Finite element exterior calculus for parabolic problems

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Content

1 Introduction

First, consider two examples[a]

Example 1. The vector Laplacian on an annulus

$$\begin{aligned} \operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f, \quad & \text{in } \Omega, \\ u \cdot n = 0, \quad \operatorname{curl} u \times n = 0, \quad & \text{on } \partial\Omega. \end{aligned}$$

Weak formulation: Find $u \in H(\operatorname{curl}) \cap \mathring{H}(\operatorname{div})$, such that

$$\langle \operatorname{curl} u, \operatorname{curl} v \rangle + \langle \operatorname{div} u, \operatorname{div} v \rangle = \langle f, v \rangle, \quad v \in H(\operatorname{curl}) \cap \mathring{H}(\operatorname{div})$$

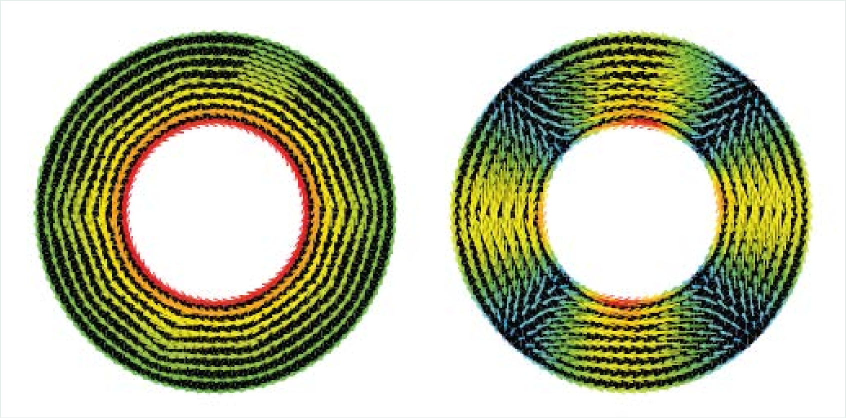
Finite element discretization: Find $u_h \in \mathcal{P}^1$, such that

$$\langle \operatorname{curl} u_h, \operatorname{curl} v \rangle + \langle \operatorname{div} u_h, \operatorname{div} v \rangle = \langle f, v \rangle, \quad v \in \mathcal{P}^1.$$

Wrong formulation, since there exists non trivial 1-harmonic function: $f = 0 \not\Rightarrow u = 0$.

[a] Arnold, Douglas N. and Falk, Richard S. and Winther, Ragnar,Finite element exterior calculus: from Hodge theory to numerical stability, Bull. Amer. Math. Soc. (N.S.),47,2010,281–354.

$f = (0, x)$



Solution: Right hand side f and the solution both should be orthogonal to 1-harmonic function, i.e.

$$\operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f (\text{mod } \mathfrak{H}^1), \quad u \perp \mathfrak{H}^1.$$

Elliptic problems should account for harmonic function.

The k-th Betti number β^k represents the dimension of the k-th cohomology, and is one of most important topology invariance in $\mathbb{R}^n$. For example, $\Omega \subset \mathbb{R}^3$,

$$\beta^k = \begin{cases} \text{no. of connected components}, & k = 0, \\ \text{no. of handles}, & k = 1, \\ \text{no. of voids}, & k = 2, \\ 0 & k = 3 \end{cases}$$

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Example 2. The vector Laplacian on a nonconvex polygon

$$\operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f, \quad \text{in } \Omega,$$
$$u \cdot n = 0, \quad \operatorname{curl} u \times n = 0, \quad \text{on } \partial\Omega.$$

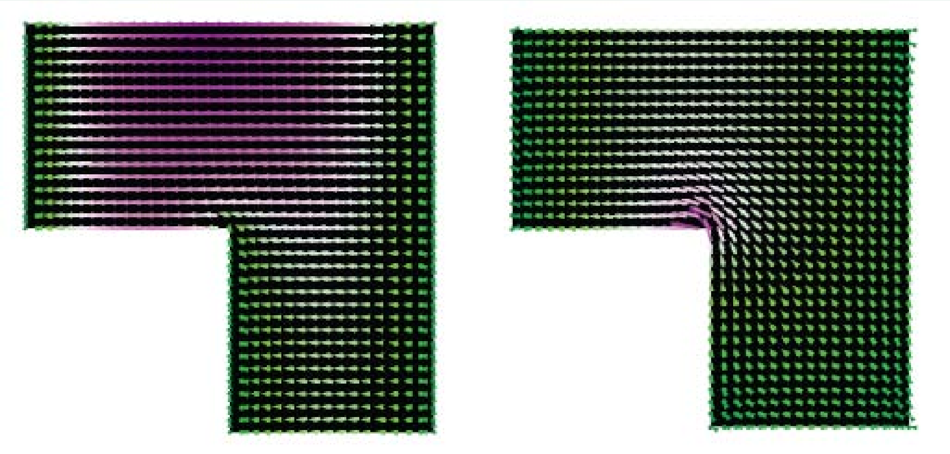
Weak formulation: Find$u \in H(\operatorname{curl}) \cap \mathring{H}(\operatorname{div})$, such that

