mathcal{L}u \leq f, \\ u - u^\star \leq 0, \\ (\mathcal{L}u - f) \cdot (u - u^\star) = 0, \end{cases} \tag{3.8}$$

where

$$\begin{cases} \mathcal{L}u = -u_t - \nabla \cdot (A\nabla u + \underline{b}u) + \underline{c}u, \\ \underline{c} = \bar{c} + \beta, u^\star = e^{\beta t}(F_0 - F^\star), f(X,Y,t) = e^{\beta t} L F_0. \end{cases} \tag{3.9}$$

- Variational inequality of the problem

Let $\Omega = [X_{min}, X_{max}] \times [Y_{min}, Y_{max}]$ and let Γ denote the boundary of Ω. Set

$$H_0^1(\Omega) = \left\{ v : v \in H^1(\Omega),\ \ v|_\Gamma = 0 \right\},$$

$$\mathcal{K} = \left\{ v(t) : v(t) \in H_0^1(\Omega), v(t) \le u^\star(t),\ \ a.e.\ \text{ in } (0,T) \right\},$$

where $u^\star(t)$ is defined by (3.9).

Now, we define the following variational inequality problem.

Problem 1 Find $u \in \mathcal{K}$ such that, for all $v \in \mathcal{K}$,

$$\left(-\frac{\partial u}{\partial t}, v - u\right) + B(u, v - u; t) \geq (f, v - u), \quad a.e. \text{ in } (0, T), \tag{3.10}$$

where B(u,v;t) is a bilinear form defined by

$$B(u, v; t) = (A\nabla u + \underline{b}u, \nabla v) + (\underline{c}u, v), \quad u, v \quad \in H_0^1(\Omega). \tag{3.11}$$

Theorem 1 *Problem 1 is the variational form of the complementarity problem* (3.8).

Lemma 1 There exist positive constants C and M, independent of v and w, such that for any $v, w \in H_0^1(\Omega)$,

$$B(v, v; t) \geq C\,\|v\|_1^2, \qquad \text{(coerciveness)}$$

$$|B(v, w; t)| \leq M\,\|v\|_1\,\|w\|_1\,. \qquad \text{(continuity)}$$

Theorem 2 *There exists a unique solution to Problem 1.*

- The power penalty approach

To derive the power penalty approach, we first consider the following nonlinear variational inequality problem:

Find $u_\lambda \in H_0^1(\Omega)$ such that, for all $v \in H_0^1(\Omega)$,

$$\left(-\frac{\partial u_\lambda}{\partial t}, v-u_\lambda\right)+B(u_\lambda, v-u_\lambda;t)+j(v)-j(u_\lambda) \geq (f, v-u_\lambda), \;\; a.e. \text{ in } (0,T) \tag{3.12}$$

where

$$j(v) = \frac{\lambda k}{k+1}(\max(v-u^\star,0))^{\frac{k+1}{k}}, \;\; k>0, \;\; \lambda>1. \tag{3.13}$$

Moreover, (3.12) is equivalent to the following problem:

Problem 2 Find $u_\lambda \in H_0^1(\Omega)$ such that, for all $v \in H_0^1(\Omega)$,

$$\left(-\frac{\partial u_\lambda}{\partial t}, v\right)+B(u_\lambda, v;t)+(j'(u_\lambda),v) = (f,v), \;\; a.e. \text{ in } (0,T), \tag{3.14}$$

where

$$j'(v) = \lambda(\max(v-u^\star,0))^{\frac{1}{k}}. \tag{3.15}$$

The strong form of (3.12)-(3.15), which defines the penalized equation approximating (3.8), is given by

$$\mathcal{L}u_\lambda + \lambda(\max(v - u^\star, 0))^{\frac{1}{k}} = f, \quad (x, y, t) \in \Omega \times [0, T), \tag{3.16}$$

with the given boundary and final conditions

$$u_\lambda(x, y, t)|_\Gamma = 0 \text{ and } u_\lambda(x, y, T) = u^\star(x, y, T). \tag{3.17}$$

- Convergent analysis

Lemma 2 Let u_λ be the solution to Problem 2. If $u_\lambda \in L^p(\Theta)$, then there exists a positive constant C, independent of u_λ and λ, such that

$$\|\max(u_\lambda - u^\star, 0)\|_{L^p(\Theta)} \leq \frac{C}{\lambda^k},$$
$$\|\max(u_\lambda - u^\star, 0)\|_{L^\infty(0,T;L^2(\Omega))} + \|\max(u_\lambda - u^\star, 0)\|_{L^2(0,T;H_0^1(\Omega))} \leq \frac{C}{\lambda^{k/2}}, \tag{3.18}$$

where k is the power of the power penalty function and $p = 1 + 1/k$, and $\Theta = \Omega \times [0, T]$.

