3rd Russian-Chinese Workshop on Numer. Math .& Sci. Computing

Numerical Methods and Solutions of Nonlinear Dirac Equation

Huazhong Tang

School of Mathematical Sciences

Peking University

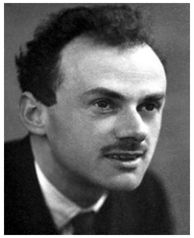
2013-09-12

Outline

1. Introduction
2. Multi-hump solitary waves
3. Runge-Kutta DG method
4. Moving mesh method
5. Numerical experiments
6. Conclusion

1、Introduction

- **Dirac equation** is a relativistic wave equation in particle physics, formulated by Paul Dirac in 1928, and describes fields corresponding to elementary spin-½ particles (such as the electron) as a vector of four complex numbers (a bi-spinor), in contrast to the Schrödinger equation which describes a field of only one complex value.



Paul Dirac shared the 1933 **Nobel Prize** for physics with Erwin Schrödinger "for the discovery of new productive forms of atomic theory."

北京大学

1、Introduction

Dirac equation in the covariant form

unified form for all inertial coordinate

$$\frac{1}{i}\gamma^\mu\partial_\mu\psi + m\psi = 0$$

where $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ are four contravariant **gamma matrices,** also known as the **Dirac matrices**

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

satisfying the anticommutation relation：

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} I_4$$

metric tensor

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1、Introduction

Dirac equation in the rest frame

$$i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t} = \left(\frac{1}{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m\right)\psi(\mathbf{x},t)$$

where m is the rest mass of spin-½ particle (electron), the reduced Planck constant is: $\hbar \equiv \frac{h}{2\pi} = 1.054\,571\,68(18)\times 10^{-34}\ \mathrm{J\cdot s},$

The matrices are all Hermitian and have squares equal to the identity matrix, and they all mutually anticommute:

$$\alpha_i^2 = \beta^2 = I_4,\ \ \alpha_i\alpha_j + \alpha_j\alpha_i = 0,\ \ \alpha_i\beta + \beta\alpha_i = 0,\ \ i \neq j$$

They are usually taken as

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

where σ_i is Pauli matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

1、Introduction

- It is consistent with both the principles of quantum mechanics and the theory of special relativity, and is the first theory to account fully for relativity in the context of quantum mechanics.
- It implies the existence of a new form of matter, ***antimatter***, hitherto unsuspected and unobserved, and actually predated its experimental discovery.

1、Introduction

- The nonlinear Dirac (NLD) system in quantum field theory is used to model extended particles by the spinor field equation.
- To make the resulting NLD model to be Lorentz invariable, the so-called self-interaction Lagrangian can be built up from the bilinear (in the spinor) covariant which are categorized into five types: scalar, pseudoscalar, vector, axial vector and tensor.
- Different self-interactions give rise to different NLD models.

Classification	Covariant Form	no. of Components
Scalar	$\bar{\psi}\psi$	1
Pseudoscalar	$\bar{\psi}\gamma_5\psi$	1
Vector	$\bar{\psi}\gamma_\mu\psi$	4
Axial Vector	$\bar{\psi}\gamma_5\gamma_\mu\psi$	4
Rank 2 antisymmetric tensor	$\bar{\psi}\sigma_{\mu\nu}\psi$	6
Total		16

北京大學

1、Introduction

- For example, (1+1)-d Soler model (based on the scalar bilinear covariant)

$$i\partial_t\psi = \left[-i\sigma_1\partial_x + m\sigma_3 \boxed{-2\lambda(\psi^\dagger\sigma_3\psi)\sigma_3} \right]\psi$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is a classical spinorial model with scalar self-interaction.

- A key feature of the NLD equation is that it allows solitary wave solutions or particle-like solutions: the stable localized solutions with finite energy and charge.
- It describes the motion of the positive & negative electrons with high-speed.

M. Soler, Phys. Rev. D 1 (1970) 2766. M.F. Rañada, Phys. Rev. D 30 (1984) 1830.

1、Introduction

- Standing wave solution

$$\psi^{sw}(x,t) \equiv \begin{pmatrix} \psi_1^{sw}(x,t) \\ \psi_2^{sw}(x,t) \end{pmatrix} = \begin{pmatrix} A(x) \\ \mathrm{i}B(x) \end{pmatrix} e^{-\mathrm{i}\Lambda t},$$

$$0 < \Lambda \leqslant m,$$

$$A(x) = \frac{\sqrt{\frac{1}{\lambda}(m^2-\Lambda^2)(m+\Lambda)}\cosh(\sqrt{(m^2-\Lambda^2)}x)}{m+\Lambda\cosh(2\sqrt{(m^2-\Lambda^2)}x)},$$

$$B(x) = \frac{\sqrt{\frac{1}{\lambda}(m^2-\Lambda^2)(m-\Lambda)}\sinh(\sqrt{(m^2-\Lambda^2)}x)}{m+\Lambda\cosh(2\sqrt{(m^2-\Lambda^2)}x)}.$$

$$E(t) = \int_{-\infty}^{\infty} \mathrm{d}x \left[\mathrm{Im}\left(\psi_1^* \partial_x \psi_2 + \psi_2^* \partial_x \psi_1\right) + m\left(|\psi_1|^2 - |\psi_2|^2\right) - \lambda\left(|\psi_1|^2 - |\psi_2|^2\right)^2\right] \equiv \int_{-\infty}^{\infty} \mathrm{d}x\, \rho_E(x,t),$$

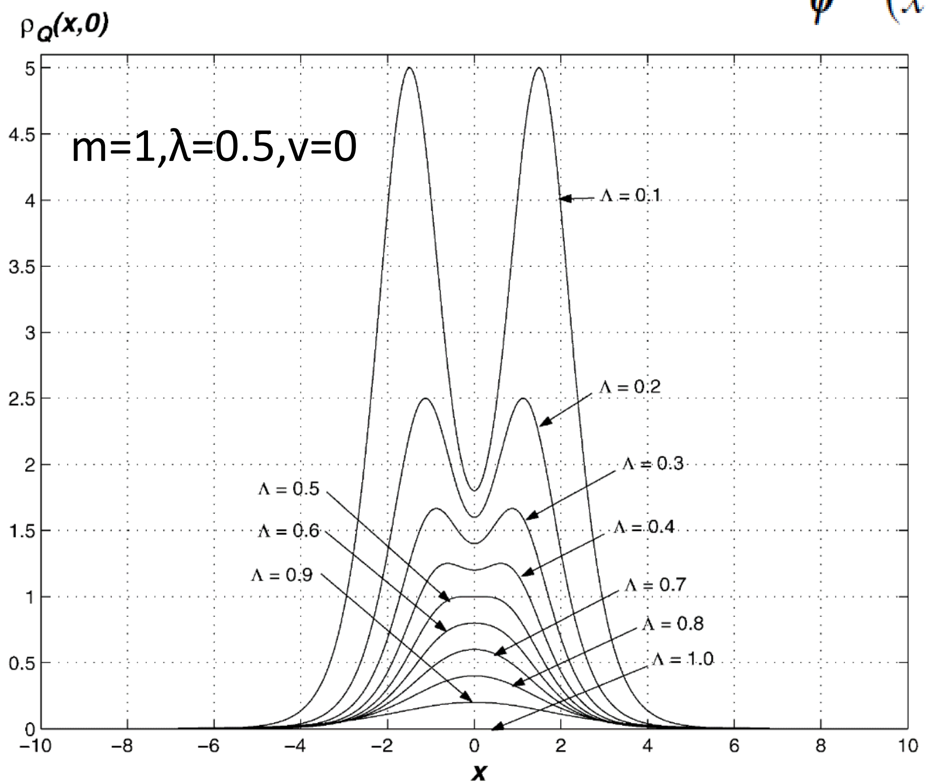
$$Q(t) = \int_{-\infty}^{\infty} \mathrm{d}x \left(|\psi_1|^2 + |\psi_2|^2\right) \equiv \int_{-\infty}^{\infty} \mathrm{d}x\, \rho_Q(x,t),$$

M. Soler, Phys. Rev. D 1 (1970) 2766.

1、Introduction

- solitary wave solution

$$\psi^{ss}(x-x_0,t)=\left(\psi_1^{ss}(x-x_0,t),\,\psi_2^{ss}(x-x_0,t)\right)^T$$



$$\psi_1^{ss}(x-x_0,t)=\sqrt{\frac{\gamma+1}{2}}\psi_1^{sw}(\tilde{x},\tilde{t})+\mathrm{sign}(v)\sqrt{\frac{\gamma-1}{2}}\psi_2^{sw}(\tilde{x},\tilde{t})$$

$$\psi_2^{ss}(x-x_0,t)=\sqrt{\frac{\gamma+1}{2}}\psi_2^{sw}(\tilde{x},\tilde{t})+\mathrm{sign}(v)\sqrt{\frac{\gamma-1}{2}}\psi_1^{sw}(\tilde{x},\tilde{t})$$

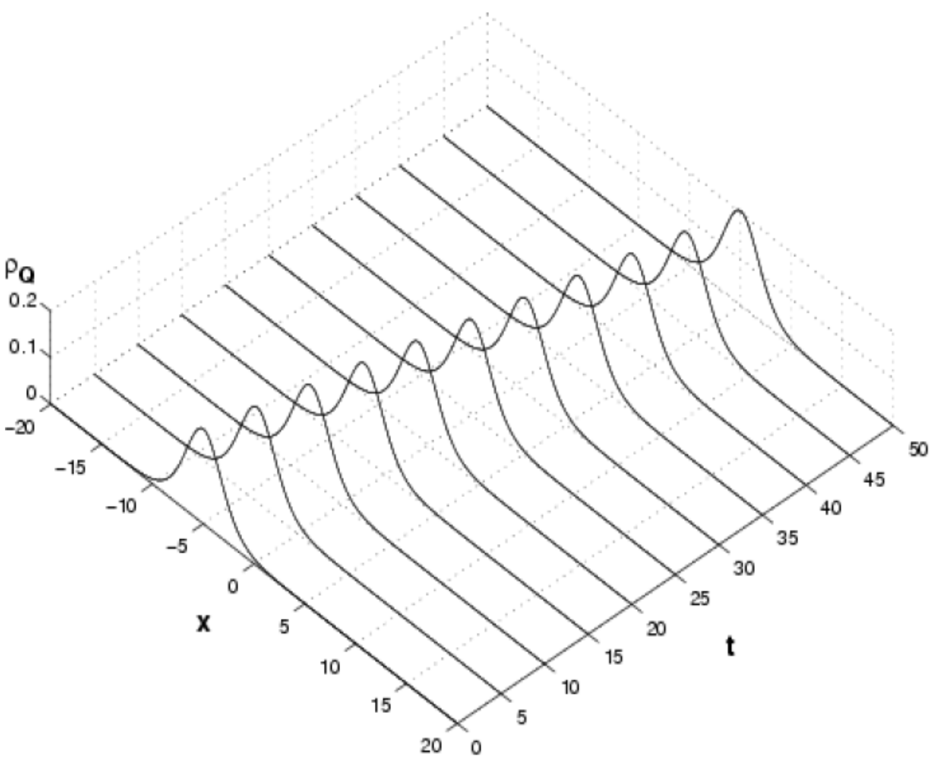
$$\gamma=1/\sqrt{1-v^2},\ \tilde{x}=\gamma(x-x_0-vt),\ \tilde{t}=\gamma(t-v(x-x_0))$$

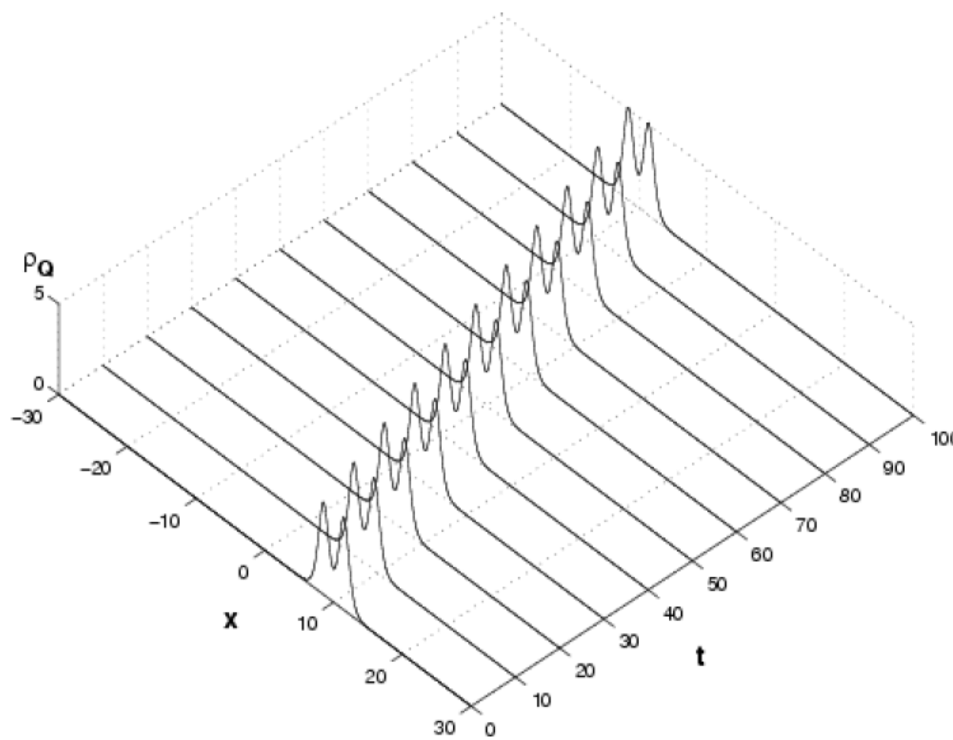
For 0<Λ<m/2, two-humped solition (with two peaks) in the charge density;
For m/2≤Λ<m, one-humped soliton;
For Λ=m, $\psi^{ss}(x-x_0,t)\equiv 0$

S.H. Shao & H.Z. Tang, Phys. Lett. A, 345(2005), 119.

1、Introduction

- Motion of Dirac solitary waves





S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

北京大學

1、Introduction

①A. Alvarez, B. Carreras, Phys. Lett. A 86 (1981) 327.
②A. Alvarez, P.Y. Kuo, L. Vázquez, Appl. Math. Comput. 13(1983) 1.
③A. Alvarez, JCP 99 (1992) 348.

Crank–Nicholson type schemes

①J. De Frutos, J.M. Sanz-serna, JCP 83 (1989) 407.

split-step spectral schemes

①Z.-Q. Wang, B.-Y. Guo, J. Comput. Math. 22 (2004) 457.

Legendre rational spectral methods

①J.L. Hong, C. Li, JCP 211(2006), 448–472.

Multi-symplectic Runge–Kutta methods

①S.H. Shao & H.Z. Tang, PLA 2005; DCDS-B 2006; CiCP 2008.
②H. Wang & H.Z. Tang, JCP 2007.
③J. Xu, S.H. Shao & H.Z. Tang, JCP 2013.

2、 Multi-hump solitary waves

The two-hump profile is first pointed out by Shao and Tang [Phys. Lett. A, 345(2005), 119] and later gotten noticed by other researchers *e.g.* [Phys. Rev. E **82**, 036604 (2010)].

Question: Is there the multi-hump profile in Dirac solitary wave?