$$\langle \operatorname{curl} u, \operatorname{curl} v\rangle + \langle \operatorname{div} u, \operatorname{div} v\rangle = \langle f, v\rangle, \quad v \in H(\operatorname{curl}) \cap \mathring{H}(\operatorname{div})$$

Finite element discretization: Find $u_h \in \mathcal{P}^1$, such that

$$\langle \operatorname{curl} u_h, \operatorname{curl} v\rangle + \langle \operatorname{div} u_h, \operatorname{div} v\rangle = \langle f, v\rangle, \quad v \in \mathcal{P}^1.$$

Wrong formulation! $f = (-1, 0)$



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Reason: $H^1 \cap \mathring{H}(\mathrm{div}) \subsetneq H(\mathrm{curl}) \cap \mathring{H}(\mathrm{div})$. Closed proper subspace, but true solution does not belong to $H^1 \cap \mathring{H}(\mathrm{div})$.

solution:

$$\sigma = -\operatorname{div} u, \quad \operatorname{grad} \sigma + \operatorname{curl} \operatorname{curl} u = f, \quad \text{in } \Omega,$$
$$u \cdot n = 0, \quad \operatorname{curl} u \times n = 0, \quad \text{on } \partial\Omega$$

Weak formulation: Find $\sigma \in H^1, u \in H(\mathrm{curl})$, such that

$$\langle \sigma, \tau \rangle - \langle \operatorname{grad} \tau, u \rangle = 0, \quad \tau \in H^1,$$
$$\langle \operatorname{grad} \sigma, v \rangle + \langle \operatorname{curl} u, \operatorname{curl} v \rangle = \langle f, v \rangle, \quad v \in H(\mathrm{curl}).$$

Finite element discretization: Find $\sigma_h \in \mathcal{P}^1, u_h \in RT^0$, such that

$$\langle \sigma_h, \tau \rangle - \langle \operatorname{grad} \tau, u_h \rangle = 0, \quad \tau \in \mathcal{P}^1,$$
$$\langle \operatorname{grad} \sigma_h, v \rangle + \langle \operatorname{curl} u_h, \operatorname{curl} v \rangle = \langle f, v \rangle, \quad v \in RT^0.$$

Right formulation! Nonconvex domain should use mixed formulation.

2 Theory framework: Exterior calculus

Let $\Omega \subset \mathbb{R}^n$. $f : \Omega \to \mathbb{R}$ is a smooth function.

- $\forall x \in \Omega$, $df_x : \mathbb{R}^n \to \mathbb{R}$ is a linear map: $df_x(\mathbf{v}) = \mathbf{v} \cdot \nabla f(x)$.
- f is 0-(differential)form, df is 1-form, $\Lambda^0(\Omega)$, $\Lambda^1(\Omega)$
- ω is a $k-$ form $\Longleftrightarrow \forall x, \omega_x$ is alternative $k-$ form on $\mathbb{R}^n$
- $dx^1, \ldots, dx^n$: is a standard dual basis on $\mathbb{R}^n$ ($dx^i(e_j) = \delta_{ij}$)$\Rightarrow$

  $dx^{i_1} \wedge \cdots \wedge dx^{i_k}, i_1 < \cdots < i_k$, is basis for alternative $k-$ form

$$\omega \in \Lambda^k(\Omega) \Longleftrightarrow \omega = \sum f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

- a k- form has $\binom{n}{k}$ coefficients ($\Lambda^k(\Omega) = 0, \quad k > n$)
- Hodge star $\star : \Lambda^k \leftrightarrow \Lambda^{n-k}$: $(j_1, \ldots, j_{n-k}) = (1, \ldots, n)/(i_1, \ldots, i_k)$

$$f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \leftrightarrow f_{i_1 \cdots i_k} dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}}$$

