The cell-centered and linearity-preserving finite volume scheme for diffusion problems on general meshes

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Joint work with J.-M. Wu and Z.-H. Dai (IAPCM)

Outline

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Diffusion equation

Let Ω be a bounded polygonal domain in $\mathbb{R}^d$, $d = 2, 3$

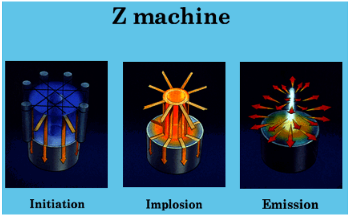
$$\begin{aligned} -\,\mathrm{div}(\Lambda \nabla u) &= f \quad \text{in } \Omega, \\ u &= g_D \quad \text{on } \partial\Omega, \end{aligned}$$

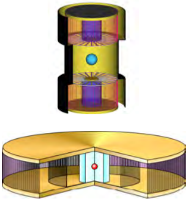
- Λ: $d \times d$ diffusion tensor.
- f: the source term.
- g_D: scalar functions which are *a.e.* defined on $\partial\Omega$.

Motivation: key applications

Equations of this kind arise in a wide range of scientific fields:

- ☞ **Z-Pinch driven ICF (Magneto-Hydrodynamics+Radiation)**: diffusion operator is coupled with Lagrangian hydrodynamics.



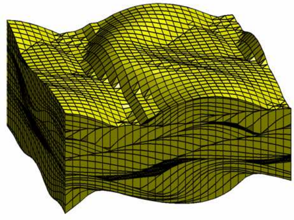


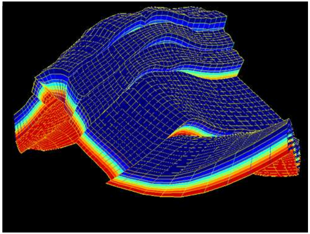
(a) A cylindrically symmetrical annulus is prepared with current to flow in the vertical (Z) direction. The implosion shows the resultant plasma shrinking and getting hotter. The plasma becomes very, very hot as it stagnates on the cylindrical axis of symmetry, releasing energy outward in the form of X-rays.
(b) Schematic of Double-ended hohlraum and dynamic hohlraum ICF concepts.
(c) Technician D. Graham uses tweezers to build an array of wires, each 1/10 the diameter of a human hair.
(courtesy of http://www.sandia.gov/)

Motivation: key applications

Equations of this kind arise in a wide range of scientific fields:

- ☞ **Z-Pinch driven ICF (Magneto-Hydrodynamics+Radiation)**: diffusion operator is coupled with Lagrangian hydrodynamics.
- ☞ **Reservoir simulation**: the complexity of the heterogeneous porous medium: layers, faults, fractures.

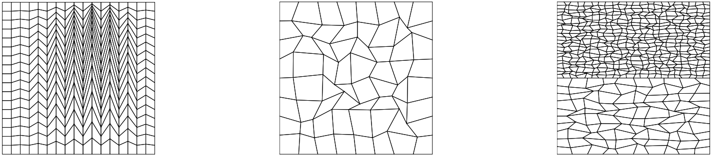




Challenges in simulations

In the above applications, we face a number of challenges:

- ☞ **Mesh issues:** large distortion, high aspect ratio, degeneration, concavity, violation of traditional mesh regular or quasi-uniform or quasi-parallelogram assumptions in FEM (rough meshes)
- ☞ **Discontinuity and nonlinearity**: multi-material, shocks (moving discontinuity), diffusion tensor being highly anisotropic or heterogeneous, strongly nonlinear



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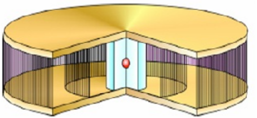

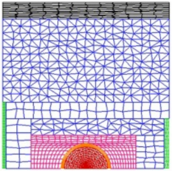

A configuration of Z-pinch driven ICF (left) and a possible 2D mesh (right).

An ultimate scheme ?

A desirable scheme must meet the above challenges and should

- be local conservative
- be cell-centered
- be linearity-preserving: exact for linear solutions
- be easy to code for both 2D and 3D cases
- lead to a symmetric and positive definite linear system
- be extremum-preserving or respect the physical bounds
- have theoretical fundations: stability, convergence, error estimates

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Incomplete lists of finite volume schemes

- Multi-point flux approximation (I. Aavatsmark et.al.,2002)
- Control volume distributed (M.Edwards, 2002)
- Mimetic finite differnece (Breil et.al., 07; Lipnikov, et.al., 09)
- Scheme Using Stabilization and Harmonic Interfaces (Eymard, Gallouet, Herbin, 2008)
- Cell-centered nine-point scheme (Li et.al.,1980; Y.Coudiere et. al. 1999; Bertolazzi-Manzini, 00s; Yuan, 2008; J.Wu et.al.,2010)
- Mixed finite volume (J.Droniou, R.Eymard, 2006)
- Hybrid finite volume (R.Eymard, T.Gallouet, R.Herbin,2008)
- Discrete Duality finite volume (F.Hermeline 2000's)
- Linearity-preserving schemes (Wu & Gao, 2005–2012)
- **Monotone schemes** (Le Potier 2005,2013; Lipnikov, Yuri, 2009; Yuan et.al., 2008; Gao & Wu, 2013)

Objective of this talk

Design a simple scheme that can be uniformly coded for both 2D and 3D problems, and maintain, as many as possible, the nice properties described before.