J. Xu, S.H. Shao, H.Z. Tang, and D.Y. Wei, Multi-hump solitary waves of nonlinear Dirac equation, submitted, 2013.

2、 Multi-hump solitary waves

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\Psi + \frac{\partial L_\mathrm{I}}{\partial\bar{\Psi}} = 0,$$ the Euler-Lagrange equation $\partial_\mu\left(\partial L/\partial(\partial_\mu\bar{\Psi})\right) - \partial L/\partial\bar{\Psi} = 0$

Lagrangian *L* reads $L = L_\mathrm{D} + L_\mathrm{I}.$

Dirac Lagrangian $L_\mathrm{D} = \frac{\mathrm{i}}{2}\left(\bar{\Psi}\gamma^\mu\partial_\mu\Psi - (\partial_\mu\bar{\Psi})\gamma^\mu\Psi\right) - m\bar{\Psi}\Psi,$

General linear combined self-interaction

$$L_\mathrm{I} = s(L_\mathrm{S})^{k+1} + p(L_\mathrm{P})^{k+1} + v(L_\mathrm{V})^{\frac{1}{2}(k+1)},$$

$$L_\mathrm{S} = \bar{\Psi}\Psi = |\Psi_1|^2 - |\Psi_2|^2 \in \mathbb{R},$$

$$L_\mathrm{P} = -\mathrm{i}\bar{\Psi}\gamma^5\Psi = 2\mathrm{Im}(\Psi_1^*\Psi_2) \in \mathbb{R},$$

$$L_\mathrm{V} = \bar{\Psi}\gamma^\mu\Psi\bar{\Psi}\gamma_\mu\Psi,$$

$$L_\mathrm{A} = \bar{\Psi}\gamma^\mu\gamma^5\Psi\bar{\Psi}\gamma_\mu\gamma^5\Psi,$$

self-interaction Lagrangian L_I is a nonlinear functional of the spinors Ψ and $\bar{\Psi}$ and is invariant under the Lorentz transformation.

$$L_\mathrm{V} = -L_\mathrm{A}$$

It is also subject to conservation laws for the current vector and the energy-momentum tensor

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi,$$

$$\partial_\mu j^\mu = 0,$$

$$\partial_\mu T^{\mu\nu} = 0,$$

$$T^{\mu\nu} = \frac{\mathrm{i}}{2}\left(\bar{\Psi}\gamma^\mu\partial^\nu\Psi - (\partial^\nu\bar{\Psi})\gamma^\mu\Psi\right) - \eta^{\mu\nu}L.$$

北京大學

2、 Multi-hump solitary waves

Consider solitary wave solution in the form

$$\mathbf{\Psi}(x,t)=\mathrm{e}^{-\mathrm{i}\omega t}\psi(x),\quad \psi(x)=\begin{pmatrix}\varphi(x)\\ \chi(x)\end{pmatrix}=R(x)\begin{pmatrix}\cos(\theta(x))\\ \mathrm{i}\sin(\theta(x))\end{pmatrix}$$

$$L_{\mathrm{I}}=(R(x))^{2(k+1)}G(x)$$

$$G(x):=s(\cos(2\theta(x)))^{k+1}+p(\sin(2\theta(x)))^{k+1}+v$$

- When $m>\omega\geq 0$

$$\theta(x)=\tan^{-1}(\alpha\tanh(k\beta x))\in\left(-\tan^{-1}(\alpha),\tan^{-1}(\alpha)\right)\subseteq\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$$

$$\alpha=\sqrt{\frac{m-\omega}{m+\omega}},\quad \beta=\sqrt{m^2-\omega^2}.$$

- When $\omega=m>0$

$$\theta(x)=\cot^{-1}(2mkx)\in(0,\pi)$$

北京大學

2、 Multi-hump solitary waves

- When $\omega > m \geq 0$

$$\theta(x) = \tan^{-1}\left(\sqrt{\frac{\omega - m}{\omega + m}} \tan\left(-k\sqrt{\omega^2 - m^2}x\right)\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$R(x) = \pm\left(\frac{m\cos(2\theta(x)) - \omega}{G(x)}\right)^{\frac{1}{2k}},$$

The physical solutions are with which the total charge Q(t) is finite. Therefore, the physical solutions may exist only in the situation:

$$k \in \mathbb{Z}^+ \text{ and } m \geq \omega \geq 0$$

We may further analyze the hump number for the above solitary waves of NLD.

J. Xu, S.H. Shao, H.Z. Tang, and D.Y. Wei, Multi-hump solitary waves of nonlinear Dirac equation, submitted, 2013.

北京大學

2、 Multi-hump solitary waves

Lemma:

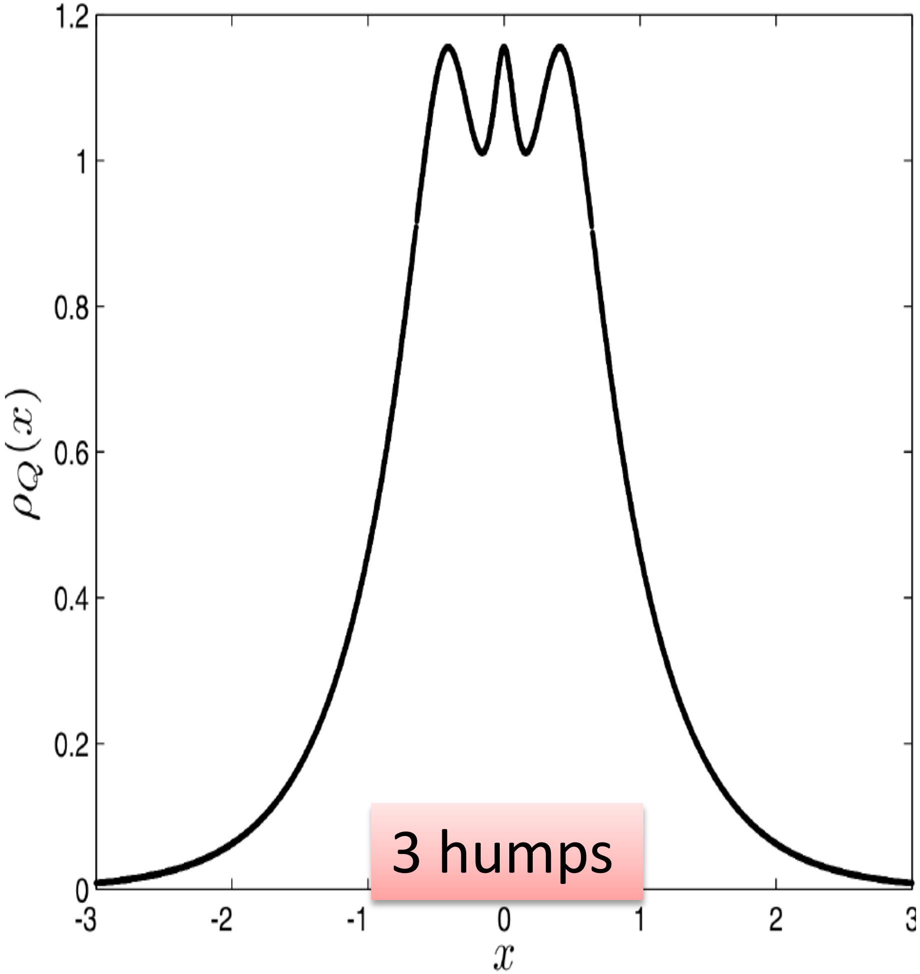
For a given integer k, the hump number in the charge density is not bigger than 4, while that in the energy density is not bigger than 3.

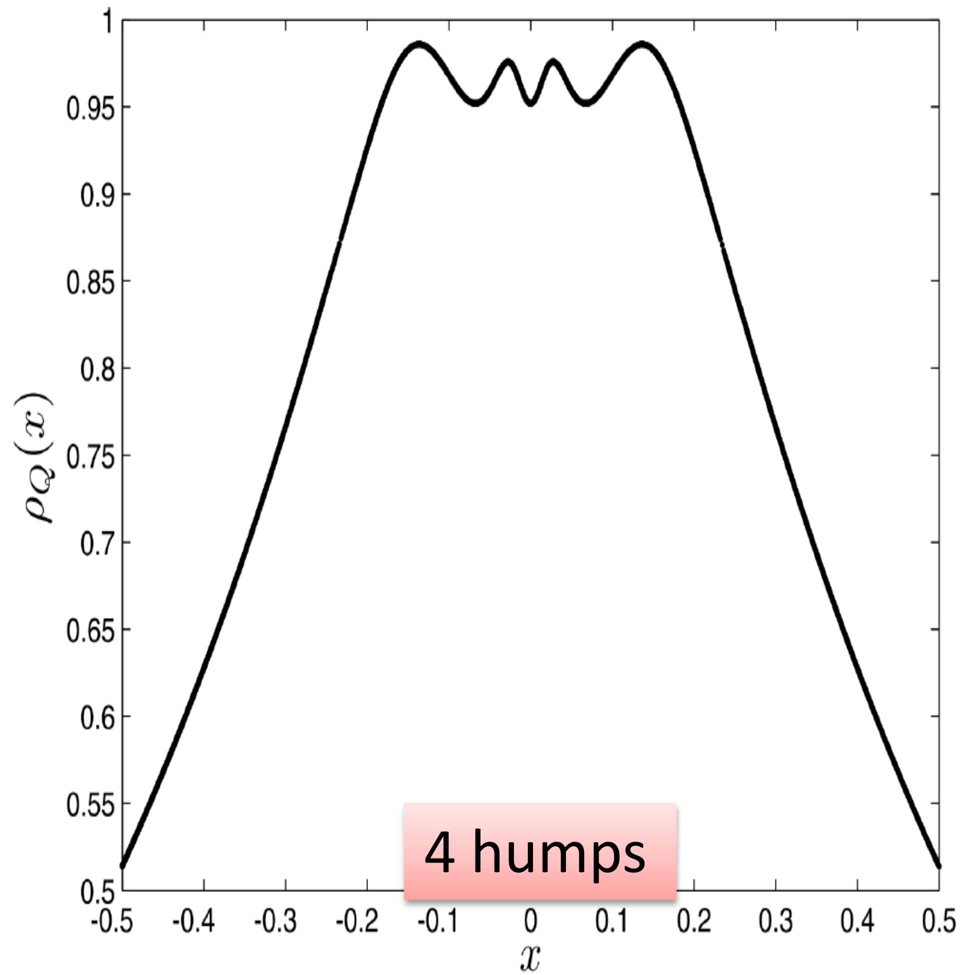
Remarks:

1. Those upper bounds can only be achieved in the situation of higher nonlinearity, namely, $k \in \{5; 6; 7; \cdots\}$ for the charge density and $k \in \{3; 5; 7; \cdots\}$ for the energy density;
2. The momentum density has the same multi-hump structure as the energy density;
3. More than two humps (resp. one hump) in the charge (resp. energy) density can only happen under the linear combination of the pseudoscalar self-interaction and at least one of the scalar and vector (or axial vector) self-interactions.

北京大學

2、 Multi-hump solitary waves





J. Xu, S.H. Shao, H.Z. Tang, and D.Y. Wei, Multi-hump solitary waves of nonlinear Dirac equation, submitted, 2013.

北京大學

3、RKDG method

① W.H. Reed and T. R. Hill, Los Alarnoo Scientific Laboratory, LA-UR-73-479, 1973.
② B. Cockburn and C.-W. Shu, *Math. Comp.*, 52(1989), 411-435.
③ B. Cockburn, S.-Y. Lin and C.-W. Shu, *JCP*, 84(1989), 90-113.
④ B. Cockburn, S. Hou and C.-W. Shu, *Math. Comp.*, 54(1990), 545-581.
⑤ B. Cockburn and C.-W. Shu, *JCP*, 141(1998), 199-224.
⑥ B. Cockburn and C.W. Shu, *J. Sci. Comput.*, 16(2001), 173-261.

The natural features of the RKDG methods are their formal high order accuracy, their nonlinear stability, their ability to capture the discontinuities or strong gradients of the exact solution without producing spurious oscillations, and their excellent parallel efficiency. Up to now, the DG methods have been successfully extended to various problems.

北京大學

3、RKDG method

- IVP of (1+1)-d Soler model

$$\partial_t \psi_1 + \partial_x \psi_2 + \mathrm{i} m \psi_1 + 2\mathrm{i}\lambda(|\psi_2|^2 - |\psi_1|^2)\psi_1 = 0$$

$$\partial_t \psi_2 + \partial_x \psi_1 - \mathrm{i} m \psi_2 + 2\mathrm{i}\lambda(|\psi_1|^2 - |\psi_2|^2)\psi_2 = 0$$

$$\psi_1(x,0) = \psi_1^0(x), \quad \psi_2(x,0) = \psi_2^0(x), \quad x \in \mathbb{R}$$

$$|\psi_i^0(x)| \to 0 \text{ as } |x| \to +\infty, \; i = 1, 2.$$

S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

3、RKDG method

Proposition 1 (Conservation laws). *Assume that* $\lim_{|x|\to+\infty}|\boldsymbol{\psi}(x,t)| = 0$ *and* $\lim_{|x|\to+\infty}|\partial_x\boldsymbol{\psi}(x,t)| < +\infty$ *hold uniformly for* $t \geq 0$. *The energy* E, *the linear momentum* P, *and the charge* Q *defined above satisfy:* $\frac{\mathrm{d}}{\mathrm{d}t}Q(t) = 0$, $\frac{\mathrm{d}}{\mathrm{d}t}P(t) = 0$, *and* $\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 0$.

$$
\begin{aligned}
E(t) = \int_{\mathbb{R}} \mathrm{d}x \, [\mathrm{Im}(\psi_1^*\partial_x\psi_2 + \psi_2^*\partial_x\psi_1) + m(|\psi_1|^2 - |\psi_2|^2) \\
- \lambda(|\psi_1|^2 - |\psi_2|^2)^2] =: \int_{\mathbb{R}} \mathrm{d}x \rho_E(x,t), \\
P(t) = \int_{\mathbb{R}} \mathrm{d}x \, [\mathrm{Im}(\psi_1^*\partial_x\psi_1 + \psi_2^*\partial_x\psi_2)] =: \int_{\mathbb{R}} \mathrm{d}x \rho_P(x,t), \\
Q(t) = \int_{\mathbb{R}} \mathrm{d}x (|\psi_1|^2 + |\psi_2|^2) =: \int_{\mathbb{R}} \mathrm{d}x \rho_Q(x,t),
\end{aligned}
$$

S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

北京大學

3、RKDG method

For any given partition of the domain

$$I_{j+\frac{1}{2}} = (x_j, x_{j+1}), h_{j+\frac{1}{2}} = x_{j+1} - x_j, x_{j+\frac{1}{2}} = (x_{j+1} + x_j)/2$$

For each t， find approximate solution

$$\psi_h = (\psi_{1,h}, \psi_{2,h})$$

where

$$\mathrm{Re}(\psi_{i,h}), \mathrm{Im}(\psi_{i,h}) \in \mathcal{V}_h^q = \left\{ \phi \middle| \ \phi(x) \in P^q(I_{j+\frac{1}{2}}) \text{ if } x \in I_{j+\frac{1}{2}}, \forall j \in \mathbb{Z} \right\}$$

$P^q(I_{j+\frac{1}{2}})$ denotes the space of the real-valued polynomials on $I_{j+\frac{1}{2}}$ of degree at most *q*.

S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

北京大學

3、RKDG method

$$\int_{I_{j+\frac{1}{2}}} \phi_1 \frac{\partial \psi_{1,h}}{\partial t}\,\mathrm{d}x + (\widehat{\psi}_2\phi_1^-)_{j+1} - (\widehat{\psi}_2\phi_1^+)_j - \int_{I_{j+\frac{1}{2}}} \psi_{2,h}\frac{\partial \phi_1}{\partial x}\,\mathrm{d}x$$

$$+ \int_{I_{j+\frac{1}{2}}} \mathrm{i}\left(m + 2\lambda f(|\psi_{1,h}|^2, |\psi_{2,h}|^2)\right)\psi_{1,h}\phi_1\mathrm{d}x = 0$$

$$\int_{I_{j+\frac{1}{2}}} \phi_2 \frac{\partial \psi_{2,h}}{\partial t}\,\mathrm{d}x + (\widehat{\psi}_1\phi_2^-)_{j+1} - (\widehat{\psi}_1\phi_2^+)_j - \int_{I_{j+\frac{1}{2}}} \psi_{1,h}\frac{\partial \phi_2}{\partial x}\,\mathrm{d}x$$

$$- \int_{I_{j+\frac{1}{2}}} \mathrm{i}\left(m - 2\lambda g(|\psi_{1,h}|^2, |\psi_{2,h}|^2)\right)\psi_{2,h}\phi_2\mathrm{d}x = 0,$$

$$(\phi_i^+)_j = \phi_i(x_j+0),\ (\phi_i^-)_{j+1} = \phi_i(x_{j+1}-0)$$

$$(\hat{\psi}_i)_j \approx \psi_{i,h}(x_j, t)$$

S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

3、RKDG method

- Numerical flux

$$(\widehat{\psi_i})_j := \widehat{h}_i\big(\boldsymbol{\psi}_h(x_j-0,t), \boldsymbol{\psi}_h(x_j+0,t)\big), \quad \widehat{h}_i(\boldsymbol{\psi}, \boldsymbol{\psi}) = \psi_i, \ \ i = 1,2$$

For example：

$$(\widehat{\psi_1})_j = \frac{1}{2}\big((\psi_1)_j^+ + (\psi_1)_j^- - (\psi_2)_j^+ + (\psi_2)_j^-\big)$$

$$(\widehat{\psi_2})_j = \frac{1}{2}\big((\psi_2)_j^+ + (\psi_2)_j^- - (\psi_1)_j^+ + (\psi_1)_j^-\big)$$

where $(\psi_i)_j^+ = \psi_{i,h}(x_j+0,t)$ and $(\psi_i)_j^- = \psi_{i,h}(x_j-0,t)$, $i = 1, 2.$

S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

3、RKDG method

Proposition 2. *If assume* $Q_h(0) = \int_{\mathbb{R}}(|\psi_1^0(x)|^2 + |\psi_2^0(x)|^2)\mathrm{d}x < \infty$, *then the solution to the weak formulation (26)-(29) satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_h(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(|\psi_{1,h}|^2 + |\psi_{2,h}|^2)\mathrm{d}x \leq 0,$$

or $Q_h(t) \leq Q_h(0)$ *for all* $t \geq 0$.