- Exterior derivative. $d^k\omega : \Lambda^k \to \Lambda^{k+1}$: choose $\omega_x(v_1, \cdots, v_k)$'s derivative in direction of v_{k+1}, and antisymmetry:

$$d(f_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_{j=1}^{n} \frac{\partial f_{i_1\cdots i_k}}{\partial x_j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

- $d^k \circ d^{k-1} = 0$, i.e. $\mathfrak{B}^k := \mathrm{range}(d^{k-1}) \subset \mathfrak{Z}^k := \ker(d^k)$
- If $F : \Omega \to \Omega'$, the pullback $F^* : \Lambda^k(\Omega') \to \Lambda^k(\Omega)$:

$$(F^*\omega)_x(\mathbf{v_1}, \ldots, \mathbf{v_k}) = \omega_{F(x)}(dF_x\mathbf{v_1}, \ldots, dF_x\mathbf{v_k})$$

- If $S \subset \Omega$, the pullback of $\mathcal{I} : S \to \Omega$ is trace operator $\mathrm{tr} : \Lambda^k(\Omega) \to \Lambda^k(S)$.

- Lebesgue space $L^p\Lambda^k$, especially $L^2\Lambda^k$. Define inner product as follows:

$$\langle \omega, \eta \rangle = \langle \omega, \eta \rangle_{L^2\Lambda^k} = \int_\Omega \langle \omega_x, \eta_x \rangle dx.$$

- Finite energy k-form is defined as:

$$H\Lambda^k(\Omega) = \{\omega \in L^2\Lambda^k(\Omega) \mid d\omega \in L^2\Lambda^{k+1}(\Omega)\}$$

Hilbert space with norm $\|\omega\|^2_{H\Lambda} = \|\omega\|^2 + \|d\omega\|^2$.

- Through exterior derivative $H\Lambda^k(\Omega)$ connect *de Rham* complex:

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d^0} H\Lambda^1(\Omega) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H\Lambda^n(\Omega) \to 0.$$

- The $k-$th de Rham cohomology: $\mathfrak{Z}^k/\mathfrak{B}^k$

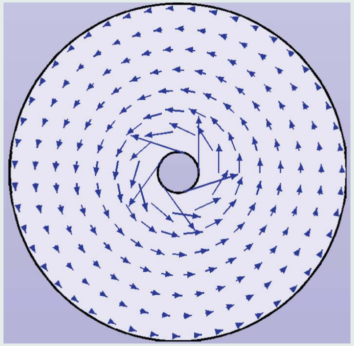
- $k-$ harmonic form $\mathfrak{H}^k := \mathfrak{Z}^k \cap \mathfrak{B}^{k\perp} \cong \mathfrak{Z}^k/\mathfrak{B}^k$. $\dim\mathfrak{H}^k$ = k-th Betti No.

$\Omega \subset \mathbb{R}^3$, *de Rham* complex is

$$0 \to H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl};\Omega) \xrightarrow{\text{curl}} H(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0.$$

$$\text{tr}: \quad |_{\partial\Omega}, \quad \times n, \quad \cdot n$$

$$\dim(\mathfrak{H}^k) = \begin{cases} \text{no. of connected compo.,} & k = 0, \\ \text{no. of handles,} & k = 1, \\ \text{no. of voids,} & k = 2, \\ 0 & k = 3 \end{cases}$$



If Ω is contractible, $\mathfrak{H}^0 = \mathbb{R}, \quad \mathfrak{H}^k = 0, k > 0$

- Hodge decomposition: $L^2\Lambda^k = \overbrace{\mathfrak{B}^k \oplus \mathfrak{H}^k}^{\mathfrak{Z}^k} \oplus \mathfrak{Z}^{k\perp}$
- Adjoint operator of d^{k-1}: $d_k^* : L^2\Lambda^k(\Omega) \to L^2\Lambda^{k-1}(\Omega)$ with domain

$$H^*\Lambda^k := \{\omega \in L^2\Lambda^k \mid d^*\omega \in L^2\Lambda^{k-1}\}$$

For example, $\Omega \subset \mathbb{R}^3, \quad d^0 = \mathrm{grad},\ d^1 = \mathrm{curl},\ d^2 = \mathrm{div}$

$$d_1^* = -\,\mathrm{div},\ d_2^* = \mathrm{curl},\ d_3^* = -\,\mathrm{grad}$$

- $\mathfrak{H}^k = \ker(d^k) \cap \mathrm{range}(d^{k-1})^\perp = \ker(d^k) \cap \ker(d_k^*)$