- **A linear version** with stability and error estimate in H_1-norm under standard assumptions.
- **A nonlinear version** having a small stencil and satisfying discrete minimum principle and maximum principle simultaneously.

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Outline

1. Problem statement
2. Cell-centered scheme
   - Construction of the scheme
   - A linear version
   - A nonlinear version
3. Numerical examples
   - Convergence analysis (P1-P2)
   - Linearity-preserving test (P3)
   - Discrete extremum principle (P4-P6)
   - 3D test
4. Discussion

Outline

Principle of finite volume schemes

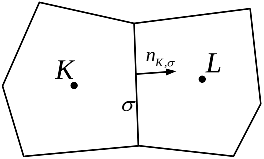
- **Flux balance**: Green's formula for diffusion equation on each cell K

$$\forall K: \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} = \int_K f$$

with $\mathcal{E}_K$ =edges/facets on ∂K and $F_{K,\sigma} = -\int_\sigma \Lambda \nabla u \cdot \boldsymbol{n}_{K,\sigma}$.

- **Flux conservativity (continuity)**

$$\forall \sigma = K|L: \; F_{K,\sigma} + F_{L,\sigma} = 0.$$



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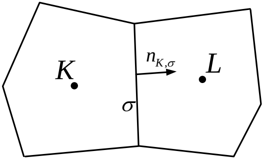

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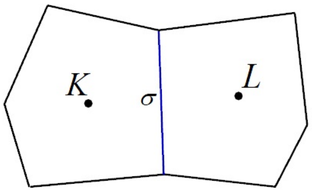
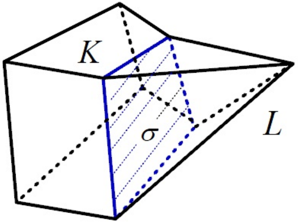
Mesh topology involved in our scheme

In our scheme, we only need to know two informations:

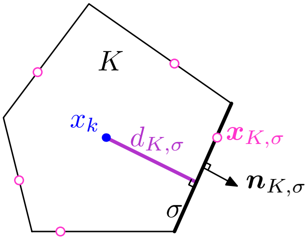
- Each cell is composed of facets: line segments (2D), polygons (3D).
- Each interior facet is shared by two cells.





Outline

Step 1: Specify cell-centers and interpolation points



- $\boldsymbol{x}_K$: cell-center of cell K.
- u_K: the primary unknown at cell center $\boldsymbol{x}_K$.
- $\boldsymbol{x}_{K,\sigma}$: **harmonic-averaging point** (L.Agelas, R.Eymard, R.Herbin, 2009).
- $u_{K,\sigma}$: the auxiliary unknown at $\boldsymbol{x}_{K,\sigma}$
- $d_{K,\sigma}$: the distance from $\boldsymbol{x}_K$ to the edge σ.
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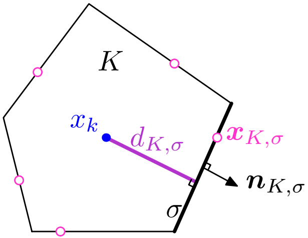

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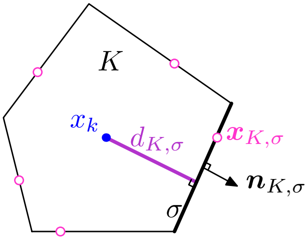

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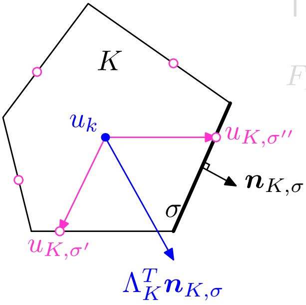
Step 2: Construct the one-sided flux

Consider the co-normal $\Lambda_K^T \boldsymbol{n}_{K,\sigma}$ and its decomposition on two vectors $\overrightarrow{\boldsymbol{x}_K \boldsymbol{x}_{K,\sigma'}}$ and $\overrightarrow{\boldsymbol{x}_K \boldsymbol{x}_{K,\sigma''}}$

$$\boxed{|\sigma|\Lambda_K^T \mathbf{n}_{K,\sigma} = \alpha(\boldsymbol{x}_{K,\sigma'} - \boldsymbol{x}_K) + \beta(\boldsymbol{x}_{K,\sigma''} - \boldsymbol{x}_K)}$$



The formula for a one-sided flux is

$$\begin{aligned}
F_{K,\sigma} &\simeq -\int_\sigma \Lambda \nabla u \cdot \boldsymbol{n}_{K,\sigma}\, \mathrm{d}s \\
&= -\int_\sigma \nabla u \cdot (\Lambda_K^T \boldsymbol{n}_{K,\sigma})\, \mathrm{d}s \\
&= -\frac{1}{|\sigma|}\int_\sigma \alpha \nabla u \cdot \overrightarrow{\boldsymbol{x}_K \boldsymbol{x}_{K,\sigma'}} + \beta \nabla u \cdot \overrightarrow{\boldsymbol{x}_K \boldsymbol{x}_{K,\sigma''}}\, \mathrm{d}s \\
&= \alpha(u_K - u_{K,\sigma'}) + \beta(u_K - u_{K,\sigma''})
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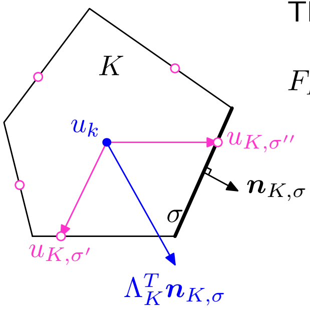