Using Lemma 2, we are able to show that the solution to Problem 2 converges to that of Problem 1 at the rate of order $\lambda^{-k/2}$, which is stated in the next theorem.

Theorem 3 *Let u and u_λ be the solutions to Problem 1 and Problem 2, respectively. If $u_\lambda \in L^p(\Theta)$ and $\frac{\partial u}{\partial t} \in L^{k+1}(\Theta)$, then there exists a positive constant C, independent of u_λ and λ, such that*

$$\|u - u_\lambda\|_{L^\infty(0,T;L^2(\Omega))} + \|u - u_\lambda\|_{L^2(0,T;H_0^1(\Omega))} \leq \frac{C}{\lambda^{k/2}}, \tag{3.19}$$

where k is the power of the power penalty function.

4 The fitted finite volume method

A fitted finite volume method will be employed to solve the above penalty problem.

Under (3.7), (3.16)-(3.17) can be rewritten as

$$-F_t - \nabla\cdot(A\nabla F + \underline{b}F) + \bar{c}F - \lambda[F^* - F]_+^{\frac{1}{k}} = 0, \quad (X, Y, t) \in \Omega\times[0, T) \quad (4.1)$$

with the corresponding boundary and final conditions, and

$$X_{min} = X_0 < X_1 < \cdots < X_{N_X} = X_{max} \text{ and } Y_{min} = Y_0 < Y_1 < \cdots < Y_{N_Y} = Y_{max}.$$

is a partition of Ω. Let

$$X_{i-\frac{1}{2}} = \frac{X_{i-1} + X_i}{2},\ X_{i+\frac{1}{2}} = \frac{X_i + X_{i+1}}{2},\ Y_{j-\frac{1}{2}} = \frac{Y_{j-1} + Y_j}{2},\ Y_{j+\frac{1}{2}} = \frac{Y_j + Y_{j+1}}{2}$$

be the second mesh.

Integrating (4.1) over $\mathcal{R}_{i,j} = [X_{i-\frac{1}{2}}, X_{i+\frac{1}{2}}]\times[Y_{j-\frac{1}{2}}, Y_{j+\frac{1}{2}}]$, we have

$$\begin{aligned}&-\int_{X_{i-\frac{1}{2}}}^{X_{i+\frac{1}{2}}}\int_{Y_{j-\frac{1}{2}}}^{Y_{j+\frac{1}{2}}}\frac{\partial F}{\partial t}dXdY - \int_{X_{i-\frac{1}{2}}}^{X_{i+\frac{1}{2}}}\int_{Y_{j-\frac{1}{2}}}^{Y_{j+\frac{1}{2}}}\nabla\cdot(A\nabla F + \underline{b}F)dXdY\\&+\int_{X_{i-\frac{1}{2}}}^{X_{i+\frac{1}{2}}}\int_{Y_{j-\frac{1}{2}}}^{Y_{j+\frac{1}{2}}}\bar{c}F\,dXdY - \lambda\int_{X_{i-\frac{1}{2}}}^{X_{i+\frac{1}{2}}}\int_{Y_{j-\frac{1}{2}}}^{Y_{j+\frac{1}{2}}}[F^* - F]_+^{\frac{1}{k}}dXdY = 0,\end{aligned}$$

for $i = 1, 2, \cdots, N_X - 1$, $j = 1, 2, \cdots, N_Y - 1$.