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Choose finite dimensional space $\Lambda_h^{k-1} \subset H\Lambda^{k-1}, \Lambda_h^k \subset H\Lambda^k$

Assume $d\Lambda_h^{k-1} \subset \Lambda_h^k, \quad d^k\Pi_h^k = \Pi_h^{k+1}d^k$

$$
\begin{array}{ccccccccc}
0 \to & H\Lambda^0(\Omega) & \xrightarrow{d^0} & H\Lambda^1(\Omega) & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & H\Lambda^n(\Omega) & \to 0 \\
 & \Pi_h^0 \downarrow & & \Pi_h^1 \downarrow & & & & \Pi_h^n \downarrow & \\
0 \to & \Lambda_h^0 & \xrightarrow{d^0} & \Lambda_h^1 & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & \Lambda_h^n & \to 0
\end{array}
$$

There exists uniformly bounded de Rham projection $\Pi_h^k : L^2\Lambda^k \to \Lambda_h^k$ such that

$$\|\Pi_h^k v\|_{L^2\Lambda^k(\Omega)} \leq C\|v\|_{L^2\Lambda^k(\Omega)}.$$

- $\mathfrak{Z}_h^k \subset \mathfrak{Z}^k \quad \mathfrak{B}_h^k \subset \mathfrak{B}^k$
- Discretized harmonic function space

$$\mathfrak{H}_h^k = \mathfrak{Z}_h^k \cap \mathfrak{B}_h^{k\perp}$$

- de Rham Theorem $\Longleftrightarrow$ dimension of discretized harmonic function space$\equiv$ k-th Betti No., $\forall h$.
- $\mathrm{gap}(\mathfrak{H}_h^k, \mathfrak{H}^k) \to 0,$ as $h \to 0$.
- Discretized Hodge decompsition

$$\Lambda_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{Z}_h^{k\perp}.$$

Let $\mathcal{T}_h$ be a simplex partition of $\Omega \subset \mathbb{R}^n$. Goal: construct $\Lambda_h^k \subset H\Lambda^k(\Omega)$ to form finite dimensional subcomplex and satisfy the above assumptions.

For general k, there exist two families of polynomial differential forms, $\mathcal{P}_r\Lambda^k, \mathcal{P}_r^-\Lambda^k$. After finite element composition, we obtain the subspace of $H\Lambda^k(\Omega)$, $\mathcal{P}_r\Lambda^k(\mathcal{T}_h), \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$.

For $\Omega \subset \mathbb{R}^n$, there totally exist 2^{n-1} finite element subcomplex. Take $n = 3$, 4 for example, the subcomplexes are as follows:

$$0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \mathcal{P}_{r-2}\Lambda^2 \xrightarrow{d} \mathcal{P}_{r-3}\Lambda^3 \to 0,$$
$$0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \mathcal{P}_{r-1}^-\Lambda^2 \xrightarrow{d} \mathcal{P}_{r-2}\Lambda^3 \to 0,$$
$$0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^2 \xrightarrow{d} \mathcal{P}_{r-2}\Lambda^3 \to 0,$$
$$0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \mathcal{P}_r^-\Lambda^2 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^3 \to 0.$$

- $\mathcal{P}_r^-\Lambda^0(\mathcal{T}_h) = \mathcal{P}_r\Lambda^0(\mathcal{T}_h) \subset H^1$ Lagrange elements
- $\mathcal{P}_r^-\Lambda^n(\mathcal{T}_h) = \mathcal{P}_{r-1}\Lambda^n(\mathcal{T}_h) \subset L^2$ discontinuous elts
- $n = 2 : \mathcal{P}_r^-\Lambda^1(\mathcal{T}_h) \subset H(\mathrm{curl})$ Raviart-Thomas elts
- $n = 2 : \mathcal{P}_r\Lambda^1(\mathcal{T}_h) \subset H(\mathrm{curl})$ Brezzi-Douglas-Marini elts
- $n = 3 : \mathcal{P}_r^-\Lambda^1(\mathcal{T}_h) \subset H(\mathrm{curl})$ Nédélec 1st kind edge elts
- $n = 3 : \mathcal{P}_r\Lambda^1(\mathcal{T}_h) \subset H(\mathrm{curl})$ Nédélec 2nd kind edge elts
- $n = 3 : \mathcal{P}_r^-\Lambda^2(\mathcal{T}_h) \subset H(\mathrm{div})$ Nédélec 1st kind face elts
- $n = 3 : \mathcal{P}_r\Lambda^2(\mathcal{T}_h) \subset H(\mathrm{div})$ Nédélec 2nd kind face elts