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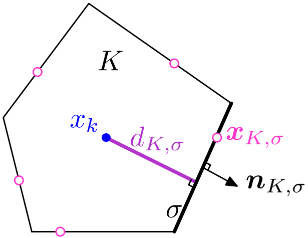
Step 3: Construct a unique edge flux

The total unique edge flux is the linear combination of one-sided flux

$$\tilde{F}_{K,\sigma} = \mu_{K,\sigma} F_{K,\sigma} - \mu_{L,\sigma} F_{L,\sigma}, \quad \tilde{F}_{L,\sigma} = \mu_{L,\sigma} F_{L,\sigma} - \mu_{K,\sigma} F_{K,\sigma}$$



Where the coefficients $\mu_{K,\sigma}$ and $\mu_{L,\sigma}$:

$$\mu_{K,\sigma} = \frac{a_{L,\sigma}}{a_{K,\sigma} + a_{L,\sigma}}, \quad \mu_{L,\sigma} = \frac{a_{K,\sigma}}{a_{K,\sigma} + a_{L,\sigma}}$$

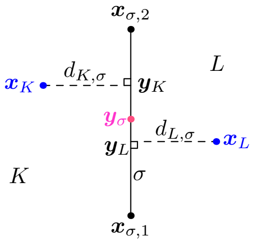
with

$$a_{K,\sigma} = \mathbf{n}_{K,\sigma}^T \Lambda_K \mathbf{n}_{K,\sigma} / d_{K,\sigma}$$

$$a_{L,\sigma} = \mathbf{n}_{L,\sigma}^T \Lambda_L \mathbf{n}_{L,\sigma} / d_{L,\sigma}$$

Step 4: Interpolation of the auxiliary unknowns

Define the interpolation points with harmonic averaging point $\boldsymbol{y}_\sigma \in \bar{\sigma}$ (L.Agelas, R.Eymard, R.Herbin, 2009)



$$\boldsymbol{y}_\sigma = \mu_{L,\sigma}\boldsymbol{x}_K + \mu_{K,\sigma}\boldsymbol{x}_L + \frac{\left(\Lambda_K^T - \Lambda_L^T\right)\boldsymbol{n}_{K,\sigma}}{a_K + a_L}$$

such that the following properties hold

- $u(\boldsymbol{x})$ is linear in K and L
- $u(\boldsymbol{x})$ is continuous at interface σ
- $u(\boldsymbol{x})$ has continuous flux across σ

The edge unknowns can be expressed as linear combinations of two cell-centered unknowns u_K and u_L

$$\boxed{u_{K,\sigma} = u_{L,\sigma} = u(\boldsymbol{y}_\sigma) \simeq \mu_{L,\sigma}u_K + \mu_{K,\sigma}u_L}$$

which gives an accurate approximation of u at the interface σ.

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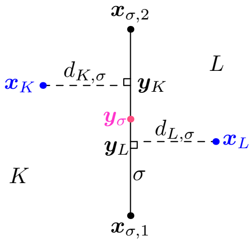

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Sketch of the linear scheme (five steps)

1. choose harmonic averaging points $\boldsymbol{y}_\sigma$ on material interfaces.
2. Define one-sided fluxes

$$F_{K,\sigma} \simeq \alpha(u_K - u_{K,\sigma'}) + \beta(u_K - u_{K,\sigma''})$$

3. Define the unique edge flux

$$\tilde{F}_{K,\sigma} = \mu_{K,\sigma} F_{K,\sigma} - \mu_{L,\sigma} F_{L,\sigma} \quad \text{with} \quad \mu_{K,\sigma} + \mu_{L,\sigma} = 1$$

4. Eliminate the auxiliary edge unknowns

$$u_{K,\sigma} = u_{L,\sigma} = u(\boldsymbol{y}_\sigma) \simeq \mu_{L,\sigma} u_K + \mu_{K,\sigma} u_L$$

5. Insert the linear fluxes into mass balance equation:

$$\sum_{\sigma \in \mathcal{E}_K} \tilde{F}_{K,\sigma} = \int_K f \, \mathrm{d}x$$

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H_1-Stability for the linear scheme in 2D

Assume that, for all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$,

- **(Mesh regularity)**

$$|K| \geq \alpha h^2, |\sigma| \geq \beta h, n_K \leq \gamma$$

- **The symmetric part of cell matrix is positive definite**

$$\mathbf{v}^T \left(\mathbb{A}_K + \mathbb{A}_K^T \right) \mathbf{v}/2 \geq \tau_K \|\mathbf{v}\|$$

- **(Geometrical assumption)**

$$|\sigma|/d_\sigma \leq q^\star, d_\sigma = d_{K,\sigma} + d_{L,\sigma} \text{ or } d_\sigma = d_{K,\sigma}.$$

Then

$$\|u_h\|_{1,\mathcal{M}} \leq \frac{C_P q^\star}{\min\{\tau_K \mu_{K,\sigma}^2\}} \|f\|_{0,\Omega}$$

Error estimate for the linear scheme in 2D

Assume that $u \in C^2(\bar{\Omega})$ and, for all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$,

- $\mathbf{v}^T \left(\mathbb{A}_K + \mathbb{A}_K^T \right) \mathbf{v}/2 \geq \tau_K \|\mathbf{v}\|$;
- $|\sigma|/d_\sigma \leq q^\star, d_\sigma = d_{K,\sigma} + d_{L,\sigma}$ or $d_\sigma = d_{K,\sigma}$;
- $\|\mathbb{A}_K \mathbf{v}\| \leq \eta \|\mathbf{v}\|$;
- $\boldsymbol{y}_\sigma \in \bar{\sigma}$.