Applying the mid-point quadrature rule to the first, the third and the last terms, we obtain from the above

$$-\frac{\partial F_{i,j}}{\partial t} R_{i,j} - \int_{\mathcal{R}_{i,j}} \nabla \cdot (A \nabla F + \underline{b} F) dX dY + \bar{c}_{i,j} F_{i,j} R_{i,j} - \lambda [F^*_{i,j} - F_{i,j}]_+^{\frac{1}{k}} R_{i,j} = 0 \quad (4.2)$$

for $i = 1, 2, \cdots, N_X - 1$, $j = 1, 2, \cdots, N_Y - 1$, where $R_{i,j} = (X_{i-\frac{1}{2}}, X_{i+\frac{1}{2}}) \times (Y_{j-\frac{1}{2}}, Y_{j+\frac{1}{2}})$, $\bar{c}_{i,j} = c(X_i, Y_j, t)$, $F_{i,j} = F(X_i, Y_j, t)$, and $F^*_{i,j} = F^*(X_i, Y_j, t)$. Consider to approximate the second term in (4.2): Integrating by parts and using the definition of flux $A \nabla F + \underline{b} F$, we have

$$\begin{aligned}
\int_{\mathcal{R}_{i,j}} \nabla \cdot (A \nabla F + \underline{b} F) dX dY &= \int_{\partial \mathcal{R}_{i,j}} (A \nabla F + \underline{b} F) \cdot n ds \\
&= \int_{(X_{i+\frac{1}{2}}, Y_{j-\frac{1}{2}})}^{(X_{i+\frac{1}{2}}, Y_{j+\frac{1}{2}})} (\frac{1}{2} \sigma_X^2 X^2 F_X + (a(Y_t - Y_0) - \lambda_X \sigma_X X - \sigma_X^2 X) F) dY \\
&- \int_{(X_{i-\frac{1}{2}}, Y_{j-\frac{1}{2}})}^{(X_{i-\frac{1}{2}}, Y_{j+\frac{1}{2}})} (\frac{1}{2} \sigma_X^2 X^2 F_X + (a(Y_t - Y_0) - \lambda_X \sigma_X X - \sigma_X^2 X) F) dY \\
&+ \int_{(X_{i-\frac{1}{2}}, Y_{j+\frac{1}{2}})}^{(X_{i+\frac{1}{2}}, Y_{j+\frac{1}{2}})} (\frac{1}{2} \sigma_Y^2 F_Y + g(Y, t) F) dX \\
&- \int_{(X_{i-\frac{1}{2}}, Y_{j-\frac{1}{2}})}^{(X_{i+\frac{1}{2}}, Y_{j-\frac{1}{2}})} (\frac{1}{2} \sigma_Y^2 F_Y + g(Y, t) F) dX. \qquad (4.3)
\end{aligned}$$

Deal with the terms above one by one: For the first term, we have

$$\int_{(X_{i+\frac{1}{2}},Y_{j-\frac{1}{2}})}^{(X_{i+\frac{1}{2}},Y_{j+\frac{1}{2}})} (a_{11}F_X + b_1F)dY \approx (a_{11}F_X + b_1F)|_{(X_{i+\frac{1}{2}},Y_j)} \cdot h_{Y_j},$$

and we approximate the term $a_{11}F_X + b_1F$ by solving the following two-point boundary value problem

$$(aX^2F_X + bF)' \equiv 0 \tag{4.4}$$

$$F(X_i, Y_j) = F_{i,j}, \quad F(X_{i+1}, Y_j) = F_{i+1,j} \tag{4.5}$$

where $a = \frac{1}{2}\sigma_X^2$, $b = \eta(Y_t - \bar{Y}_0) - \lambda_X\sigma_X X - \sigma_X^2 X$ and $b_{i+\frac{1}{2},j} = b(x_{i+\frac{1}{2}}, j)$, and

$$F = e^{\frac{\alpha_{i,j}}{X}} \left(\int e^{-\frac{\alpha_{i,j}}{X}} \frac{C_1}{aX^2} dX + C_2 \right) = \frac{C_1}{b_{i+\frac{1}{2},j}} + C_2 e^{\frac{\alpha_{i,j}}{X}}. \tag{4.6}$$