3 Finite element exterior calculus for parabolic problems

3.1. Mixed formulation for parabolic problems

Let Ω be a bounded domain in $\mathbb{R}^n$ with piecewise smooth and Lipschitz continuous boundary. Consider the following initial boundary problem:

$$\begin{aligned} u_t + (dd^* + d^*d)u = f, &\quad \text{in } \Omega \times (0, T], \\ \operatorname{tr}(\star u) = 0,\ \operatorname{tr}(\star du) = 0, &\quad \text{on } \partial\Omega \times (0, T], \\ u(\cdot, 0) = u_0, &\quad \text{in } \Omega, \end{aligned}$$

where the unknown u is time dependent differential $k-$form in Ω, u_t denotes the time derivative of u.

For example, $\Omega \subset \mathbb{R}^3$, $\quad d^0 = \mathrm{grad},\ d^1 = \mathrm{curl},\ d^2 = \mathrm{div}$

$$d_1^* = -\mathrm{div},\ d_2^* = \mathrm{curl},\ d_3^* = -\mathrm{grad}$$

- $k = 0$: $u_t - \Delta u = f$ and $\partial u/\partial n = 0$ Neumann problem of heat equation
- $k = 1$: $u_t + \mathrm{curl}\,\mathrm{curl}\, u - \mathrm{grad}\,\mathrm{div}\, u = f$ and $u \cdot n = 0,\ \mathrm{curl}\, u \times n = 0$
- $k = 2$: $u_t + \mathrm{curl}\,\mathrm{curl}\, u - \mathrm{grad}\,\mathrm{div}\, u = f$ and $u \times n = 0,\ \mathrm{div}\, u = 0$
- $k = 3$: $u_t - \Delta u = f$ and $u = 0$ Dirichlet problem of heat equation

Let $\sigma = d^* u$. Weak formulation: Find $(\sigma, u) : (0, T] \to H\Lambda^{k-1} \times H\Lambda^k$, such that $u(0) = u_0$ and

$$\langle \sigma, \tau \rangle - \langle d\tau, u \rangle = 0, \quad \tau \in H\Lambda^{k-1},\ t \in (0, T],$$
$$\langle u_t, v \rangle + \langle d\sigma, v \rangle + \langle du, dv \rangle = \langle f, v \rangle, \quad v \in H\Lambda^k.\ t \in (0, T],$$

Hodge Laplacian: unbounded operator $L = dd^* + d^*d : D(L) \subset L^2\Lambda^k \to L^2\Lambda^k$, where

$$D(L) = \{v \in H\Lambda^k \cap \mathring{H}^*\Lambda^k \mid d^*v \in H\Lambda^{k-1}, dv \in \mathring{H}^*\Lambda^{k+1}\}.$$

$$u_t + Lu = f, t \in (0, T] \quad u(0) = u_0.$$

$$\begin{array}{c} u_t + Lu = 0, t \in (0, T] \\ u(0) = u_0 \end{array} \xrightarrow{\text{Duhamel's principle}} \begin{array}{c} u_t + Lu = f, t \in (0, T] \\ u(0) = 0 \end{array}$$

$$u(t) = S(t)u_0 \qquad\qquad u(t) = \int_0^t S(t-s)f(s)ds$$

Theorem 1. Hodge Laplacian L is maximal monotone, i.e., satisfying

$$\langle Lv, v\rangle \geq 0, \quad \forall v \in D(L),$$

and for all $f \in L^2\Lambda^k$, there exists $u \in D(L)$, such that $u + Lu = f$.