Then we have the following H_1 error estimate

$$\|\Pi_h u - u_h\|_{1,\mathcal{M}} \leq Ch.$$

where Π_h is a piecewise constant interpolation operator.

Outline

A nonlinear extremum-preserving FV scheme

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$$F_{K,\sigma} \simeq \alpha(u_K - u_{K,\sigma'}) + \beta(u_K - u_{K,\sigma''}), \quad \alpha > 0, \beta > 0$$

3. Define unique edge flux

$$\tilde{F}_{K,\sigma} = \mu_{K,\sigma} F_{K,\sigma} - \mu_{L,\sigma} F_{L,\sigma} \quad \text{with} \quad \mu_{K,\sigma} + \mu_{L,\sigma} = 1$$

For the previous linear scheme, we have chosen

$$\mu_{K,\sigma} = \frac{a_{L,\sigma}}{a_{K,\sigma} + a_{L,\sigma}}, \quad a_{K,\sigma} = \frac{1}{d_{K,\sigma}} \mathbf{n}_{K,\sigma}^T \Lambda_K \mathbf{n}_{K,\sigma}$$

A nonlinear extremum-preserving FV scheme

1. choose harmonic averaging points $\boldsymbol{y}_\sigma$ on material interfaces.
2. Define one-sided fluxes

$$F_{K,\sigma} \simeq \alpha(u_K - u_{K,\sigma'}) + \beta(u_K - u_{K,\sigma''}), \quad \alpha > 0, \beta > 0$$

3. Define unique edge flux **with nonlinear coefficients**

$$\tilde{F}_{K,\sigma} = \tilde{\mu}_{K,\sigma}(u)F_{K,\sigma} - \tilde{\mu}_{L,\sigma}(u)F_{L,\sigma} \quad \text{with} \quad \tilde{\mu}_{K,\sigma}(u) + \tilde{\mu}_{L,\sigma}(u) = 1$$

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The nonlinear coefficients

$$\tilde{\mu}_{K,\sigma}(u) = \frac{|\bar{F}_{L,\sigma}| + \epsilon}{|\bar{F}_{K,\sigma}| + |\bar{F}_{L,\sigma}| + 2\epsilon} \text{ with } F_{K,\sigma} = \underbrace{\gamma a_{\sigma\sigma}^K (u_K - u_\sigma)}_{\text{"good"}} + \underbrace{\bar{F}_{K,\sigma}}_{\text{"bad"}}$$

☞ A small parameter $\epsilon > 0$, $\gamma \in [0, 1]$.

A nonlinear extremum-preserving FV scheme

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4. Eliminate the auxiliary edge unknowns

$$u(\boldsymbol{y}_\sigma) \simeq \mu_{L,\sigma} u_K + \mu_{K,\sigma} u_L, \quad \sigma = K|L.$$

5. Insert the **nonlinear fluxes** into mass balance equation:

$$\sum_{\sigma \in \mathcal{E}_K} \tilde{F}_{K,\sigma} = \int_K f \, \mathrm{d}x$$

Characteristics of the linear scheme

- locally conservative and cell-centered
- **linearity-preserving**: exact for linear solutions
- reliable on unstructured anisotropic meshes that may be distorted;
- allow heterogeneous full diffusion tensors;
- have higher than the first-order accuracy for smooth solutions;
- **2D & 3D implementation almost in an uniform code**
- **Stability and error estimates in H_1-norm.**

Characteristics of the nonlinear scheme

- locally conservative and cell-centered
- **linearity-preserving**: exact for linear solutions
- reliable on unstructured anisotropic meshes that may be distorted;
- allow heterogeneous full diffusion tensors;
- have higher than the first-order accuracy for smooth solutions;
- **2D & 3D implementation almost in an uniform code**
- **satisfy discrete minimum principle and maximum principle**
- **small stencil**: 4-point stencil for triangle mesh, 5-point for structured quad mesh, 7-point for structured hexahedral meshes.

Outline

Methodology for convergence analysis

- The discrete L_2-norm of solution errors

$$E_u = \left(\sum_K |K| (u(x_K) - u_K) \right)^{1/2}.$$

The L_2-norm of its gradient errors E_q can be defined analogously.

- The rate of convergence

$$R_\alpha = \frac{\log[E_\alpha(h_2)/E_\alpha(h_1)]}{\log(h_2/h_1)}, \quad \alpha = u, q,$$

where h_1, h_2 denote the mesh sizes of the two successive meshes.

- Most of tests can be found in the 5th and 6th Conference on Finite Volumes for Complex Applications - FVCA 5 & 6.