Therefore,

$$(a_{11}F_X + b_1F)|_{(X_{i+\frac{1}{2}},Y_j)} \cdot h_{Y_j} \approx \left(b_{i+\frac{1}{2},j} \frac{e^{-\frac{\alpha_{i,j}}{X_{i+1}}} F_{i+1,j} - e^{-\frac{\alpha_{i,j}}{X_i}} F_{i,j}}{e^{-\frac{\alpha_{i,j}}{X_{i+1}}} - e^{-\frac{\alpha_{i,j}}{X_i}}} \right) \cdot h_{Y_j}. \tag{4.7}$$

Similarly, the other three terms in (4.3) can be approximated by

$$(a_{11}F_X+b_1F)|_{(X_{i-\frac{1}{2}},Y_j)}\cdot h_{Y_j}\approx\left(b_{i-\frac{1}{2},j}\frac{e^{-\frac{\alpha_{i-1,j}}{X_i}}F_{i,j}-e^{-\frac{\alpha_{i-1,j}}{X_{i-1}}}F_{i-1,j}}{e^{-\frac{\alpha_{i-1,j}}{X_i}}-e^{-\frac{\alpha_{i-1,j}}{X_{i-1}}}}\right)\cdot h_{Y_j}, \quad (4.8)$$

$$(\frac{1}{2}\sigma_Y^2F_Y+g(Y)F)|_{(X_i,Y_{j+\frac{1}{2}})}\cdot h_{X_i}\approx\left(\bar{b}_{i,j+\frac{1}{2}}\frac{e^{\bar{\alpha}_{i,j}Y_{j+1}}F_{i,j+1}-e^{\bar{\alpha}_{i,j}Y_j}F_{i,j}}{e^{\bar{\alpha}_{i,j}Y_{j+1}}-e^{\bar{\alpha}_{i,j}Y_j}}\right)\cdot h_{X_i}, \quad (4.9)$$

$$(\frac{1}{2}\sigma_Y^2F_Y+g(Y)F)|_{(X_i,Y_{j+\frac{1}{2}})}\cdot h_{X_i}\approx\left(\bar{b}_{i,j-\frac{1}{2}}\frac{e^{\bar{\alpha}^{i,j-1}Y_j}F_{i,j}-e^{\bar{\alpha}_{i,j-1}Y_{j-1}}F_{i,j-1}}{e^{\bar{\alpha}_{i,j-1}Y_j}-e^{\bar{\alpha}_{i,j-1}Y_{j-1}}}\right)\cdot h_{X_i}, \quad (4.10)$$

for $i=0,1,\cdots,N_X-1$ and $j=0,1,\cdots,N_Y-1$, where $\bar{a}=a_{22}$, $\bar{b}_{i,j+\frac{1}{2}}=b_2(X_i,Y_{j+\frac{1}{2}})$ and $\bar{\alpha}_{i,j}=\frac{\bar{b}_{i,j+\frac{1}{2}}}{a_{22}}$.

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Using (4.3), and (4.7), (4.8), (4.9), (4.10), we obtain the following equation:

$$
\begin{aligned}
&-\frac{\partial F_{i,j}}{\partial t} R_{i,j} + e^{i,j}_{i-1,j} F_{i-1,j} + e^{i,j}_{i,j-1} F_{i,j-1} + e^{i,j}_{i,j} F_{i,j} + e^{i,j}_{i,j+1} F_{i,j+1} \\
&+ e^{i,j}_{i+1,j} F_{i+1,j} - \lambda [F^*_{i,j} - F_{i,j}]_+^{\frac{1}{k}} R_{i,j} = 0,
\end{aligned}
\tag{4.11}
$$

for $i = 1, 2, \cdots, N_X - 1$ and $j = 1, 2, \cdots, N_Y - 1$, where

$$
e^{i,j}_{i-1,j} = -\frac{b_{i-\frac{1}{2},j} e^{-\frac{\alpha_{i-1,j}}{X_{i-1}}} h_{Y_j}}{e^{-\frac{\alpha_{i-1,j}}{X_i}} - e^{-\frac{\alpha_{i-1,j}}{X_{i-1}}}}, \quad e^{i,j}_{i,j-1} = -\frac{\bar{b}_{i,j-\frac{1}{2}} e^{\bar{\alpha}_{i,j-1} Y_{j-1}} h_{X_i}}{e^{\bar{\alpha}_{i,j-1} Y_j} - e^{\bar{\alpha}_{i,j-1} Y_{j-1}}},
\tag{4.12}
$$