S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

3、RKDG method

- In practical computation, the solution may be written as

$$\psi_i(x,t) = \psi_i^r(x,t) + \mathrm{i}\psi_i^s(x,t), \quad i = 1, 2.$$

satisfying

$$\partial_t\psi_1^r + \partial_x\psi_2^r - m\psi_1^s - 2\lambda f(|\psi_1|^2, |\psi_2|^2)\psi_1^s = 0,$$
$$\partial_t\psi_1^s + \partial_x\psi_2^s + m\psi_1^r + 2\lambda f(|\psi_1|^2, |\psi_2|^2)\psi_1^r = 0,$$
$$\partial_t\psi_2^r + \partial_x\psi_1^r + m\psi_2^s - 2\lambda g(|\psi_1|^2, |\psi_2|^2)\psi_2^s = 0,$$
$$\partial_t\psi_2^s + \partial_x\psi_1^s - m\psi_2^r + 2\lambda g(|\psi_1|^2, |\psi_2|^2)\psi_2^r = 0,$$

S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

北京大學

3、RKDG method

$$\int_{I_{j+\frac{1}{2}}} \phi_1^r \partial_t \psi_1^r \,\mathrm{d}x + (\widehat{\psi_2^r}\phi_1^{r,-})_{j+1} - (\widehat{\psi_2^r}\phi_1^{r,+})_j - \int_{I_{j+\frac{1}{2}}} \psi_2^r \partial_x \phi_1^r \,\mathrm{d}x$$
$$- \int_{I_{j+\frac{1}{2}}} \left(m + 2\lambda f(|\psi_1|^2, |\psi_2|^2)\right) \psi_1^s \phi_1^r \,\mathrm{d}x = 0,$$
$$\int_{I_{j+\frac{1}{2}}} \phi_1^s \partial_t \psi_1^s \,\mathrm{d}x + (\widehat{\psi_2^s}\phi_1^{s,-})_{j+1} - (\widehat{\psi_2^s}\phi_1^{s,+})_j - \int_{I_{j+\frac{1}{2}}} \psi_2^s \partial_x \phi_1^s \,\mathrm{d}x$$
$$+ \int_{I_{j+\frac{1}{2}}} \left(m + 2\lambda f(|\psi_1|^2, |\psi_2|^2)\right) \psi_1^r \phi_1^s \,\mathrm{d}x = 0,$$
$$\int_{I_{j+\frac{1}{2}}} \phi_2^r \partial_t \psi_2^r \,\mathrm{d}x + (\widehat{\psi_1^r}\phi_2^{r,-})_{j+1} - (\widehat{\psi_1^r}\phi_2^{r,+})_j - \int_{I_{j+\frac{1}{2}}} \psi_1^r \partial_x \phi_2^r \,\mathrm{d}x$$
$$+ \int_{I_{j+\frac{1}{2}}} \left(m - 2\lambda g(|\psi_1|^2, |\psi_2|^2)\right) \psi_2^s \phi_2^r \,\mathrm{d}x = 0,$$
$$\int_{I_{j+\frac{1}{2}}} \phi_2^s \partial_t \psi_2^s \,\mathrm{d}x + (\widehat{\psi_1^s}\phi_2^{s,-})_{j+1} - (\widehat{\psi_1^s}\phi_2^{s,+})_j - \int_{I_{j+\frac{1}{2}}} \psi_1^s \partial_x \phi_2^s \,\mathrm{d}x$$
$$- \int_{I_{j+\frac{1}{2}}} \left(m - 2\lambda g(|\psi_1|^2, |\psi_2|^2)\right) \psi_2^r \phi_2^s \,\mathrm{d}x = 0.$$

S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

3、RKDG method

choose the Legendre polynomials $P_l(\xi)$ as local basis functions

$$\int_{-1}^{1} P_l(\xi)P_k(\xi)\,d\xi = \frac{2}{2l+1}\delta_{l,k}, \quad l \le k,$$

$$P_l(1) = 1 \qquad P_l(-1) = (-1)^l$$

$$\psi_i^z(x,t) = \sum_{l=0}^{q} \boxed{\psi_{i,j+\frac{1}{2}}^{z,(l)}(t)}\,\phi_{i,j+\frac{1}{2}}^{z,(l)}(x) \; = \psi_{i,j+\frac{1}{2}}^{z}(x,t), \quad \text{if } x \in I_{j+\frac{1}{2}}$$

where $i = 1, 2$, the superscript $z = r$ or s, and

$$\phi_{i,j+\frac{1}{2}}^{z,(l)}(x) = P_l(\xi_{j+\frac{1}{2}}), \quad \xi_{j+\frac{1}{2}} := \frac{2(x - x_{j+\frac{1}{2}})}{h_{j+\frac{1}{2}}}.$$

S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

北京大學

3、RKDG method

$$\left(\frac{h_{j+\frac{1}{2}}}{2l+1}\right)\frac{\mathrm{d}}{\mathrm{d}t}\psi^{r,(l)}_{1,j+\frac{1}{2}} + \widehat{\psi}^{r}_{2,j+1} - (-1)^{l}\widehat{\psi}^{r}_{2,j} - \int_{I_{j+\frac{1}{2}}} \psi^{r}_{2,j+\frac{1}{2}}\partial_x\phi^{r,(l)}_{1,j+\frac{1}{2}}\ \mathrm{d}x$$
$$-\int_{I_{j+\frac{1}{2}}} (m + 2\lambda f(|\psi_{1,j+\frac{1}{2}}|^2, |\psi_{2,j+\frac{1}{2}}|^2))\psi^{s}_{1,j+\frac{1}{2}}\phi^{r,(l)}_{1,j+\frac{1}{2}}\ \mathrm{d}x = 0,$$
$$\left(\frac{h_{j+\frac{1}{2}}}{2l+1}\right)\frac{\mathrm{d}}{\mathrm{d}t}\psi^{s,(l)}_{1,j+\frac{1}{2}} + \widehat{\psi}^{s}_{2,j+1} - (-1)^{l}\widehat{\psi}^{s}_{2,j} - \int_{I_{j+\frac{1}{2}}} \psi^{s}_{2,j+\frac{1}{2}}\partial_x\phi^{s,(l)}_{1,j+\frac{1}{2}}\ \mathrm{d}x$$
$$+\int_{I_{j+\frac{1}{2}}} (m + 2\lambda f(|\psi_{1,j+\frac{1}{2}}|^2, |\psi_{2,j+\frac{1}{2}}|^2))\psi^{r}_{1,j+\frac{1}{2}}\phi^{s,(l)}_{1,j+\frac{1}{2}}\ \mathrm{d}x = 0,$$
$$\left(\frac{h_{j+\frac{1}{2}}}{2l+1}\right)\frac{\mathrm{d}}{\mathrm{d}t}\psi^{r,(l)}_{2,j+\frac{1}{2}} + \widehat{\psi}^{r}_{1,j+1} - (-1)^{l}\widehat{\psi}^{r}_{1,j} - \int_{I_{j+\frac{1}{2}}} \psi^{r}_{1,j+\frac{1}{2}}\partial_x\phi^{r,(l)}_{2,j+\frac{1}{2}}\ \mathrm{d}x$$
$$+\int_{I_{j+\frac{1}{2}}} (m - 2\lambda g(|\psi_{1,j+\frac{1}{2}}|^2, |\psi_{2,j+\frac{1}{2}}|^2))\psi^{s}_{2,j+\frac{1}{2}}\phi^{r,(l)}_{2,j+\frac{1}{2}}\ \mathrm{d}x = 0,$$
$$\left(\frac{h_{j+\frac{1}{2}}}{2l+1}\right)\frac{\mathrm{d}}{\mathrm{d}t}\psi^{s,(l)}_{2,j+\frac{1}{2}} + \widehat{\psi}^{s}_{1,j+1} - (-1)^{l}\widehat{\psi}^{s}_{1,j} - \int_{I_{j+\frac{1}{2}}} \psi^{s}_{1,j+\frac{1}{2}}\partial_x\phi^{s,(l)}_{2,j+\frac{1}{2}}\ \mathrm{d}x$$
$$-\int_{I_{j+\frac{1}{2}}} (m - 2\lambda g(|\psi_{1,j+\frac{1}{2}}|^2, |\psi_{2,j+\frac{1}{2}}|^2))\psi^{r}_{2,j+\frac{1}{2}}\phi^{s,(l)}_{2,j+\frac{1}{2}}\ \mathrm{d}x = 0,$$

S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

北京大學

3、RKDG method

$$\psi^{z,(l)}_{i,j+\frac{1}{2}}(0) = \frac{2l+1}{h_{j+\frac{1}{2}}}\int_{I_{j+\frac{1}{2}}} \psi^z_i(x,0)\phi^{z,(l)}_{i,j+\frac{1}{2}}(x)\mathrm{d}x$$

integrals will be computed numerically, e.g. by using Gaussian quadrature.

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = L(U)$$

$$\begin{aligned} U^{(1)} &= U^n + \Delta t L(U^n), \\ U^{(2)} &= \frac{3}{4}U^n + \frac{1}{4}U^{(1)} + \frac{1}{4}\Delta t L(U^{(1)}), \\ U^{n+1} &= \frac{1}{3}U^n + \frac{2}{3}U^{(2)} + \frac{2}{3}\Delta t L(U^{(2)}), \end{aligned}$$

$$\frac{\Delta t}{\Delta x} \le \frac{\mu}{2q+1}$$

$$\begin{aligned} U^{(1)} &= U^n + \frac{1}{2}\Delta t L(U^n), \\ U^{(2)} &= U^n + \frac{1}{2}\Delta t L(U^{(1)}), \\ U^{(3)} &= U^n + \Delta t L(U^{(2)}), \\ U^{n+1} &= \frac{1}{3}\left(U^{(1)} + 2U^{(2)} + U^{(3)} - U^n + \frac{1}{2}\Delta t L(U^{(3)})\right) \end{aligned}$$

S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

3、RKDG method

	Real part of ψ_1^{ss}				Imaginary part of ψ_1^{ss}			
N	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
280	1.95e-02	-	9.68e-03	-	7.30e-03	-	4.08e-03	-
P^1 560	2.46e-03	2.99	1.27e-03	2.93	9.50e-04	2.94	5.76e-04	2.82
1120	3.08e-04	2.99	1.79e-04	2.88	1.26e-04	2.92	9.81e-05	2.69
2240	3.84e-05	3.00	2.94e-05	2.78	1.89e-05	2.86	2.13e-05	2.53
140	5.35e-04	-	5.13e-04	-	3.18e-04	-	2.51e-04	-
P^2 280	2.25e-05	4.57	4.57e-05	3.49	1.80e-05	4.14	2.36e-05	3.41
560	2.14e-06	3.98	5.21e-06	3.31	1.88e-06	3.70	3.35e-06	3.11
1120	2.59e-07	3.67	6.34e-07	3.22	2.32e-07	3.47	4.30e-07	3.06
70	2.10e-04	-	3.03e-04	-	1.50e-04	-	1.26e-04	-
P^3 140	6.73e-06	4.96	1.44e-05	4.40	4.99e-06	4.91	1.14e-05	3.47
280	4.13e-07	4.50	9.36e-07	4.17	3.05e-07	4.47	7.27e-07	3.72
560	2.77e-08	4.30	5.93e-08	4.11	2.01e-08	4.29	4.70e-08	3.80
	Real part of ψ_2^{ss}				Imaginary part of ψ_2^{ss}			
N	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
280	4.06e-03	-	3.45e-03	-	3.40e-03	-	1.75e-03	-
P^1 560	5.35e-04	2.92	4.98e-04	2.79	4.67e-04	2.86	2.89e-04	2.60
1120	7.48e-05	2.88	8.66e-05	2.66	7.50e-05	2.75	7.26e-05	2.30
2240	1.25e-05	2.78	1.83e-05	2.52	1.50e-05	2.61	1.95e-05	2.16
140	1.96e-04	-	2.45e-04	-	1.57e-04	-	2.50e-04	-
P^2 280	1.61e-05	3.61	3.48e-05	2.82	1.69e-05	3.22	2.91e-05	3.10
560	1.91e-06	3.34	4.75e-06	2.84	2.10e-06	3.11	3.61e-06	3.06
1120	2.38e-07	3.23	6.06e-07	2.89	2.63e-07	3.07	4.50e-07	3.04
70	1.18e-04	-	1.73e-04	-	8.54e-05	-	1.75e-04	-
P^3 140	6.10e-06	4.27	1.29e-05	3.75	5.18e-06	4.04	1.19e-05	3.88
280	3.81e-07	4.14	8.84e-07	3.81	3.26e-07	4.02	7.45e-07	3.94
560	2.39e-08	4.09	5.62e-08	3.86	2.10e-08	4.00	4.78e-08	3.95

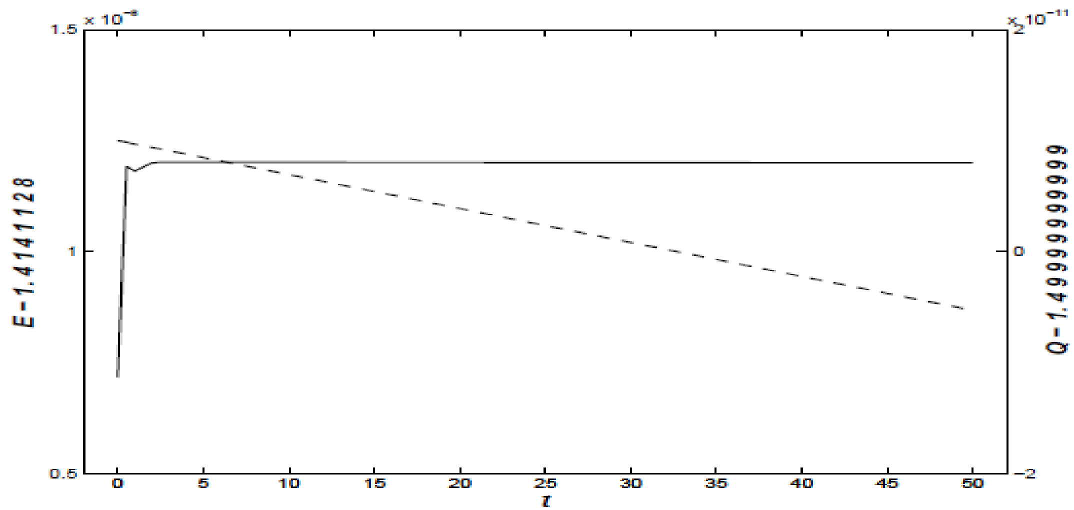
S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