Theorem 2. Assume $u_0 \in L^2\Lambda^k$, $f \in C([0,T]; L^2\Lambda^k) \cap L^1((0,T); D(L))$. Then there exists unique solution $\sigma \in C((0,T]; H\Lambda^{k-1})$,

$$u \in C([0,T]; L^2\Lambda^k) \cap C((0,T]; D(L)) \cap C^1((0,T]; L^2\Lambda^k)$$

satisfying the original mixed problem and the initial condition $u(0) = u_0$. Moreover, if $f \in C((0,T]; D(L)) \cap L^1((0,T); D(L^2))$,

$$u \in C^1((0,T]; D(L)).$$

3.2. Finite element discretization for parabolic problems

Choose $\Lambda_h^{k-1}, \Lambda_h^k$ to discretize the weak formulation: Find $(\sigma_h, u_h) \in C([0,T];\Lambda_h^{k-1}) \times C^1([0,T];\Lambda_h^k)$, such that

$$\begin{aligned} \langle \sigma_h, \tau\rangle - \langle d\tau, u_h\rangle &= 0, \quad \tau \in \Lambda_h^{k-1}, \\ \langle u_{h,t}, v\rangle + \langle d\sigma_h, v\rangle + \langle du_h, dv\rangle &= \langle f, v\rangle, \quad v \in \Lambda_h^k. \end{aligned}$$

Error estimate:

$$\begin{aligned} \sigma - \sigma_h &= (\sigma - \hat{\sigma}_h) + (\hat{\sigma}_h - \sigma_h) \\ u - u_h &= (u - \hat{u}_h) + (\hat{u}_h - u_h) \end{aligned}$$

Choosing proper test function and by energy inequality, $\hat{\sigma}_h - \sigma_h, \hat{u}_h - u_h$ can be bounded by $\sigma - \hat{\sigma}_h, u - \hat{u}_h$.

Elliptic projection of true solution: Given $u \in D(L)$, let $\sigma = d^*u$. Define the elliptic projection of (σ, u) by $(\hat{\sigma}_h, \hat{u}_h, \hat{p}_h) \in \Lambda_h^{k-1} \times \Lambda_h^k \times \mathfrak{H}_h^k$, such that

$$\langle \hat{\sigma}_h, \tau \rangle - \langle d\tau, \hat{u}_h \rangle = 0, \quad \tau \in \Lambda_h^{k-1},$$
$$\langle d\hat{\sigma}_h, v \rangle + \langle d\hat{u}_h, dv \rangle + \langle \hat{p}_h, v \rangle = \langle d\sigma, v \rangle + \langle du, dv \rangle, \quad v \in \Lambda_h^k,$$
$$\langle \hat{u}_h, q \rangle = \langle u, q \rangle, \quad q \in \mathfrak{H}_h^k.$$

Due to the definition of $\mathfrak{H}, \mathfrak{H}_h$,

$$\|\sigma - \hat{\sigma}_h\|, \|d(\sigma - \hat{\sigma}_h)\|, \|d(u - \hat{u}_h)\|$$

can be obtained from the error estimates of elliptic equation. It only remains to estimate $\|u - \hat{u}_h\|, \|\hat{p}_h\|$.

Introduce the optimal order error estimates under L^2 norm:

$$E(w) = \inf_{v \in \Lambda_h^k} \|w - v\|, \forall w \in L^2\Lambda^k, \ k = 0, \ldots, n.$$

Error estimates. $P_{\mathfrak{H}} u - P_{\mathfrak{H}_h} \hat{u}_h = P_{\mathfrak{H}} u - P_{\mathfrak{H}_h} u = (I - P_{\mathfrak{H}_h}) P_{\mathfrak{H}} u + P_{\mathfrak{H}_h}(P_{\mathfrak{H}} u - u)$,

$$\|(I - P_{\mathfrak{H}_h}) P_{\mathfrak{H}} u\| \leq \|(I - \Pi_h) P_{\mathfrak{H}} u\| \leq CE(P_{\mathfrak{H}} u).$$

$$u - P_{\mathfrak{H}} u = \overbrace{u_b}^{\mathfrak{B}^k} + \overbrace{u_\perp}^{\mathfrak{Z}^{k\perp}}.$$

Let $q = P_{\mathfrak{H}_h}(P_{\mathfrak{H}} u - u)/\|P_{\mathfrak{H}_h}(P_{\mathfrak{H}} u - u)\| \in \mathfrak{H}_h^k$.

$$\begin{aligned}\|P_{\mathfrak{H}_h}(P_{\mathfrak{H}} u - u)\| &= (P_{\mathfrak{H}_h}(P_{\mathfrak{H}} u - u) \overbrace{, q) = (P_{\mathfrak{H}_h}}^{Hodge\ decomp.} (\Pi_h u_b - u_b), q) = (\Pi_h u_b - u_b, q) \\ &= (\Pi_h u_b - u_b, q - P_{\mathfrak{H}} q) \leq \|(I - \Pi_h) u_b\| \sup_{r \in \mathfrak{H}^k, \|r\|=1} \|(I - \Pi_h) P_{\mathfrak{H}} r\| \\ &\leq C\|(I - \Pi_h) P_{\mathfrak{H}}\| \cdot E(P_{\mathfrak{B}} u).\end{aligned}$$