$$
\begin{aligned}
e^{i,j}_{i,j} =& \frac{b_{i+\frac{1}{2},j} e^{-\frac{\alpha_{i,j}}{X_i}} h_{Y_j}}{e^{-\frac{\alpha_{i,j}}{X_{i+1}}} - e^{-\frac{\alpha_{i,j}}{X_i}}} + \frac{b_{i-\frac{1}{2},j} e^{-\frac{\alpha_{i-1,j}}{X_i}} h_{Y_j}}{e^{-\frac{\alpha_{i-1,j}}{X_i}} - e^{-\frac{\alpha_{i-1,j}}{X_{i-1}}}} \\
&+ \frac{\bar{b}_{i,j+\frac{1}{2}} e^{\bar{\alpha}_{i,j} Y_j} h_{X_i}}{e^{\bar{\alpha}_{i,j} Y_{j+1}} - e^{\bar{\alpha}_{i,j} Y_j}} + \frac{\bar{b}_{i,j-\frac{1}{2}} e^{\bar{\alpha}_{i,j-1} Y_j} h_{X_i}}{e^{\bar{\alpha}_{i,j-1} Y_j} - e^{\bar{\alpha}_{i,j-1} Y_{j-1}}} \\
&+ \bar{c}_{i,j} R_{i,j},
\end{aligned}
\tag{4.13}
$$

$$
e^{i,j}_{i,j+1} = -\frac{\bar{b}_{i,j+\frac{1}{2}} e^{\bar{\alpha}_{i,j} Y_{j+1}} h_{X_i}}{e^{\bar{\alpha}_{i,j} Y_{j+1}} - e^{\bar{\alpha}_{i,j} Y_j}}, \quad e^{i,j}_{i+1,j} = -\frac{b_{i+\frac{1}{2},j} e^{-\frac{\alpha_{i,j}}{X_{i+1}}} h_{Y_j}}{e^{-\frac{\alpha_{i,j}}{X_{i+1}}} - e^{-\frac{\alpha_{i,j}}{X_i}}},
\tag{4.14}
$$

and $e^{i,j}_{m,n} = 0$ if $m \neq i-1, i, i+1$ and $n \neq j-1, j, j+1$.

Let $T = t_0 > t_1, \cdots, > t_M = 0$ be a partition of $[0, T]$. Applying the two-level implicit time-stepping method with a splitting parameter $\theta \in [\frac{1}{2}, 1]$ to (4.11) on this mesh, we get the following full discrete system:

$$(\theta E^{m+1} + G^m)F^{m+1} + \theta D(F^{m+1}) = (G^m - (1-\theta)E^m)F^m - (1-\theta)D(F^m). \tag{4.15}$$

Theorem 4 *For any given $m = 1, 2, \cdots, M-1$, if $|\Delta t_m|$ is sufficiently small, $\bar{c} \geq 0$, and (4.15) is solved by Newton's method, then its system matrix is an M-matrix.*

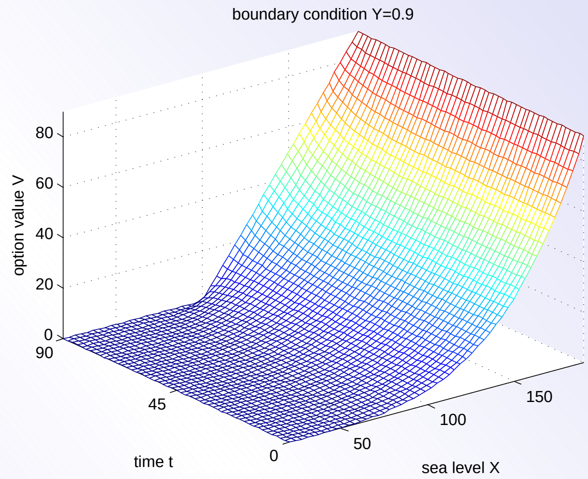
Theorem 5 *Let F and F_h be the exact and the semi-discrete solutions. Then we have*