$t = 50$

3、RKDG method

	P^1		P^2		P^3	
N	ΔQ_{50}	ΔE_{50}	ΔQ_{50}	ΔE_{50}	ΔQ_{50}	ΔE_{50}
140	-	-	-4.86e-05	-6.45e-05	-1.35e-07	6.84e-07
280	-1.99e-03	-2.42e-03	-1.39e-06	-2.35e-06	-1.08e-09	5.32e-08
560	-2.52e-04	-3.64e-04	-4.90e-08	-1.05e-07	-1.02e-11	3.42e-09
1120	-3.15e-05	-5.98e-05	-1.54e-09	-4.95e-09	-	-

where $\Delta Q_{50} = (Q_h(50) - Q_h(0))/Q_h(0) \quad \Delta E_{50} = (E_h(50) - E_h(0))/E_h(0)$

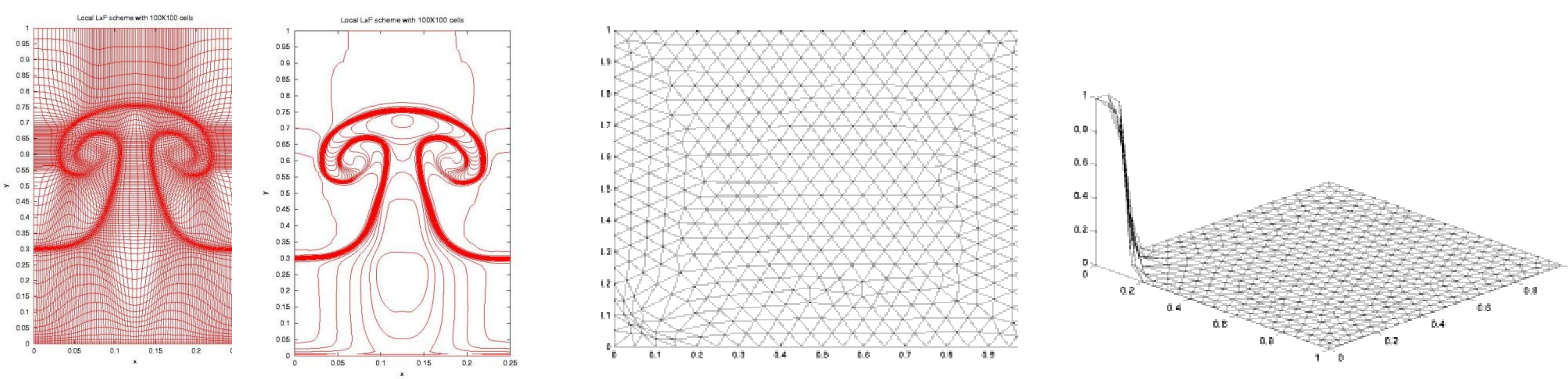


S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

$t = 50$

4、Moving mesh method

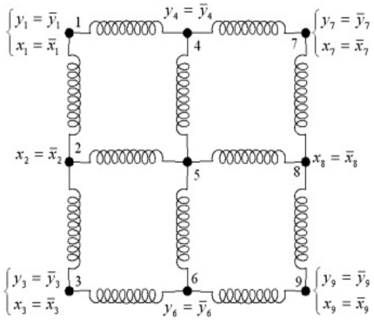
- It is also known as r-method (redistribution), and relocates mesh / gird points having a fixed number of nodes in such a way that the nodes remain concentrated in regions of rapid variation of the solution.



http://www.math.hkbu.edu.hk/~ttang/MMmovie/harmonic/Shock.html

4、Moving mesh method

- Equi-distribution principle [C. de Boor, In Lecture Notes in Mathematics, vol.363, Springer-Verlag, 1973]: an appropriately chosen mesh should equally distribute some measure of the solution variation or computational error over the entire domain.
- Spring analogy scheme[J.T. Batina, AIAA 89-0115]: each edge of the mesh is represented by a spring whose stiffness is proportional to the reciprocal of the length of the edge.
- Grid generation based on the variational method[A. Winslow, JCP, 1973].
- ……
- Lagrange method in CFD.



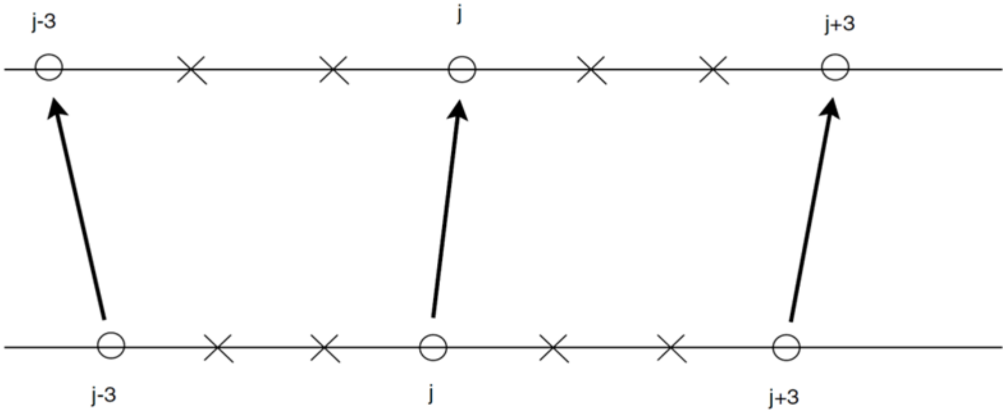
C.J. Budd, W. Huang & R.D. Russell, Acta Numerica 18 (2009), 111-241.
W. Huang and R. D. Russell, Adaptive Moving Mesh Methods, Springer, 2011.
T. Tang & J.C. Xu, Adaptive Computations: Theory and Algorithms, Science Pub., 2007.

北京大學

4、Moving mesh method

Two decoupled steps：

- Redistribute the mesh points iteratively
  - Solve coarse mesh equation an iterative step
  - Divide the coarse mesh cell into several uniform fine cells
  - Remap solution from the “old” fine mesh to the “new”
- Solve NLD eq. on a fixed fine mesh



H. Wang, H.Z. Tang, JCP, 222(2007)

H.Z. Tang & T. Tang, SINUM, 41(2003)

4、Moving mesh method

- (1+1)-d Dirac eq.

$$\frac{\partial \boldsymbol{u}}{\partial t}+\frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x}=\boldsymbol{s}(\boldsymbol{u})$$

(quasi-linear) balance law

$$\boldsymbol{f}(\boldsymbol{u})=\boldsymbol{A}\boldsymbol{u}\equiv\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}\psi_1^r\\\psi_1^s\\\psi_2^r\\\psi_2^s\end{pmatrix}$$

$$\boldsymbol{u}=(\psi_1^r,\psi_1^s,\psi_2^r,\psi_2^s)^T,\boldsymbol{s}(\boldsymbol{u})=g(x,t)(\psi_1^s,-\psi_1^r,-\psi_2^s,\psi_2^r)^T$$

$$\psi_i(x,t)=\psi_i^r(x,t)+\mathrm{i}\psi_i^s(x,t),\quad i=1,2$$

H. Wang, H.Z. Tang, JCP, 222(2007)

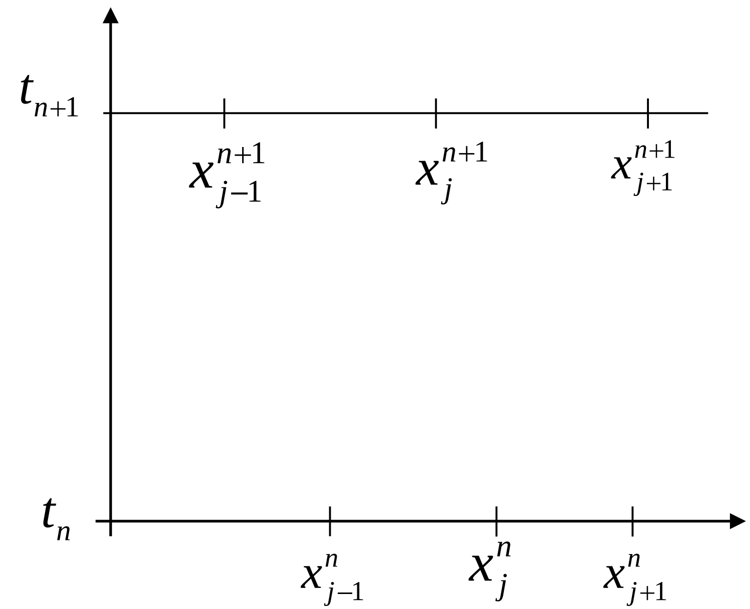
4、Moving mesh method

- 1D (quasi-linear) balance law

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=s(u)$$

- "Initial" mesh & data

$$\begin{cases} t_n,\quad x_j^n \\ u_{j+1/2}^n \approx \dfrac{1}{h_{j+1/2}^n}\displaystyle\int_{x_j^n}^{x_{j+1}^n} u(x,t_n)\,dx \end{cases}$$



H.Z. Tang & T. Tang, SINUM, 41(2003)

4、Moving mesh method

- **Question:** How to get $\{x_j^{n+1}, u_{j+1/2}^{n+1}\}$?
- Redistribute the mesh points iteratively

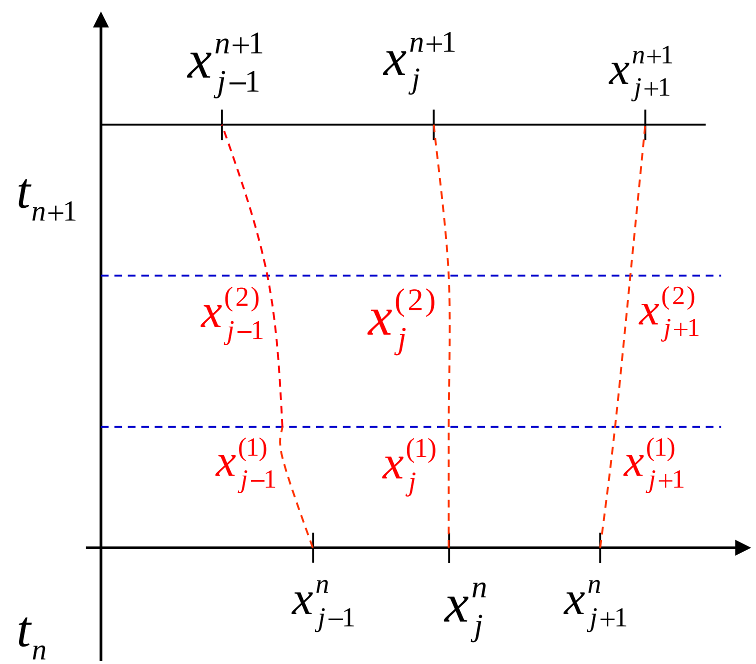
$$\frac{\partial}{\partial \xi}\left(w\frac{\partial x}{\partial \xi}\right)=0$$

$$\boxed{w_{j+1/2}^{(\nu)}\left(x_{j+1}^{(\nu)}-x_j^{(\nu+1)}\right)-w_{j+1/2}^{(\nu)}\left(x_j^{(\nu+1)}-x_{j-1}^{(\nu)}\right)=0}$$

$$\nu=0,1,2,\dots,\mu$$

$$x_j^{(0)}:=x_j^n,\ u_{j+1/2}^{(0)}:=u_{j+1/2}^n;$$

$$x_j^{n+1}:=x_j^{(\mu)},\ u_{j+1/2}^n:=u_{j+1/2}^{(\mu)}$$



H.Z. Tang & T. Tang, SINUM, 41(2003)

4、 Moving mesh method

- Remap the solution, that is, get $u_{j+1/2}^{(\nu+1)}$

$$x_j^{(\nu+1)} = x_j^{(\nu)} - c_j^{(\nu)}, x^{(\nu+1)}(\xi) = x^{(\nu)}(\xi) - c^{(\nu)}(\xi)$$

$$\begin{aligned}\int_{\tilde{x}_j}^{\tilde{x}_{j+1}} \tilde{u}(\tilde{x})\, d\tilde{x} &= \int_{x_j}^{x_{j+1}} u(x - c(x))(1 - c'(x))\, dx \\ &\approx \int_{x_j}^{x_{j+1}} (u(x) - c(x)u_x(x))(1 - c'(x))\, dx \\ &\approx \int_{x_j}^{x_{j+1}} (u(x) - (cu)_x)\, dx \\ &= \int_{x_j}^{x_{j+1}} u(x)\, dx - ((cu)_{j+1} - (cu)_j),\end{aligned}$$

$$\Rightarrow \quad \boxed{\begin{aligned} h_{j+1/2}^{(\nu+1)} u_{j+1/2}^{(\nu+1)} &= h_{j+1/2}^{(\nu)} u_{j+1/2}^{(\nu)} \\ &\quad - (cu)_{j+1}^{(\nu)} + (cu)_j^{(\nu)} \end{aligned}}$$

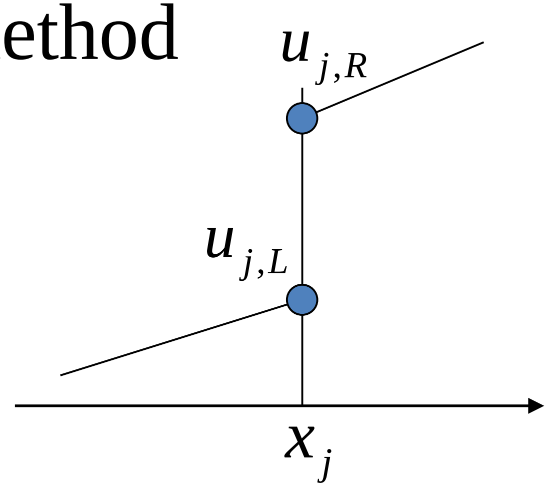
Conservative remap

$$\boxed{\{x_j^{(\nu)}, u_{j+1/2}^{(\nu)}\} \Rightarrow \{x_j^{(\nu+1)}, u_{j+1/2}^{(\nu)}\} \Rightarrow \{x_j^{(\nu+1)}, u_{j+1/2}^{(\nu+1)}\}}$$

H.Z. Tang & T. Tang, SINUM, 41(2003)

4、 Moving mesh method

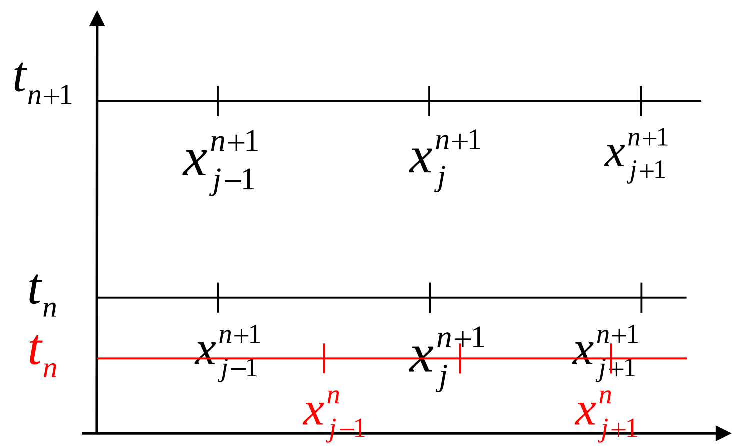
- Solve Dirac eq. by finite volume method



$$h_{j+1/2}^{n+1}u_{j+1/2}^{n+1} = h_{j+1/2}^{n+1}u_{j+1/2}^{n} - \tau_n(\hat{f}_{j+1}^{n} - \hat{f}_{j}^{n})$$

$$+\tau_n h_{j+1/2}^{n+1} s_{j+1/2}^{n}$$

$$\hat{f}_j^n = \hat{f}(u_{j,L}^n, u_{j,R}^n),\ \hat{f}(u,u) = f(u)$$



$$x_j^{n+1} := x_j^{(\mu)}, u_{j+1/2}^{n} := u_{j+1/2}^{(\mu+1)} \Rightarrow u_{j+1/2}^{n+1}$$

H. Wang, H.Z. Tang, JCP, 222(2007)

H.Z. Tang & T. Tang, SINUM, 41(2003)

北京大學

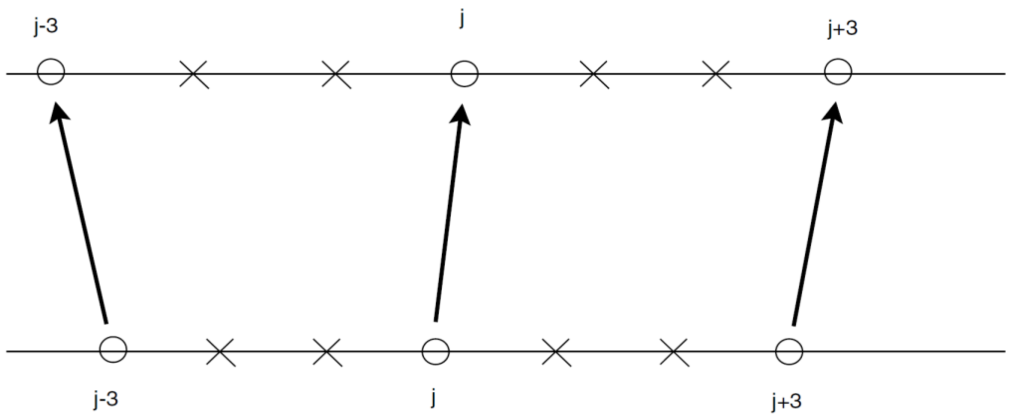
4、Moving mesh method

- Move the coarse mesh points “o” by solving iteratively the mesh equation：

$$\omega\left(x^{[\nu]}_{j+\frac{1}{2}}\right)\left(x^{[\nu]}_{j+1}-x^{[\nu+1]}_{j}\right)-\omega\left(x^{[\nu]}_{j-\frac{1}{2}}\right)\left(x^{[\nu+1]}_{j}-x^{[\nu+1]}_{j-1}\right)=0 \qquad \nu=0,1,\cdots,\mu-1$$

$$\omega=\sqrt{1+\alpha|\boldsymbol{u}|^2+\beta|\boldsymbol{u}_x|^2}$$

- Move fine mesh points “X” by uniformly dividing each coarse mesh cell.