Combining them, $\|P_{\mathfrak{H}} u - P_{\mathfrak{H}_h} \hat{u}_h\| \leq C[E(P_{\mathfrak{H}} u) + \|(I - \Pi_h) P_{\mathfrak{H}}\| \cdot E(P_{\mathfrak{B}} u)] \Rightarrow \|u - \hat{u}_h\|$.

$$\|\hat{p}_h\| = \|P_{\mathfrak{H}_h}(d\sigma + d^* du)\| = \|P_{\mathfrak{H}_h}(d\sigma)\| \leq C\|(I - \Pi_h) P_{\mathfrak{H}}\| \cdot E(d\sigma).$$

Rather than give a complicated statement of the results, covering all the possible cases, in the following theorem and below we restrict to a particular choice of spaces from among the possibilities in two families of finite element differential forms. Choose $\Lambda_h^{k-1} = \mathcal{P}_r^- \Lambda^{k-1}(\mathcal{T}_h)$, $\Lambda_h^k = \mathcal{P}_r^- \Lambda^k(\mathcal{T}_h)$ and the auxiliary space $\Lambda_h^{k+1} = \mathcal{P}_r^- \Lambda^{k+1}(\mathcal{T}_h)$(where $r > 1$). Then, we can get the error estimates of elliptic projection:

$$
\begin{aligned}
\|d(\sigma - \hat{\sigma}_h)\| &\le Ch^r \|d\sigma\|_r, \\
\|\sigma - \hat{\sigma}_h\| &\le Ch^r \|\sigma\|_r, \\
\|\hat{p}_h\| &\le Ch^r \|d\sigma\|_{r-2}, \\
\|d(u - \hat{u}_h)\| &\le Ch^r (\|du\|_r + \|d\sigma\|_{r-1}), \\
\|u - \hat{u}_h\| &\le Ch^r \|u\|_{\bar{H}^r},
\end{aligned}
$$

where $\bar{H}^r = \{u \in H^r | P_{\mathfrak{H}} u \in H^r, P_{\mathfrak{B}} u \in H^{r-2}\}$.

Theorem 3. Assume $f \in C((0,T];D(L)) \cap L^1((0,T);D(L^2)$, $u_0 \in D(L)$. (σ,u) is the solution of parabolic problems. Choose $\Lambda_h^{k-1} = \mathcal{P}_r^- \Lambda^{k-1}(\mathcal{T}_h)$, $\Lambda_h^k = \mathcal{P}_r^- \Lambda^k(\mathcal{T}_h)$ and auxiliary space $\Lambda_h^{k+1} = \mathcal{P}_r^- \Lambda^{k+1}(\mathcal{T}_h)$(where $r > 1$). (σ_h, u_h) is finite element solution, and choose $u_h(0)$ as the elliptic projection of u_0. Then we have the following error estimates:

$$
\begin{aligned}
\|\sigma - \sigma_h\|_{L^2(L^2)} &\le Ch^r(\|u_t\|_{L^1(\bar{H}^r)} + \|d^*u\|_{L^2(H^r)}), \\
\|\sigma - \sigma_h\|_{L^\infty(L^2)} &\le Ch^r(\|u_t\|_{L^2(\bar{H}^r)} + \|d^*u\|_{L^\infty(H^r)}), \\
\|d(\sigma - \sigma_h)\|_{L^2(L^2)} &\le Ch^r(\|u_t\|_{L^2(\bar{H}^r)} + \|dd^*u\|_{L^2(H^r)}), \\
\|u - u_h\|_{L^\infty(L^2)} &\le Ch^r(\|u\|_{L^\infty(\bar{H}^r)} + \|u_t\|_{L^1(\bar{H}^r)}), \\
\|d(u - u_h)\|_{L^2(L^2)} &\le Ch^r(\|u_t\|_{L^1(\bar{H}^r)} + \|du\|_{L^2(H^r)} + \|dd^*u\|_{L^2(H^{r-1})}).
\end{aligned}
$$