$$\|F - F_h\|_h \leq Ch,$$

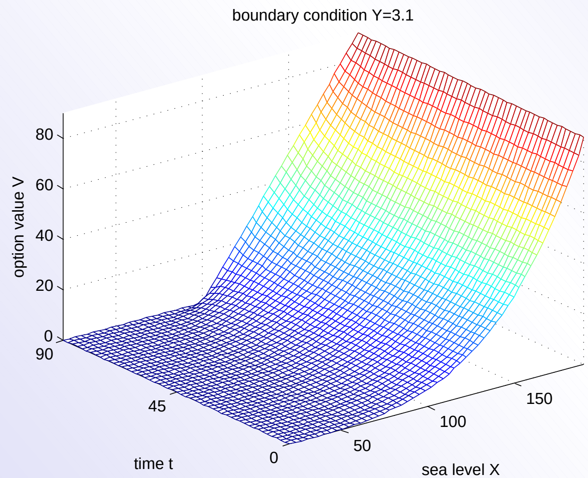
$\|\cdot\|_h$ *is a weighted discrete H^1-norm.*

5 The numerical example

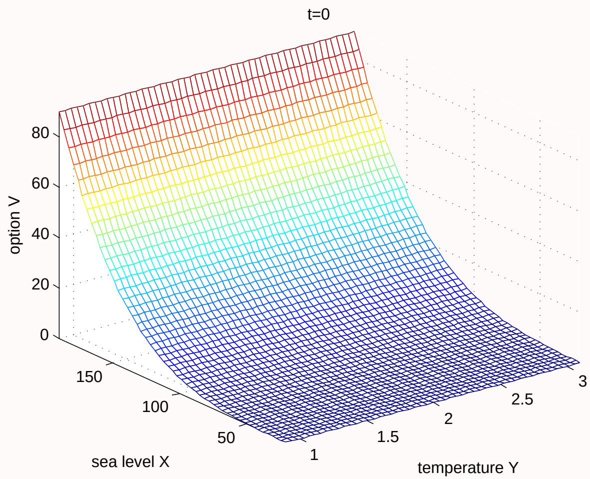
American option with the parameters: $X_{min} = 20$, $X_{max} = 190$, $Y_{min} = 0.9$, $Y_{max} = 3.1$, $\eta = 0.3$, $\theta = 0.01$, $T = 90$, $r = 0.1$, $Y_0 = 0.5$, $Y_1 = 1.5$, $\sigma_X = 0.2$, $\sigma_Y = 0.3$, $K = 100$, $\lambda = 10$.


boundary condition Y=0.9
option value V
80
60
40
20
0
90
45
0
time t
50
100
150
sea level X

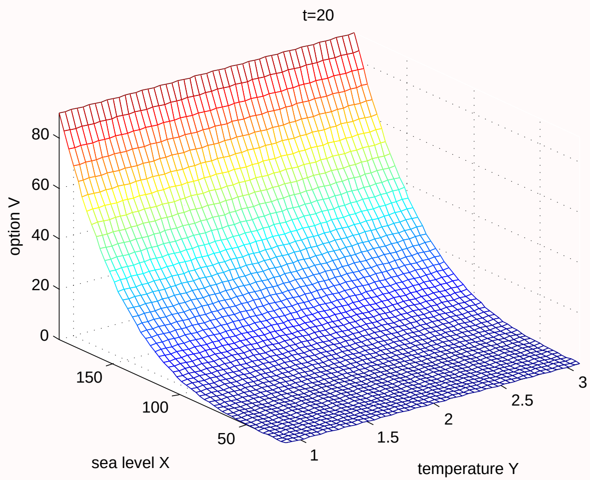
(a)


boundary condition Y=3.1
option value V
80
60
40
20
0
90
45
0
time t
50
100
150
sea level X