H. Wang, H.Z. Tang, JCP, 222(2007)

北京大學

4、Moving mesh method

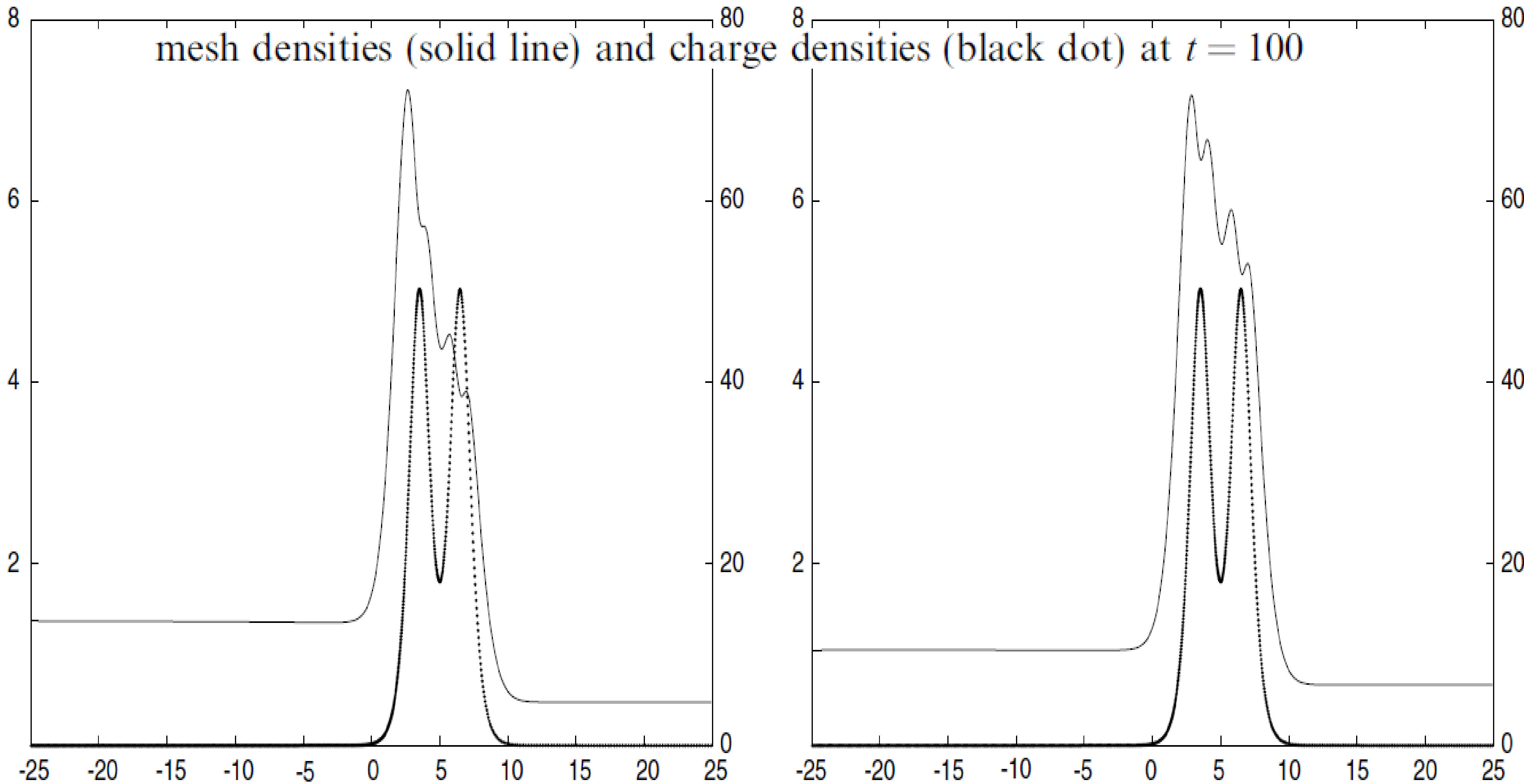
Accuracy test:

N	100	200	400	800	1600	3200
l^1-error order	3.782	4.725e−1	5.848e−2	7.331e−3	9.293e−4	1.184e−04
	–	3.00	3.01	3.00	2.98	2.97
l^2-error order	2.989	3.715e−1	4.504e−2	5.463e−3	6.597e−4	7.775e−05
	–	3.01	3.04	3.04	3.05	3.08
l^∞-error order	1.003	1.156e−1	1.435e−2	1.775e−3	2.177e−4	2.858e−05
	–	3.12	3.01	3.02	3.03	2.93
CPU time (s)	1.04	3.43	12.20	48.89	183.70	707.75

H. Wang, H.Z. Tang, JCP, 222(2007)

北京大學

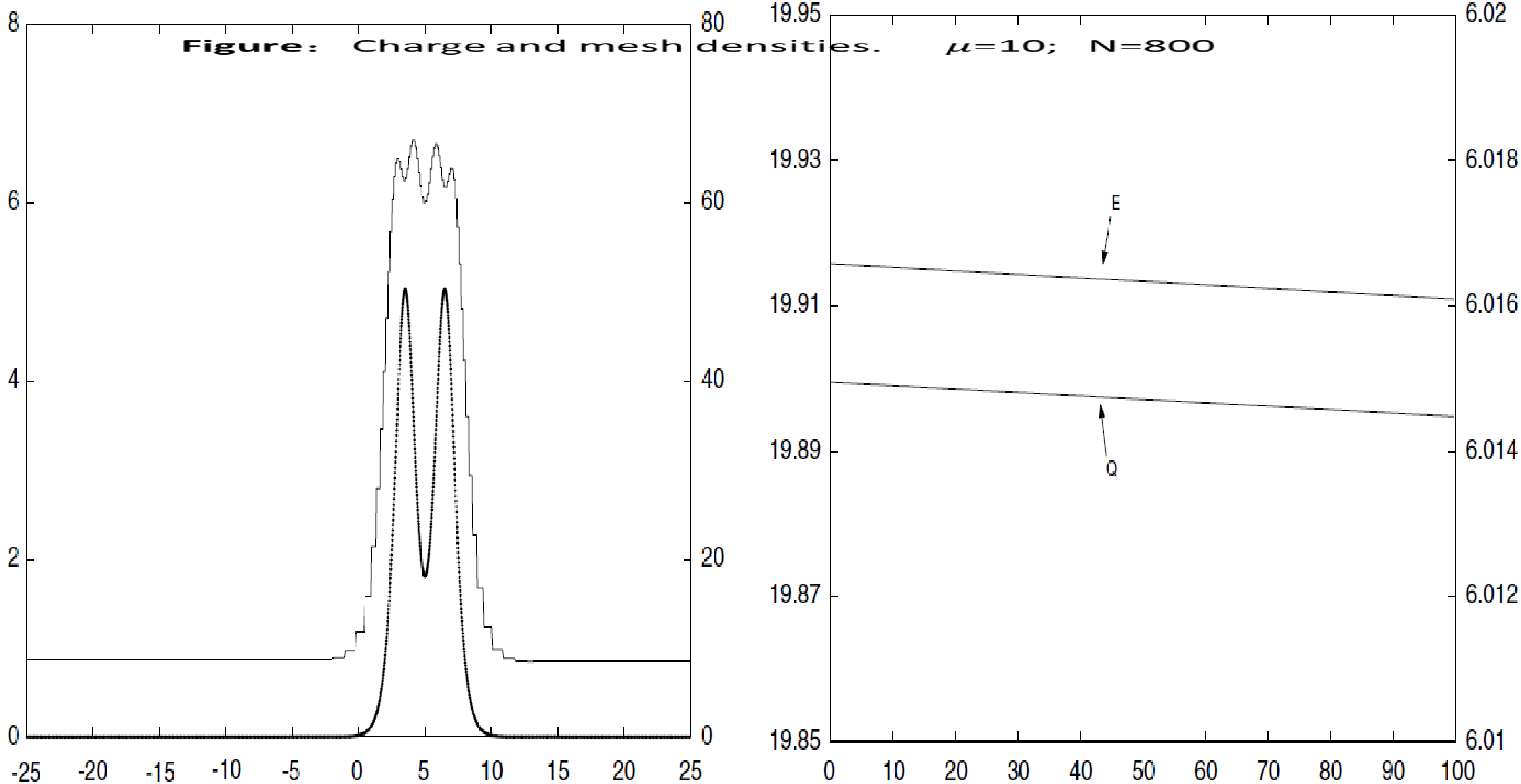
4、Moving mesh method



Figure： Charge and mesh densities. μ=10, 30; N=800

H. Wang, H.Z. Tang, JCP, 222(2007)

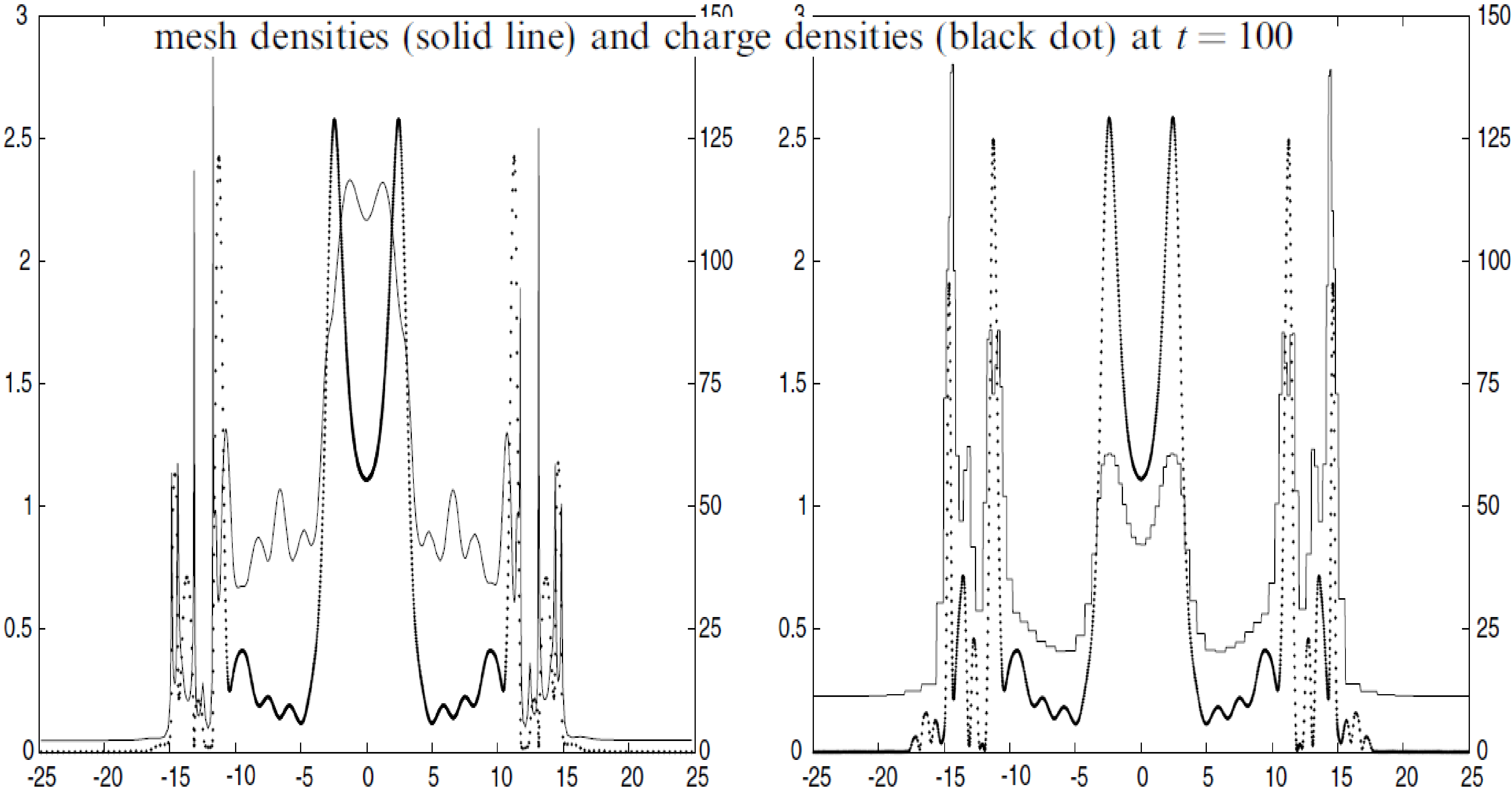
4、Moving mesh method



Figure： Charge and mesh densities. μ=10; N=800

H. Wang, H.Z. Tang, JCP, 222(2007)

4、Moving mesh method



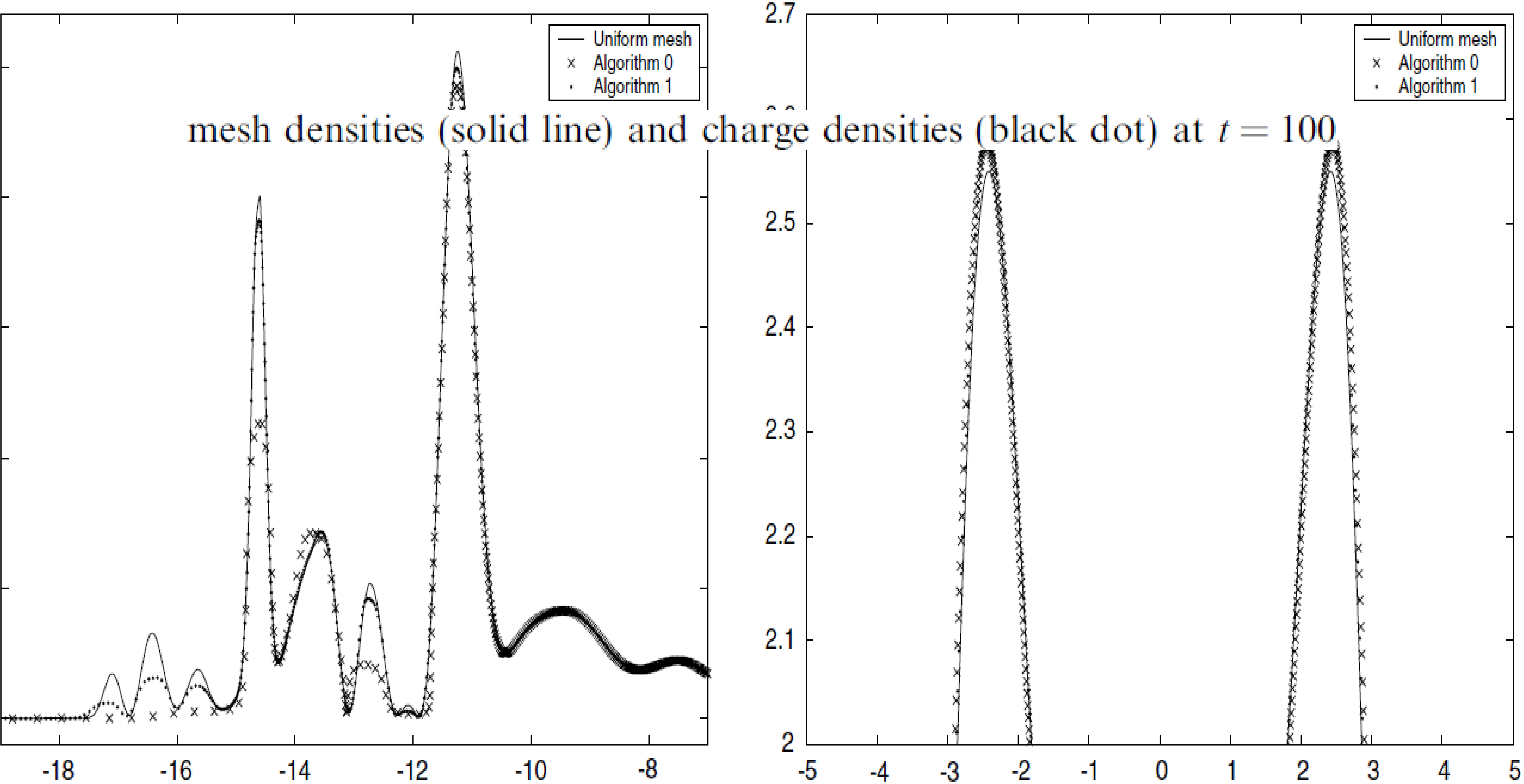
Figure： Charge and mesh densities. N=1600