Outline

1. Problem statement
2. Cell-centered scheme
   - Construction of the scheme
   - A linear version
   - A nonlinear version
3. Numerical examples
   - Convergence analysis (P1-P2)
   - Linearity-preserving test (P3)
   - Discrete extremum principle (P4-P6)
   - 3D test
4. Discussion

(P1) Smooth solution: description

- Dirichlet B.V.P. in $\Omega = [0,1]^2$
- diffusion tensor

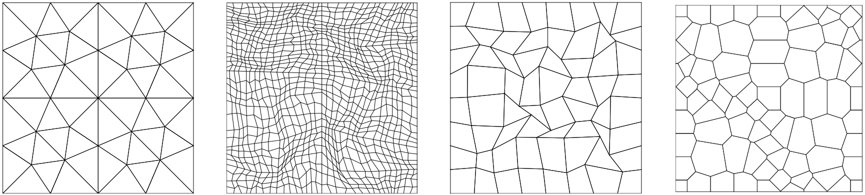
$$K = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$$

- exact solution $0 \le u(x,y) \le 1$:

$$u(x,y) = \frac{1}{2}\left[\frac{\sin((1-x)(1-y))}{\sin(1)} + (1-x)^3(1-y)^2\right],$$

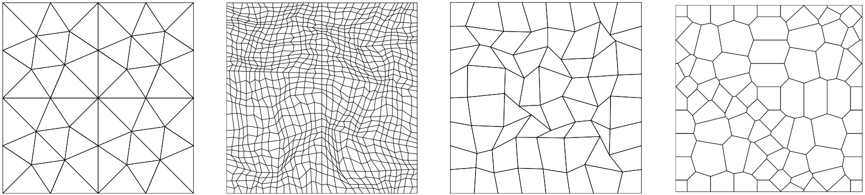
- Slight modification of Test 1.2 in FVCA5.

(P1) Smooth solution: convergence analysis



Mesh	Scheme	umin	umax	Stencil	Rates(u)	Rates(∇u)
Mesh1	nonlinear	2.80E-03	0.754	4	1.929	1.025
	linear	3.06E-03	0.754	10	1.983	1.089
Mesh2	nonlinear	1.17E-03	0.825	5	2.305	1.046
	linear	1.32E-03	0.825	13	2.525	1.070
Mesh3	nonlinear	1.84E-03	0.810	5	1.986	1.027
	linear	1.93E-03	0.810	13	1.984	1.030
Mesh4	nonlinear	4.19E-04	0.906	7	1.814	1.240
	linear	4.42E-04	0.906	19	1.810	1.236

(P1) Smooth solution: convergence analysis



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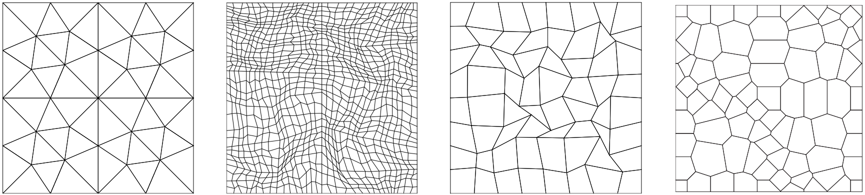


Percentage of harmonic averaging points that are outside mesh edges

Mesh	h_1	h_2	h_3	h_4	h_5
Mesh1	0	0	0	0	0
Mesh2	0	0.18%	0.19%	0.18%	0.15%
Mesh3	0	0	0	0.02%	0.03%
Mesh4	8.61%	10.71%	7.56%	5.44%	3.36%

(P2) Discontinuous anisotropy: description

- Dirichlet B.V.P. in the unit square domain
- Λ changes the eigenvalues and orientation of eigenvectors across the line $x = 0.5$,

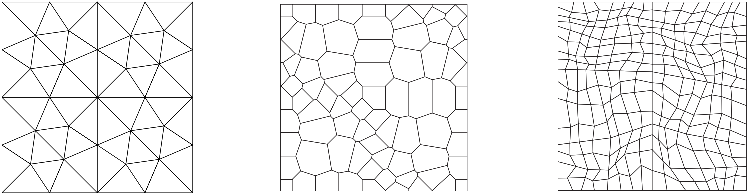
$$\Lambda = \begin{pmatrix} 10 & 2 \\ 2 & 5 \end{pmatrix}, x \leq 0.5; \qquad K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, x > 0.5.$$

- exact solution

$$u(x,y) = \begin{cases} \left[1 + (x - 0.5)\left(.1 + 8\pi(y-0.5)\right)\right] e^{-20\pi(y-0.5)^2}, & x \leq 0.5, \\ e^{x-0.5} e^{-20\pi(y-0.5)^2}, & x > 0.5. \end{cases}$$

☞*Ref.:* Younes A, Fontaine V., Int. J. for Numer. Meth. in Engrg. 2008 (76) 314-336.

(P2) Discontinuous anisotropy: convergence analysis

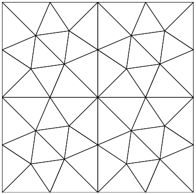


triangular mesh polygonal mesh Shestakov mesh

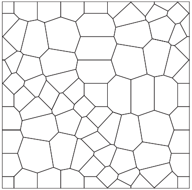
Linear scheme: L_2-errors of solution and its gradient

h	triangular mesh		polygonal mesh		Shestakov mesh	
	E_u	E_q	E_u	E_q	E_u	E_q
h_1	2.16E-1	2.01E0	3.08E-1	3.64E0	2.62E-1	3.07E0
h_2	2.33E-2	6.31E-1	5.47E-2	9.06E-1	4.72E-2	1.09E0
h_3	4.72E-3	2.63E-1	9.39E-3	2.38E-1	1.13E-2	3.73E-1
h_4	1.17E-3	1.13E-1	2.53E-3	9.22E-2	3.87E-3	1.99E-1
h_5	2.93E-4	5.34E-2	1.29E-3	4.87E-2	1.30E-3	1.06E-1
Rates	2.381	**1.309**	2.064	**1.588**	2.780	**1.776**