4 Numerical experiments

Example 4.1 The vectorial heat equation on a square annulus

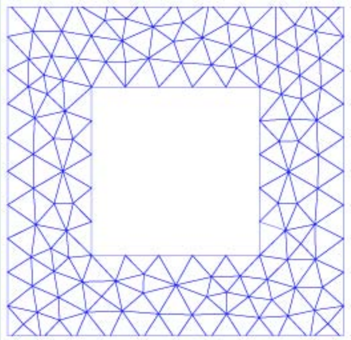
$\Omega \subset \mathbb{R}^2$. Using vector proxies, we may write 1-form parabolic equations as

$$\begin{aligned} u_t + \operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f, &\quad \text{in } \Omega \times (0, T], \\ u \cdot n = 0, \quad \operatorname{curl} u = 0, &\quad \text{on } \partial\Omega \times (0, T], \\ u(\cdot, 0) = u_0, &\quad \text{in } \Omega, \end{aligned}$$

Choose f, u_0 such that the solution is

$$u = \begin{pmatrix} 100x(x-1)(x-0.25)(x-0.75)t \\ 100y(y-1)(y-0.25)(y-0.75)t \end{pmatrix}.$$

Choose time step as $\Delta t = 0.0001$ and compute the error at $T = 0.01$(after 100 time steps).



$$\mathcal{P}_1\Lambda^0 \times \mathcal{P}_1^-\Lambda^1 (h = 0.1)$$

mesh size	$\|\sigma - \sigma_h\|$	rate	$\|\nabla(\sigma - \sigma_h)\|$	rate	$\|u - u_h\|$	rate
h	8.491E-4	1.99	1.026E-1	1.01	1.059E-3	0.96
h/2	2.132E-4	1.99	5.128E-2	1.00	5.341E-4	0.99
h/4	5.341E-5	2.00	2.565E-2	1.00	2.678E-4	1.00
h/8	1.332E-5	2.00	1.283E-2	1.00	1.340E-4	1.00

$$\mathcal{P}_2\Lambda^0 \times \mathcal{P}_2^-\Lambda^1$$

mesh size	$\|\sigma - \sigma_h\|$	rate	$\|\nabla(\sigma - \sigma_h)\|$	rate	$\|u - u_h\|$	rate
h	9.392E-6	3.03	1.651E-3	2.03	7.051E-5	1.97
h/2	1.173E-6	3.00	4.119E-4	2.00	1.766E-5	1.99
h/4	1.466E-7	3.00	1.030E-4	2.00	4.415E-6	1.99
h/8	1.830E-8	3.04	2.574E-5	2.00	1.104E-6	2.00

Example 4.2 Vectorial heat equation on a cube $\Omega \subset \mathbb{R}^3$. Using vector proxies, we may write 1-form parabolic equations as

$$\begin{aligned} u_t + (\operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div})u = f, \quad & \text{in } \Omega \times (0,T] \\ u \cdot n = 0, \quad \operatorname{curl} u \times n = 0, \quad & \text{on } \partial\Omega \times (0,T] \\ u(\cdot, 0) = u_0, \quad & \text{in } \Omega. \end{aligned}$$

Ω is a cube $[0,1] \times [0,1] \times [0,1]$. Choose u_0, f such that the solution is

$$u = \begin{pmatrix} \sin(\pi x_1)t \\ \sin(\pi x_2)t \\ \sin(\pi x_3)t \end{pmatrix}.$$

Choose time step as $\Delta t = 0.0001$ and compute the error at $T = 0.01$ (after 100 time steps).

$$P_1\Lambda^0 \times P_1^-\Lambda^1 (h = 0.25)$$

mesh size	$\|\sigma - \sigma_h\|$	rate	$\|\nabla(\sigma - \sigma_h)\|$	rate	$\|u - u_h\|$	rate
h	2.333E-3	2.06	2.602E-2	1.02	2.602E-3	1.00
h/2	5.735E-4	2.02	1.348E-2	0.95	1.350E-3	0.95
h/4	1.429E-4	2.01	6.817E-3	0.98	6.879E-4	0.97

5 Conclusion

We consider the extension of the finite element exterior calculus from elliptic problems to parabolic problems. The numerical discretization and analysis of parabolic problems presented here not only cover the scalar heat equation (which can be viewed as the special case of n-forms), but also the general k-form case. Moreover, unlike in the elliptic case, harmonic forms do not enter the weak formulation or the Galerkin method for the parabolic problem. Finally, the method can be similarly extended to variable coefficients and other boundary conditions.

Thank you!