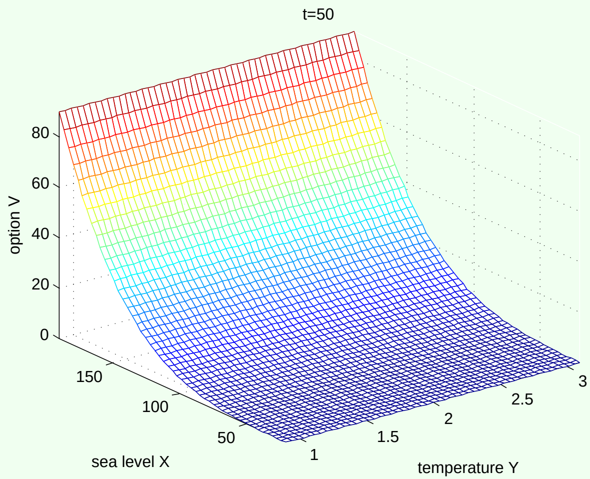
(b)


t=0
option V
80
60
40
20
0
150
100
50
sea level X
1
1.5
2
2.5
3
temperature Y

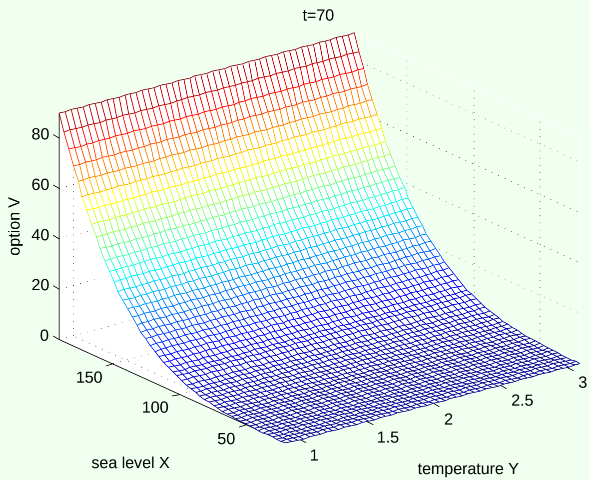
(a) $t = 0$


t=20
option V
80
60
40
20
0
150
100
50
sea level X
1
1.5
2
2.5
3
temperature Y

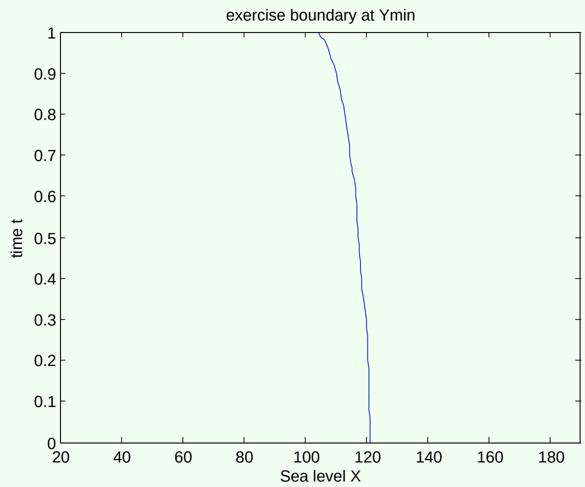
(b) $t = 20$


t=50
option V
80
60
40
20
0
150
100
50
sea level X
1
1.5
2
2.5
3
temperature Y

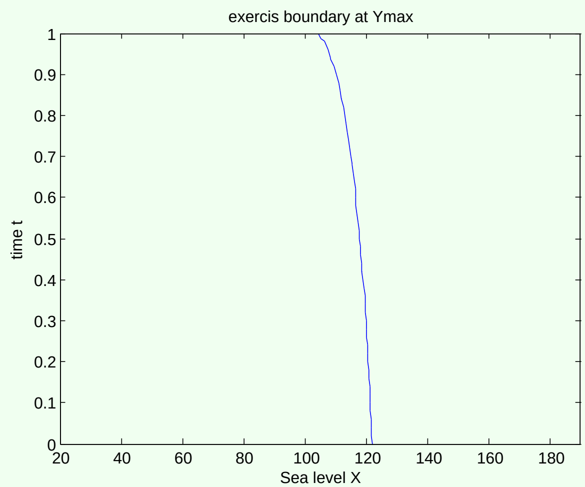
(c) $t = 50$


t=70
option V
80
60
40
20
0
150
100
50
sea level X
1
1.5
2
2.5
3
temperature Y

(d) $t = 70$


exercise boundary at Ymin
time t
Sea level X
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
20
40
60
80
100
120
140
160
180

(a)


exercis boundary at Ymax
time t
Sea level X
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
20
40
60
80
100
120
140
160
180

(b)

Thank you very much !