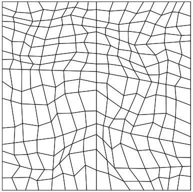
(P2) Discontinuous anisotropy: convergence analysis



triangular mesh



polygonal mesh



Shestakov mesh

Nonlinear scheme: L_2-errors of solution and its gradient

h	triangular mesh		polygonal mesh		Shestakov mesh	
	E_u	E_q	E_u	E_q	E_u	E_q
h_1	2.51E-1	2.18E0	3.07E-1	3.70E0	2.70E-1	3.18E0
h_2	4.23E-2	8.58E-1	5.13E-2	8.98E-1	5.22E-2	1.21E0
h_3	1.36E-2	4.28E-1	8.49E-3	2.41E-1	1.75E-2	5.47E-1
h_4	3.82E-3	1.70E-1	2.32E-3	9.65E-2	8.66E-3	3.26E-1
h_5	1.03E-3	6.88E-2	1.12E-3	5.31E-2	3.44E-3	1.88E-1
Rates	1.981	**1.246**	2.068	**1.563**	2.295	**1.491**

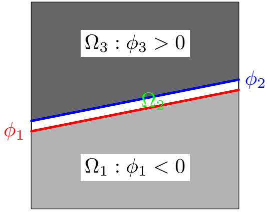
Outline

(P3) Oblique barrier (Test 7 in FVCA5)

☞ The domain Ω is composed of three subdomains

☞ Exact solution (parallel to boundaries of subdomains)

$$u(x,y) = \begin{cases} -\phi_1(x,y) & \text{on } \Omega_1, \\ -100\phi_1(x,y) & \text{on } \Omega_2, \\ -\phi_2(x,y) - 5 & \text{on } \Omega_3, \end{cases}$$

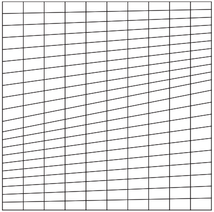


$\phi_1 = y - 0.2(x - 0.5) - 0.475$

$\phi_2 = \phi_1 - 0.05$

☞ The permeability tensor Λ

$$\Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \text{ with } \alpha = \begin{cases} 1 & \text{on } \Omega_1 \cup \Omega_3, \\ 0.01 & \text{on } \Omega_2. \end{cases}$$



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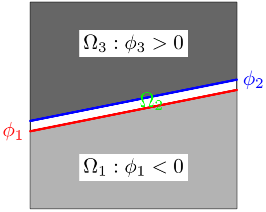

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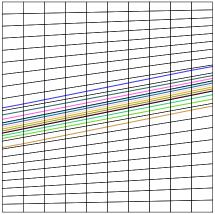

☞ Relative error of **solution** and gradient

Scheme	uemin	uemax	umin	umax	Sol.(L_2)	Sol.(H_1)
Linear	-5.575	0.575	-5.537	0.537	**7.96E-16**	3.02E-15
Nonlinear	-5.575	0.575	-5.537	0.537	**6.51E-16**	2.98E-15

Outline

(P4) oblique flow (Test 3 in FVCA5)

- Unit square $\Omega = [0,1]^2$
- Anisotropic diffusion tensor

$$\Lambda = R_\theta \begin{pmatrix} 1 & 0 \\ 0 & 10^{-4} \end{pmatrix} R_\theta^{-1}, \quad \theta = 40^\circ$$

- Source term $f = 0$
- Dirichlet B. C.: g_D continuous and piecewise linear

$$g_D(x,y) = \begin{cases} 1 \text{ on } (0,0.2) \times \{0\} \cup \{0\} \times (0,0.2), \\ 0 \text{ on } (0.8,1) \times \{1\} \cup \{1\} \times (0.8,1), \\ 0.5 \text{ on } (0.3,1) \times \{0\} \cup \{0\} \times (0.3,1), \\ 0.5 \text{ on } (0,0.7) \times \{1\} \cup \{1\} \times (0,0.7). \end{cases}$$

(P4) oblique flow (Test 3 in FVCA5)

- Unit square $\Omega = [0,1]^2$
- Anisotropic diffusion tensor

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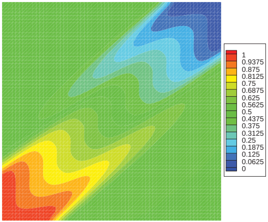

- Source term $f = 0$
- Dirichlet B. C.: g_D continuous and piecewise linear

$$g_D(x,y) = \begin{cases} 1 \text{ on } (0,0.2)\times\{0\}\cup\{0\}\times(0,0.2), \\ 0 \text{ on } (0.8,1)\times\{1\}\cup\{1\}\times(0.8,1), \\ 0.5 \text{ on } (0.3,1)\times\{0\}\cup\{0\}\times(0.3,1), \\ 0.5 \text{ on } (0,0.7)\times\{1\}\cup\{1\}\times(0,0.7). \end{cases}$$

☞ This test represents a flow with boundary conditions such that the pressure driven flow “would like" to go from vertex (0,0) to (1,1) ,and the solution features a Z across the $y = x$ axis.