H. Wang, H.Z. Tang, JCP, 222(2007)

北京大學

4、Moving mesh method

mesh densities (solid line) and charge densities (black dot) at $t = 100$



Figure： Close-up of the charge densities. N=1600 vs 10000

H. Wang, H.Z. Tang, JCP, 222(2007)

4、Moving mesh method

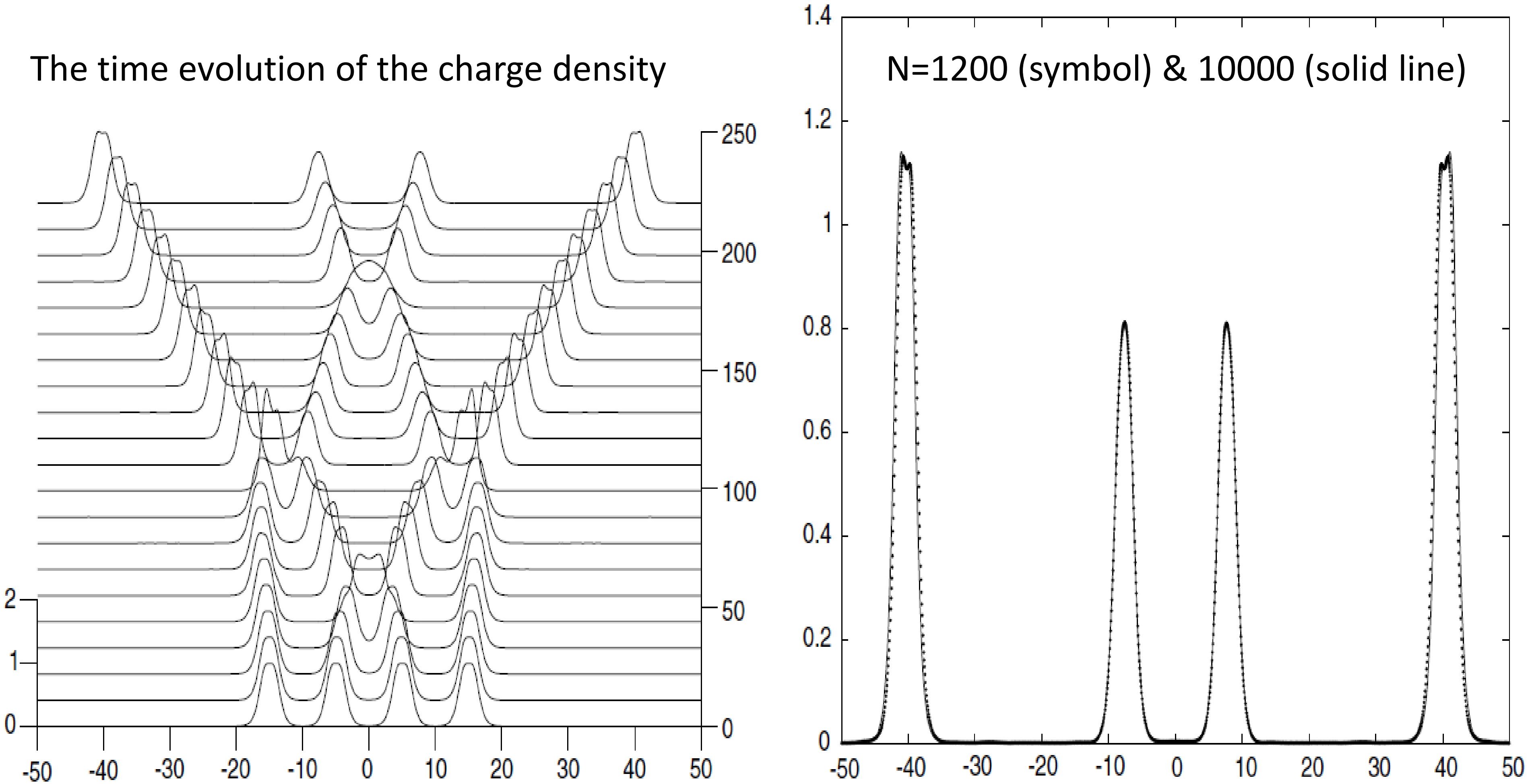


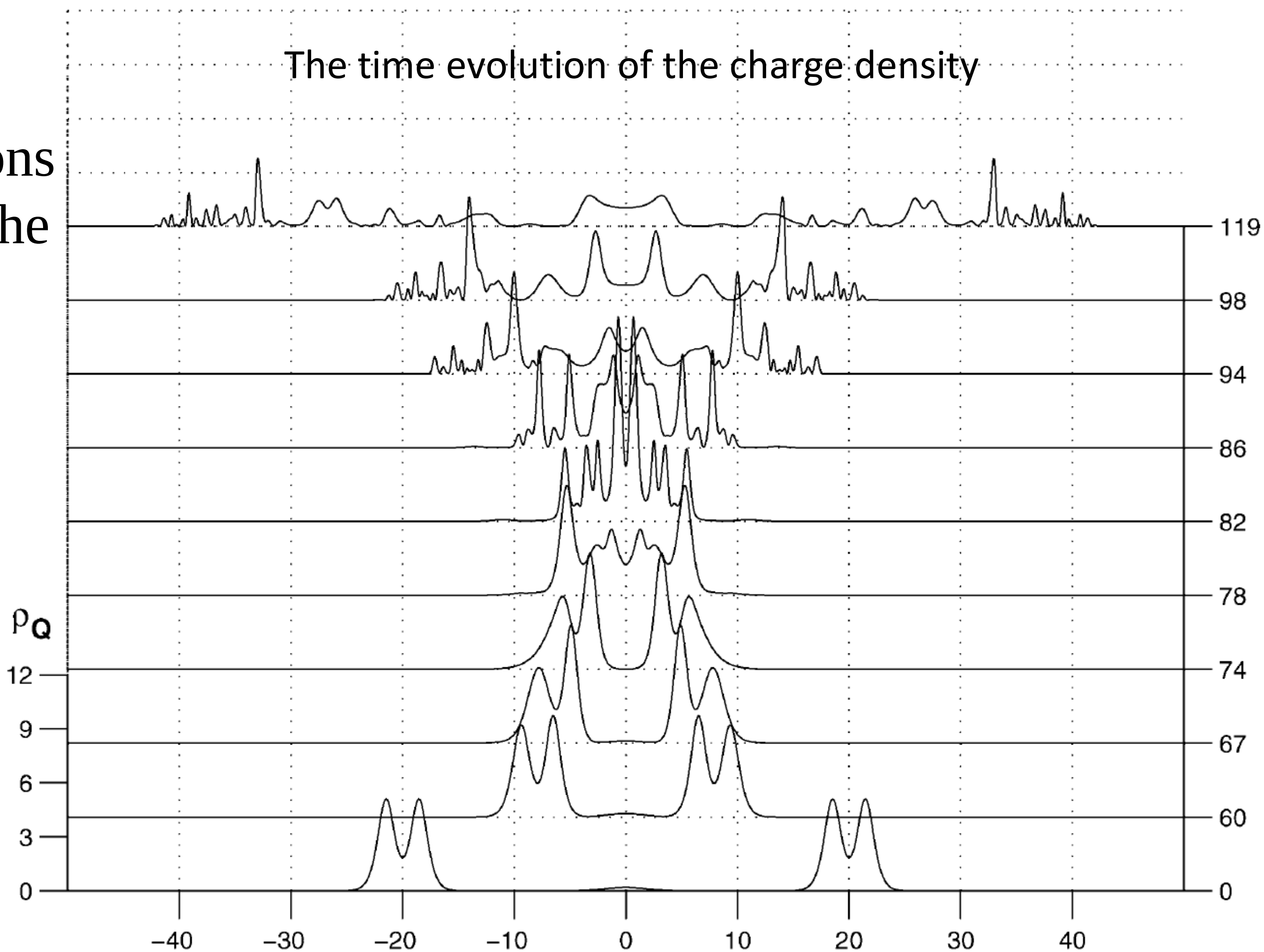
Figure: Charge densities. μ=10.

H. Wang, H.Z. Tang, JCP, 222(2007)

5、 Numerical experiments

Test 1: left & right two-humped solitons interact with the one-humped

Collapsing phenomenon



S.H. Shao & H.Z. Tang, Phys. Lett. A, 345(2005), 119.

$\Lambda_l = \Lambda_r = 0.1; \Lambda_m = 0.9$

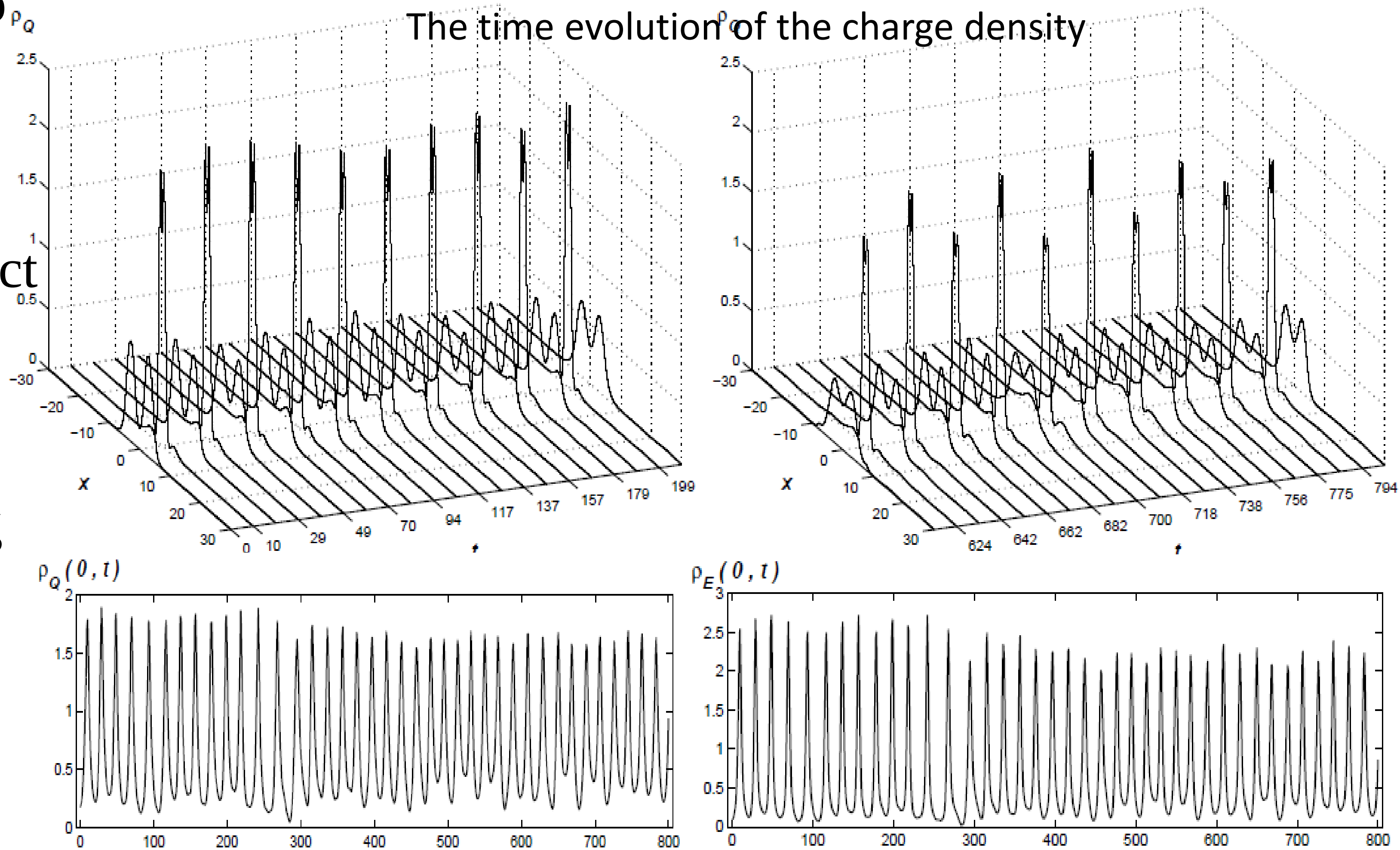
5、 Numerical experiments

Test 2: two one-humped solitons at rest interact each other

↓

oscillating state with a long lifetime

The time evolution of the charge density



Charge and enegy densities at *x* = 0 as a function of time.

S.H. Shao & H.Z. Tang, DCDS. B, 6(2006), 623.

$$\Lambda_l = \Lambda_r = 0.6$$

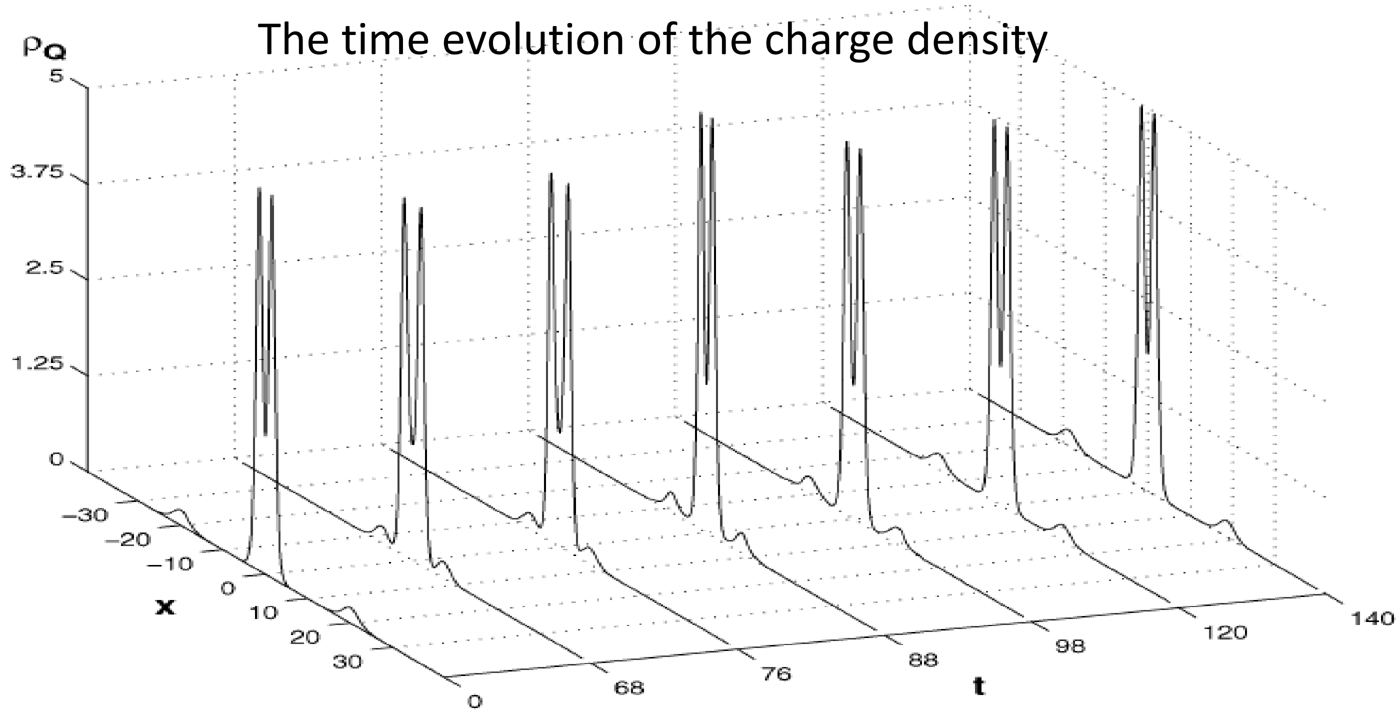
5、 Numerical experiments

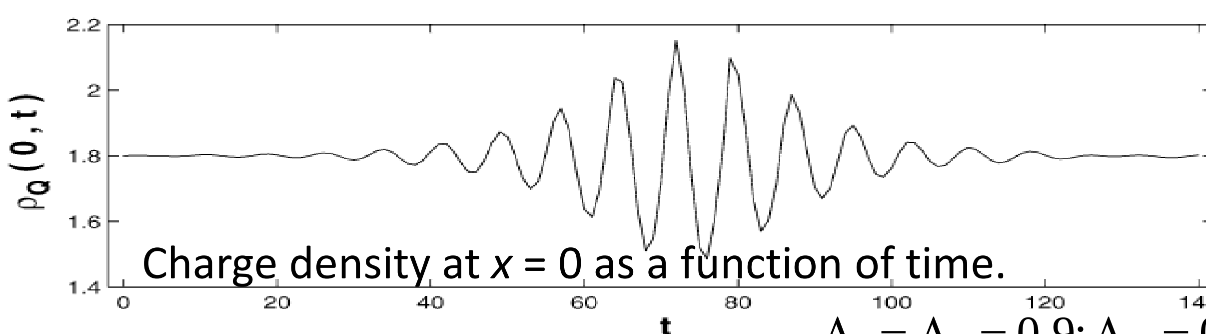
Test 3: left & right one-humped solitons interact with the two-humped

↓

A short-lived bound state in the ternary collisions

The time evolution of the charge density





Charge density at x = 0 as a function of time.

$\Lambda_l = \Lambda_r = 0.9; \Lambda_m = 0.1$

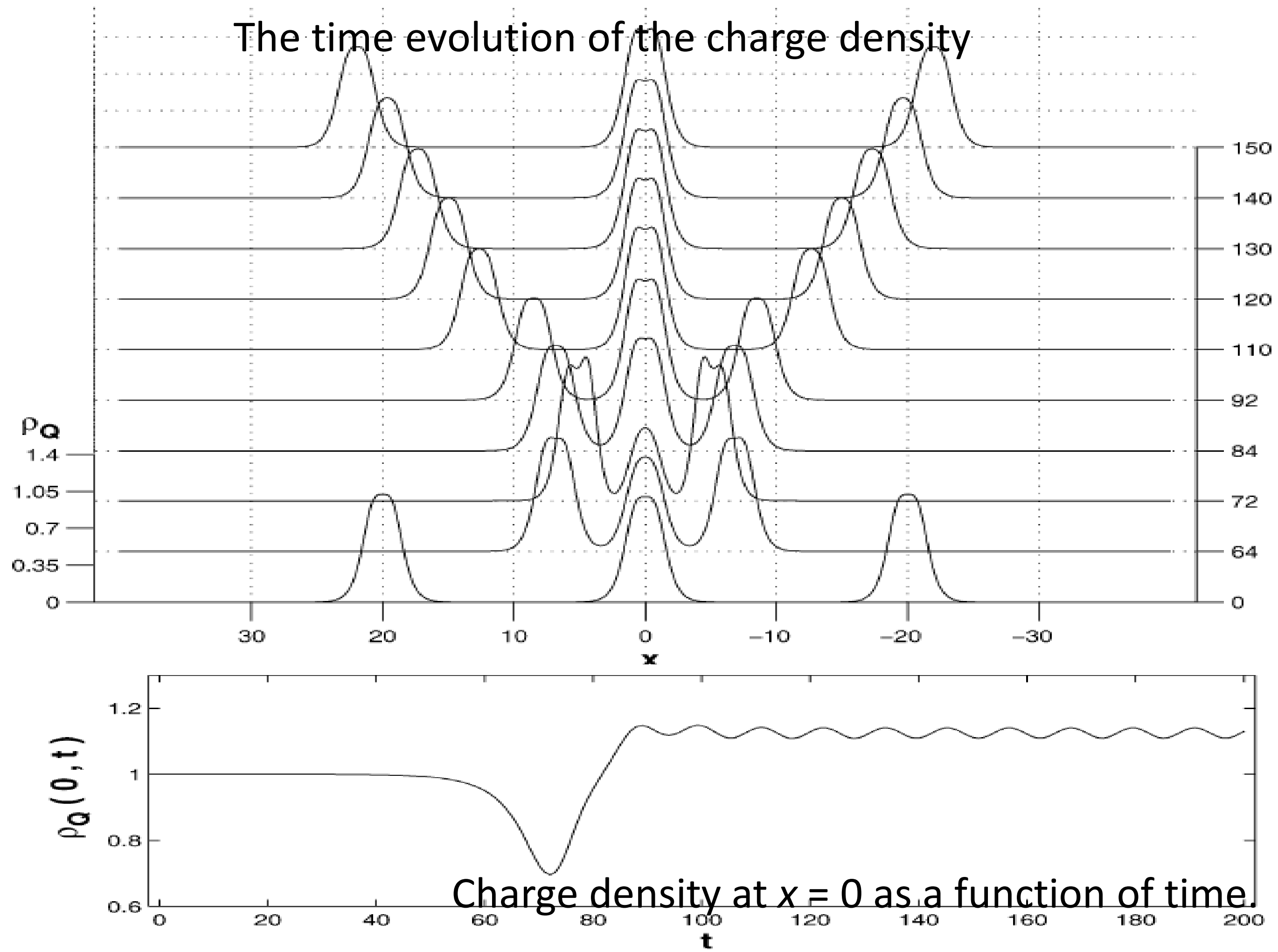
S.H. Shao & H.Z. Tang, Phys. Lett. A, 345(2005), 119.

5、 Numerical experiments

Test 4: Three one-humped solitons interact each other

↓

A long-lived bound state in the ternary collisions



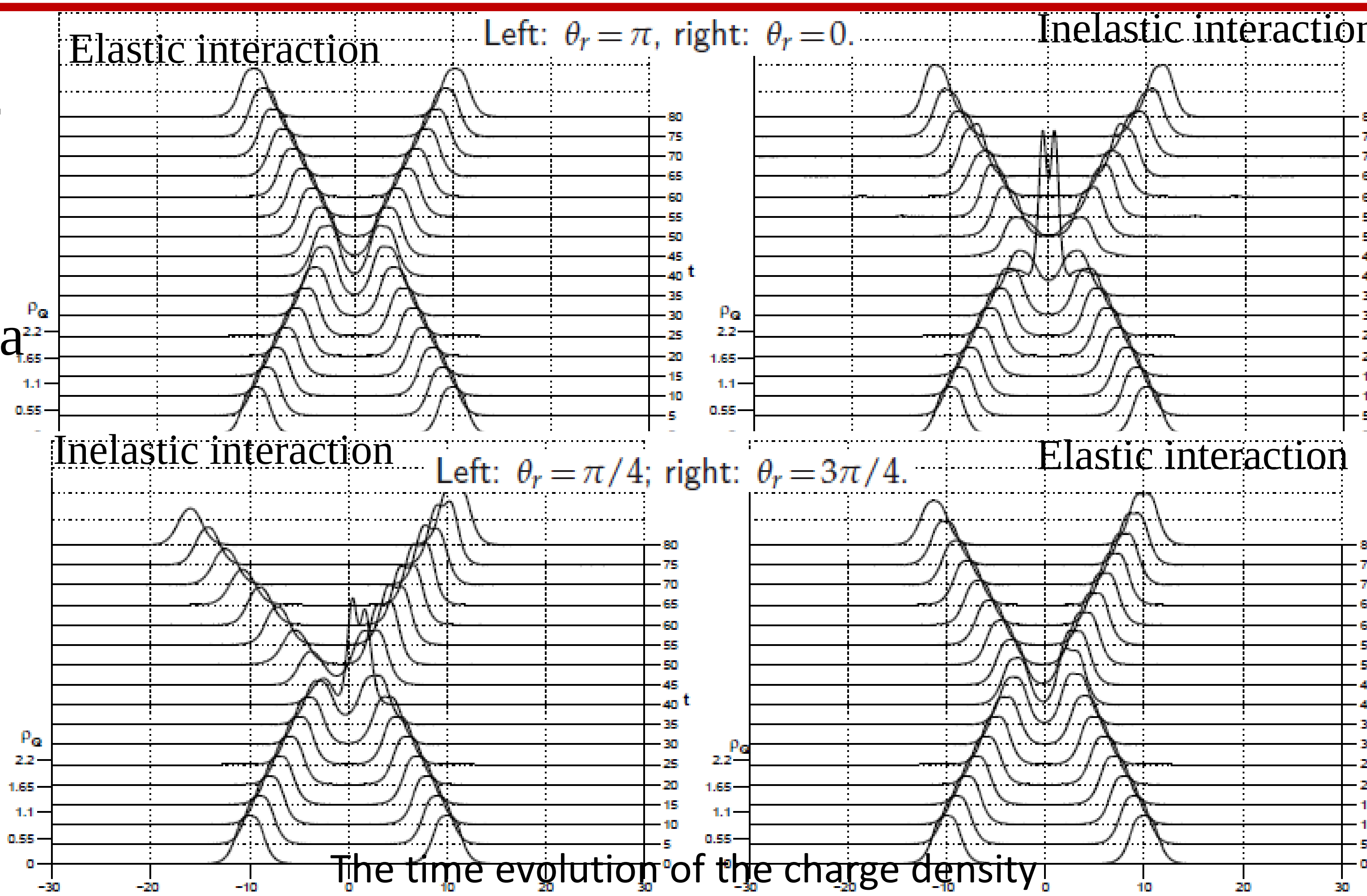
S.H. Shao & H.Z. Tang, Phys. Lett. A, 345(2005), 119.

$$\Lambda_l = \Lambda_m = \Lambda_r = 0.5$$

5、 Numerical experiments

Test 5:

Interaction of two one-humped solitons with a phase shift



The time evolution of the charge density

$$\Lambda_l=\Lambda_r=0.5,\ \ v_l=-v_r=0.2,\ \ \theta_l=0,$$

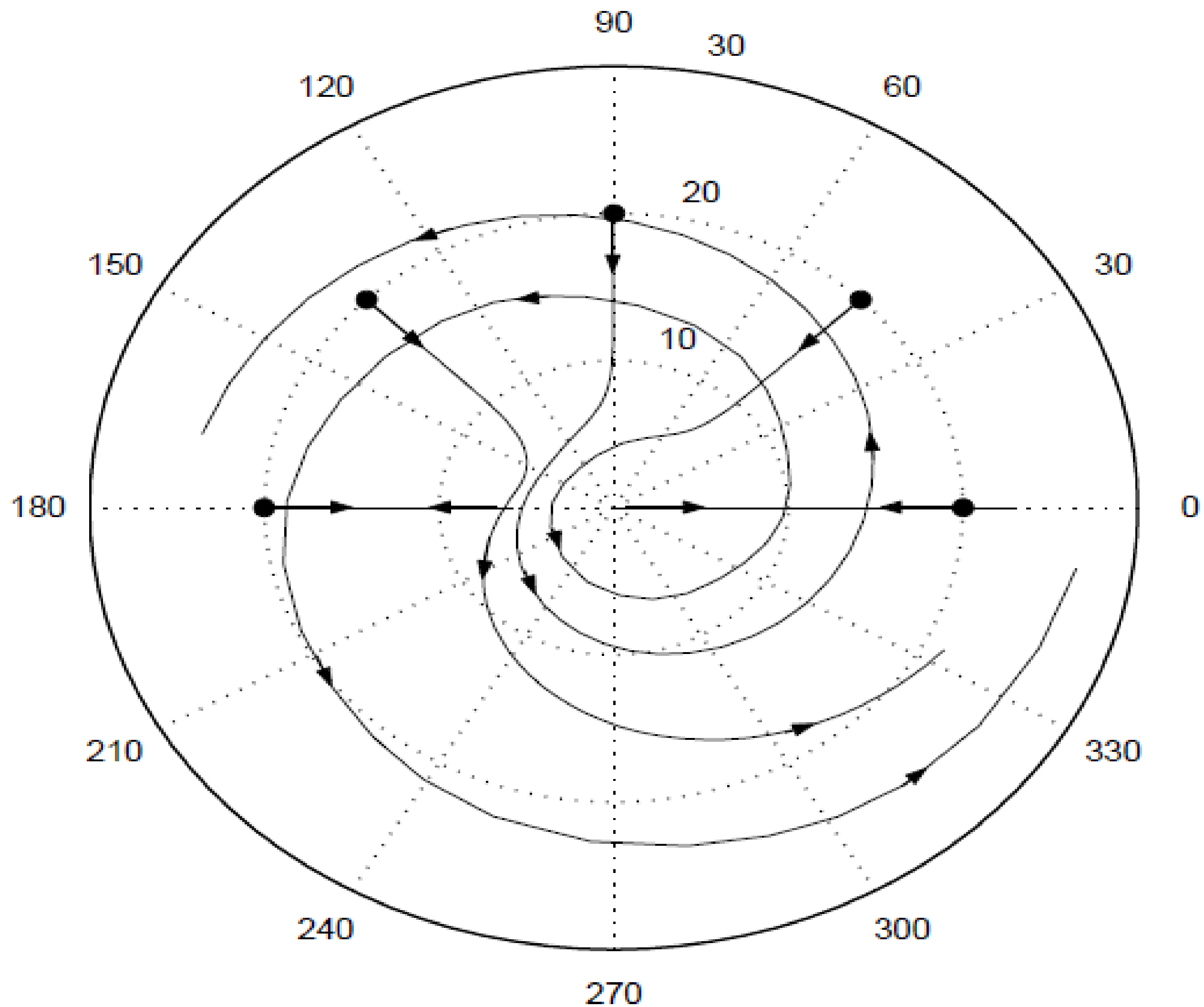
S.H. Shao & H.Z. Tang, CiCP., 3 (2008), 950.

5、 Numerical experiments

Test 5:
Interaction of two one-humped solitons with a phase shift

phase plane method

their relative phase may vary with the interaction



S.H. Shao & H.Z. Tang, CiCP., 3 (2008), 950.