(P4) oblique flow (Test 3 in FVCA5)

- Unit square $\Omega = [0,1]^2$
- Anisotropic diffusion tensor

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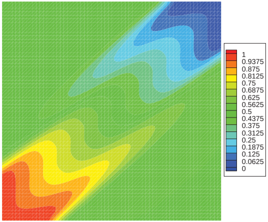
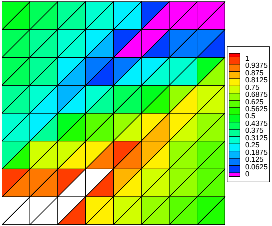


- Source term $f = 0$
- Dirichlet B. C.: g_D continuous and piecewise linear

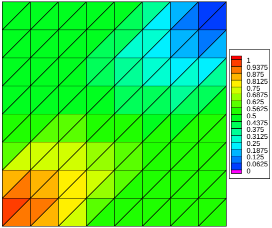
$$g_D(x,y) = \begin{cases} 1 \ \text{on}\ (0,0.2)\times\{0\}\cup\{0\}\times(0,0.2), \\ 0 \ \text{on}\ (0.8,1)\times\{1\}\cup\{1\}\times(0.8,1), \\ 0.5 \ \text{on}\ (0.3,1)\times\{0\}\cup\{0\}\times(0.3,1), \\ 0.5 \ \text{on}\ (0,0.7)\times\{1\}\cup\{1\}\times(0,0.7). \end{cases}$$

Maximum principle is not always easy to verify for such a solution !

(P4) oblique flow: solution profiles



(a) MPFA

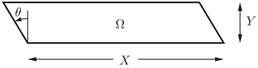


(b) nonlinear scheme

- 128 cells.
- white color: $u > 1$.
- magenta color: $u < 0$.

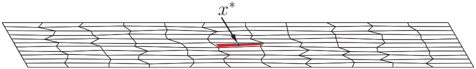
Scheme	min. sol.	Max. sol.
MPFA	-0.22	1.22
Nonlinear scheme	0.0302	0.97

(P5) Perturbed parallelograms (Test 8 in FVCA5)



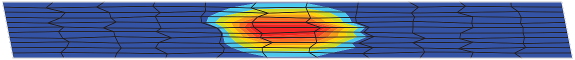
- $X = 1$, $Y = 1/30$, $\theta = 30°$
- Diffusion tensor $\Lambda = \mathsf{Id}$
- $f = 0$ in all cells except cell $x^* = (6, 6)$ where

$\int_{\text{cell}(6,6)} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 1.$



- perturbed parallelogram mesh with 11×11 cells. The height is 3 times the real height of the mesh.
- The solution should be a function with a maximum in cell x^*. If the solution shows internal oscillations or is negative, Hopf's first lemma is violated.

(P5) Perturbed parallelograms: min. and max. solutions [Herbin-Hubert, Table 12 in FVCA5]



Scheme	umin	umax	Scheme	umin	umax
Fine grid	**1.07E-24**	**4.10E-01**	FVHYB	-3.38E-02	1.12E-01
CMPFA	-2.31E-02	1.03E-01	FVSYM	-7.21E-02	1.52E-01
CVFE	-1.23E-03	4.24E-02	FVPMMD	1.22E-09	3.99E-01
DDFV-BHU	-1.25E-03	8.22E-02	MFD-BLS	-1.03E-01	1.85E-01
DDFV-HER	-1.61E-03	8.99E-02	MFD-FHE	-6.54E-02	1.44E-01
DDFV-MNI	-1.46E-03	6.69E-02	MFD-MAR	-2.62E-02	9.07E-02
DDFV-OMN	-1.77E-03	8.36E-02	MFV	-8.08E-03	5.81E-02
DG-C	-7.33E-03	1.05E-01	NMFV	3.05E-15	9.42E-02
DG-W	-9.03E-03	6.57E-02	SUSHI-NP	-1.19E-03	5.65E-02
FEQ1	-4.17E-03	4.90E-02	SUSHI-P	3.26E-06	6.77E-03
Linear	-7.51E-04	7.19E-02	Nonlinear	1.58E-09	8.03E-02

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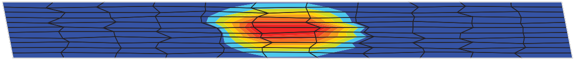

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DDFV-OMN	-1.77E-03	8.36E-02	MFV	-8.08E-03	5.81E-02
DG-C	-7.33E-03	1.05E-01	NMFV	3.05E-15	9.42E-02
DG-W	-9.03E-03	6.57E-02	SUSHI-NP	-1.19E-03	5.65E-02
FEQ1	-4.17E-03	4.90E-02	SUSHI-P	3.26E-06	6.77E-03
Linear	-7.51E-04	7.19E-02	Nonlinear	1.58E-09	8.03E-02

(P6) Heterogeneous diffusion tensor

- Dirichlet B.V.P. in $\Omega = [0,1]^2$ with four square subdomains Ω_i
- Diffusion tensor $\Lambda = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.
- Source term $f = \frac{81}{4}$ in $\boldsymbol{x} \in \left[\frac{7}{18}, \frac{11}{18}\right]^2$; $f = 0$ otherwise.
- **Results on randomly distorted quadrilateral mesh in two cases.**

(P6) Heterogeneous diffusion tensor

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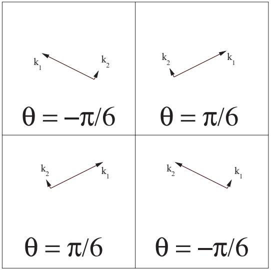
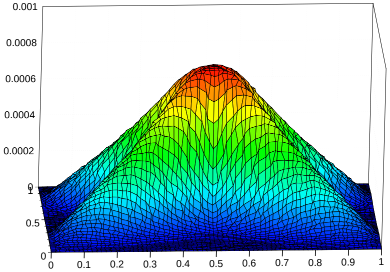
- (1) various angle θ and $k_1 = 1000$, $k_2 = 1$: $\min u = 1.18\text{E-}10$