5、 Numerical experiments

Test 6:

Compara. of several methods

SI=semi-implicit; HP=Hopscotch; LF=leapfrog; OS=exponential operator splitting

Method	Charge	Energy	Linear momentum	time
CN	Conserved	Conserved	Not conserved	Reversible
LCN1	Not conserved	Not conserved	Not conserved	Not reversible
LCN2	Conserved	Not conserved	Not conserved	Not reversible
SI	Not conserved	Not conserved	Not conserved	Reversible
HP	Not conserved	Not conserved	Not conserved	Reversible
LF	Not conserved	Not conserved	Not conserved	Reversible
WENO	Not conserved	Not conserved	Not conserved	Not reversible
DG	Not conserved	Not conserved	Not conserved	Not reversible
OS	Conserved	Not conserved	Not conserved	Reversible

Method	linearized stability	Truncation errer	L^∞ error	Scheme
CN	Stable	$\mathcal{O}(\tau^2+h^2)$	Linear increasing	Implicit,nonlinear
LCN1	Conditional stable	$\mathcal{O}(\tau^2+h^2)$	Linear increasing	Implicit
LCN2	Stable	$\mathcal{O}(\tau^2+h^2)$	Linear increasing	Implicit
SI	Conditional stable	$\mathcal{O}(\tau^2+h^2)$	Linear increasing	Implicit
HP	Conditional stable	$\mathcal{O}(\tau^2+h^2)$	Linear increasing	Explicit
LF	Conditional stable	$\mathcal{O}(\tau^2+h^2)$	Linear increasing	Explicit
WENO	Conditional stable	$\mathcal{O}(\tau^4+h^5)$	Linear increasing	Explicit
DG	Conditional stable	$\mathcal{O}(\tau^4+h^4)$	Oscillating	Explicit
OS	Conditional stable	$\mathcal{O}(\tau^4)$	Linear increasing	Explicit

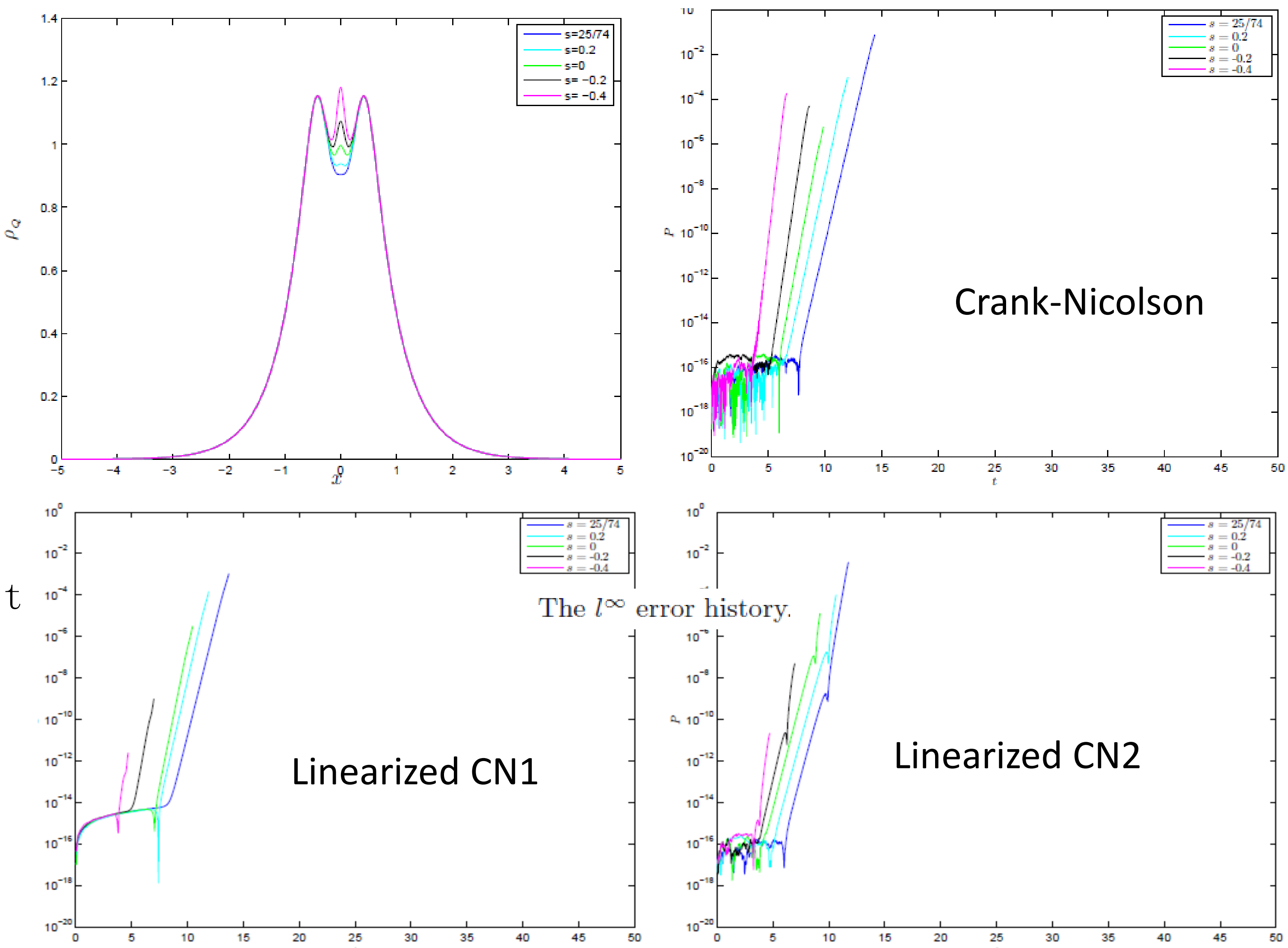
J. Zhao, S.H. Shao & H.Z. Tang, JCP, 245(2013), 131.

5、 Numerical experiments

Test 6:

Compara. of several methods

l^∞ errors of all schemes increase almost linearly with the time.



J. Zhao, S.H. Shao & H.Z. Tang, JCP, 245(2013), 131.

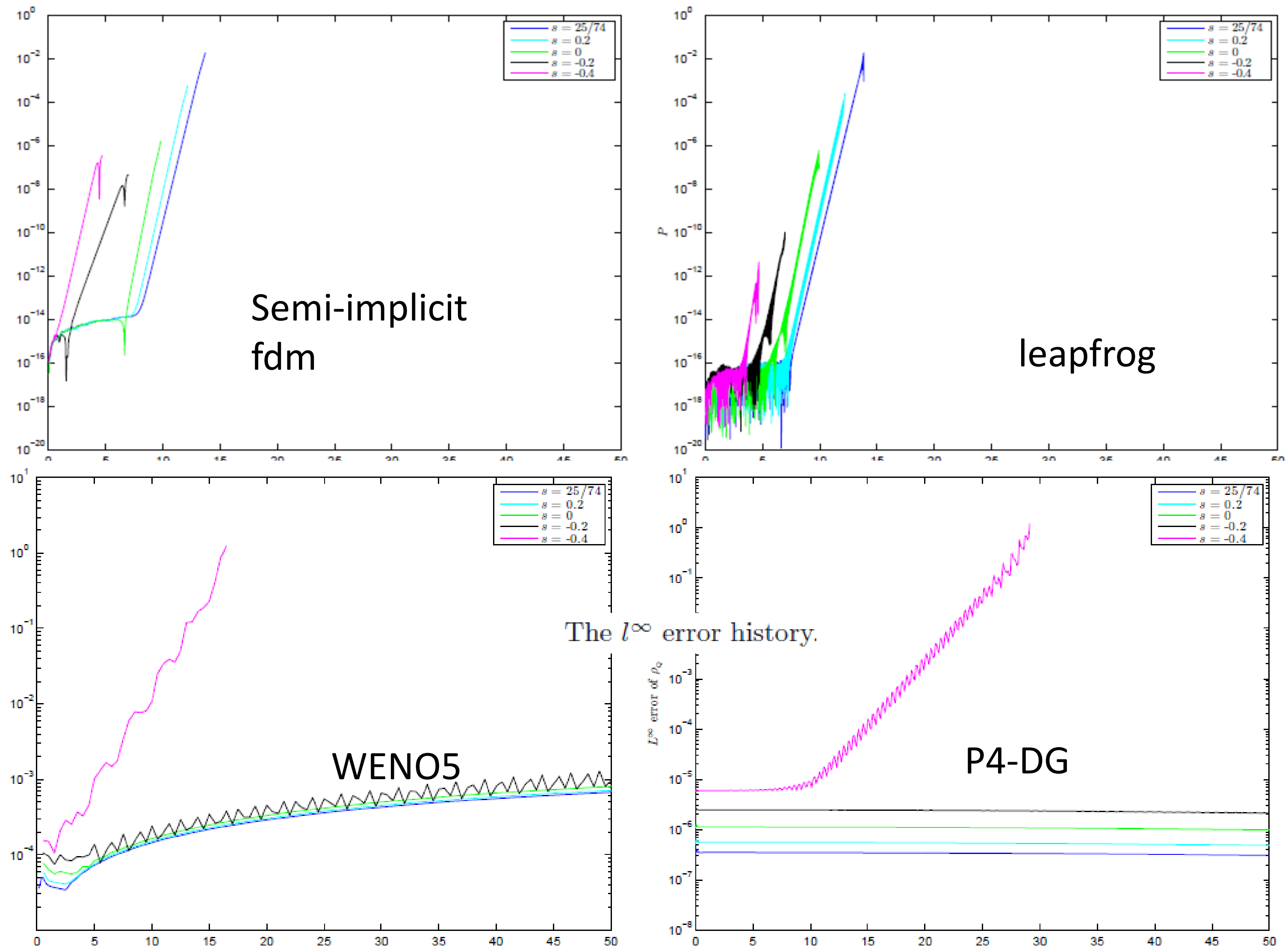
5、 Numerical experiments

Test 6:

Compara. of several methods

l^∞ errors of all schemes increase with t too.

The smaller the slope is, the longer time the scheme could simulate to.



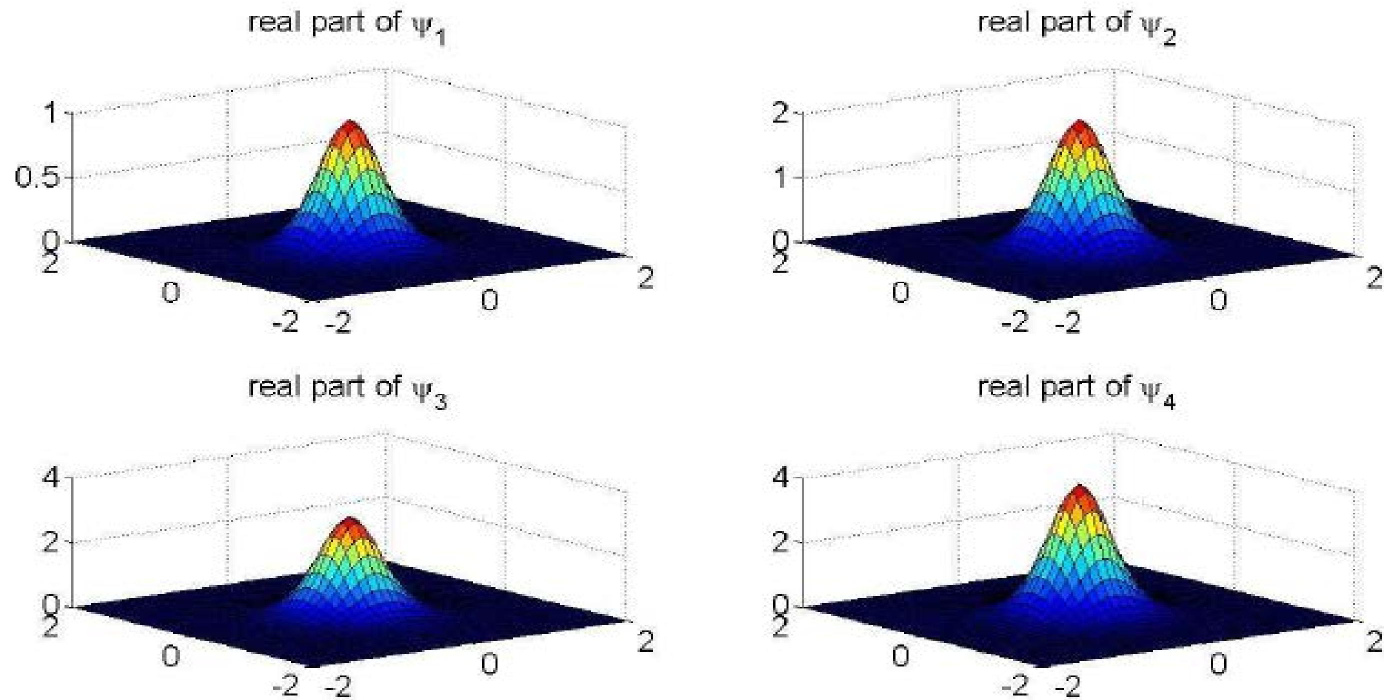
J. Zhao, S.H. Shao & H.Z. Tang, JCP, 245(2013), 131.

北京大學

5、 Numerical experiments

Test 7:
RKDG for (1+2)-d Dirac eq.

	Real part of ψ_1						
k	N	L^∞–error	order	L^1–error	order	L^2–error	order
4	10×10	5.02e-03	–	1.48e-03	–	1.11e-03	–
	20×20	2.15e-04	4.54	5.41e-05	4.78	4.72e-05	4.56
	40×40	7.80e-06	4.79	1.72e-06	4.98	1.56e-06	4.92
	80×80	2.21e-07	5.14	5.36e-08	5.01	4.90e-08	4.99



X. Ji, S.H. Shao & H.Z. Tang, preprint, 2012.

6、Conclusions

- For (1+1)-d NLD eq. with *a general self-interaction, a linear combination of the scalar, pseudoscalar, vector and axial vector self-interactions to the power of the integer k*, its soliton solutions are analytically derived, and the number of soliton humps in the charge and energy densities is proved in theory: *the number of soliton humps in charge (or energy / momentum) density is not bigger than 4 (or 3)*.
- Several numerical methods are discussed and compared. Interaction dynamics for Dirac solitons is studied. Some new phenomena are observed: *(a) a new quasi-stable long-lived oscillating bound state from binary collisions of a single-humped soliton & a two-humped soliton; (b) collapse in binary & ternary collisions; (c) strongly inelastic interaction in ternary collisions; and (d) bound states with a short or long lifetime from ternary collisions.* Phase plane method reveals that the relative phase of those waves may vary with the interaction.

Acknowledgements

- Collaborators：

  Sihong Shao (Assoc. Prof., PKU)

  Han Wang (Dr., Freie Universität Berlin)

  Xia Ji (Assoc. Prof., Inst. Comput. Math., CAS)

  Jian Xu (Mr., PKU)

  Dongyi Wei (Mr., PKU)

- Grant

  NSFC

Thank you for your attention!

The 8th International Congress on

INDUSTRIAL AND APPLIED MATHEMATICS

August 10-14, 2015, Beijing,China

ICIAM 2015 ICIAM

ICIAM 2015

The International Congress on Industrial and Applied Mathematics (ICIAM) is the premier international congress in the field of applied mathematics held every four years under the auspices of the International Council for Industrial and Applied Mathematics. From August 10 to 14, 2015, mathematicians from around the world will gather in Beijing, China for the 8th ICIAM to be held at China National Convention Center inside the Beijing Olympic Green.

Home
Committees
Organizers
Sponsors

CFP/Submissions
Important Dates
Invited Speakers
Conference Program

Conference Poster
Conference Venue
Latest News
Press/Media

Registration
Travel/Visa
Tours
Hotel Reservation

北京大學

In 2004 readers of 《*Physics World*》 voted for their favourite equation "The greatest equations":

No.1 The Maxwell's Eqs.

$$\nabla \cdot \mathbf{D} = \rho_f \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

电的高斯定律、磁的高斯定律、法拉第定律以及安培定律

No.2 Euler's Identity

$$e^{i\pi} + 1 = 0$$

No.3 Newton's 2nd Law of Motion

$$\mathbf{F} = m\mathbf{a}$$

No.4 Pythagorean Theorem

$$a^2 + b^2 = c^2$$

No.5 Mass–energy Equivalence

$$E_0 = mc^2$$

No.6 The Schrodinger Eq.

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}, t) = \hat{H}\Psi(\mathbf{r}, t)$$

No.7
1+1=2

No.8 The de Broglie Relations

$$p = \hbar k$$

$$E = \hbar\omega$$

No.9 The Fourier Transform

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx,$$

No.10 The Length of the Circumference of a Circle

$$c = 2\pi r$$