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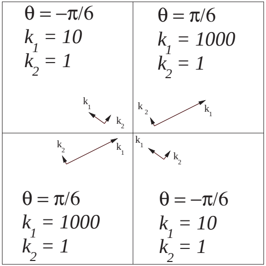
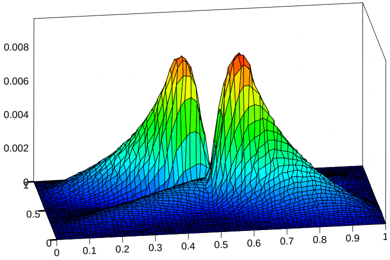
- (2) various angle θ, chess board distribution of k_1, k_2: $\min u$=6.2E-10





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Our nonlinear FV scheme can handle strong jumps of full diffusion tensor with different couples of principle directions and eigenvalues !

Outline

1. Problem statement

2. Cell-centered scheme
   - Construction of the scheme
   - A linear version
   - A nonlinear version

3. Numerical examples
   - Convergence analysis (P1-P2)
   - Linearity-preserving test (P3)
   - Discrete extremum principle (P4-P6)
   - 3D test

4. Discussion

Linear scheme (3D): meshes with planar facets

- Dirichlet B.V.P. in $\Omega = [0,1]^3$
- diffusion tensor

$$\Lambda = \begin{pmatrix} 1.5 & 0.5 & 0 \\ 0.5 & 1.5 & 0.5 \\ 0 & 0.5 & 1.5 \end{pmatrix}$$

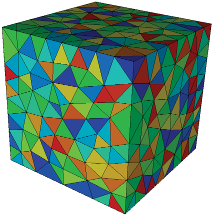
- exact solution

$$u(x,y,z) = 1 + \sin(\pi x)\sin\left(\pi\left(y+\frac{1}{2}\right)\right)\sin\left(\pi\left(z+\frac{1}{3}\right)\right)$$

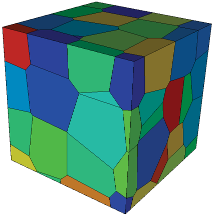
- meshes: all six typical mesh types in the sixth conference on finite volumes for complex applications

W. W. Sun, J. M. Wu, X.P. Zhang, A family of linearity-preserving schemes for anisotropic diffusion problems on arbitrary polyhedral grids, 2013, CMAME.

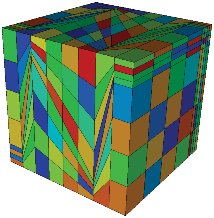
Linear scheme (3D): meshes with planar facets



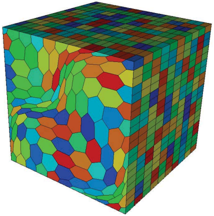
(c) 1.93(0.96)



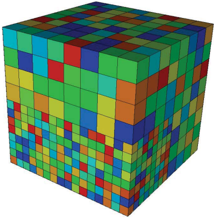
(d) 2.06(1.48)



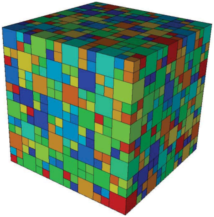
(e) 1.38(0.96)



(f) 2.06(1.58)



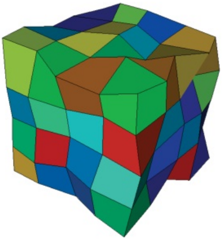
(g) 1.99(1.43)

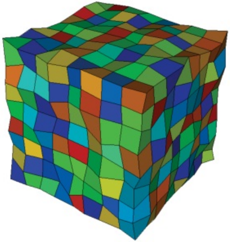


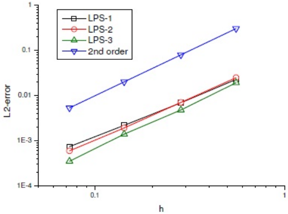
(h) 2.05(1.13)

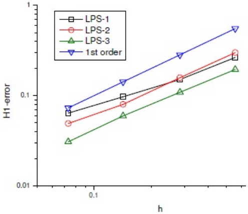
Linear scheme (3D): meshes with non-planar facets

- $\Lambda = \mathrm{diag}(1, 1, 1000)$, a strongly anisotropic tensor
- exact solution $u(x, y, z) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$









Outline

Summary and perspectives

Cell-centered FV scheme: linear version, nonlinear version.

- cell-centered and locally conservative
- robust, reliable for strongly anisotropic heterogeneous diffusion.
- 2nd order accuracy on most polygonal or polyhedral meshes
- uniformly coding for both 2D and 3D models
- Linear scheme: stability and error estimate in H_1-norm under general and standard assumptions on polygonal meshes.
- Nonlinear scheme: small stencil and extremum-preserving

☞ Construction: How to solve $\boldsymbol{y}_\sigma \notin \bar{\sigma}$?

☞ Application: Z-pinch ICF simulation and some related fields

Two Key References

Z.-M. Gao, J.-M. Wu
A small stencil and extremum-preserving scheme for anisotropic diffusion problems on arbitrary 2D and 3D meshes.
Journal of Computational Physics, 250 (2013), 308-331.

J.-M. Wu, Z.-M. Gao, Z.-H. Dai
A stabilized linearity -preserving scheme for the heterogeneous and anisotropic diffusion problems on polygonal meshes.
Journal of Computational Physics, 231 (2012), 7152-7169.

Thank you !

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