The Third Russian-Chinese Workshop on Numerical Mathematics and Scientific Computing

Postprocessing solutions of Hermitian Finite Element Methods

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$$\begin{aligned} Lu &= f, \ x \in \ \Omega, & (1) \\ u(x) &= 0, \ x \in \ \partial\overline{\Omega}, & (2) \end{aligned}$$

where

$$Lu = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}(a_i \frac{\partial}{\partial x_i} u).$$

$\Omega = (0, 1) \times (0, 1)$ and $\partial\overline{\Omega}$ its boundary.

Galerkin Approach

$$(Lu, v) = (f, v), \forall v \in H^1.$$

where

$$(u, v) = \int_\Omega u(x)v(x)d\Omega,$$

$$||u||_{L_2} = \sqrt{(u, u)}.$$

Galerkin Approach

For Laplace operator, Galerkin formulation is written as follows:

$$\mathcal{L}(u, v) = \int_{\Omega} u'_x v'_x + u'_y v'_y d\Omega = (f, v), \forall v \in H^1_0.$$

1D Numerical Solution

$$u^h \in S^h$$

.

$$u^h(x) = \sum_{i=1}^{N} u_i \Phi_i(x).$$

$$(Lu^h, v^h) = (f, v^h), \forall v^h \in S^h.$$

1D Hermitian element

One-dimensional cubic Hermitian element $s(x)$ is defined at segment $[-1, 1]$ by four values

$$f(-1), f'(-1), f(1), f'(1),$$

so that

$$s(-1) = f(-1), s'(-1) = f'(-1), s(1) = f(1), s'(1) = f'(1).$$

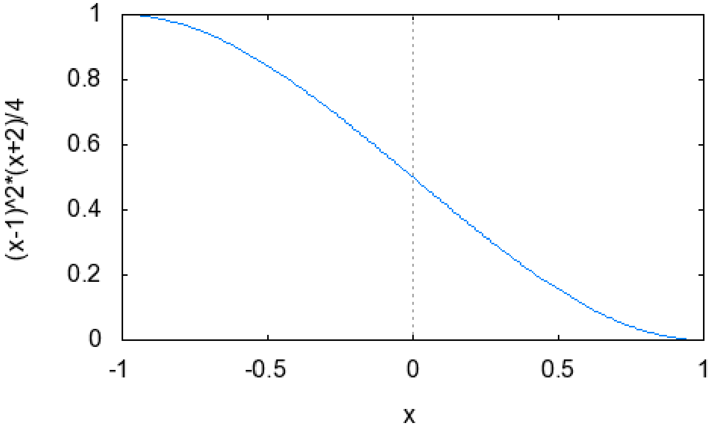
To interpolate function f it is convenient to represent this interpolant by following basis functions:

$$s(x) = f(-1)\phi_{-1}(x) + f(1)\phi_1(x) + f'(-1)\psi_{-1}(x) + f'(1)\psi_1(x).$$

Construct basis functions by following way. Basis function ϕ_{-1} takes the value 1 at -1 and value 0 at 1:
$\phi_{-1}(-1) = 1$, $\phi'_{-1}(-1) = 0$, $\phi_{-1}(1) = 0$, $\phi_{-1}(1) = 0$.
So,

$$\phi_{-1}(x) = \frac{(x-1)^2\,(x+2)}{4},$$

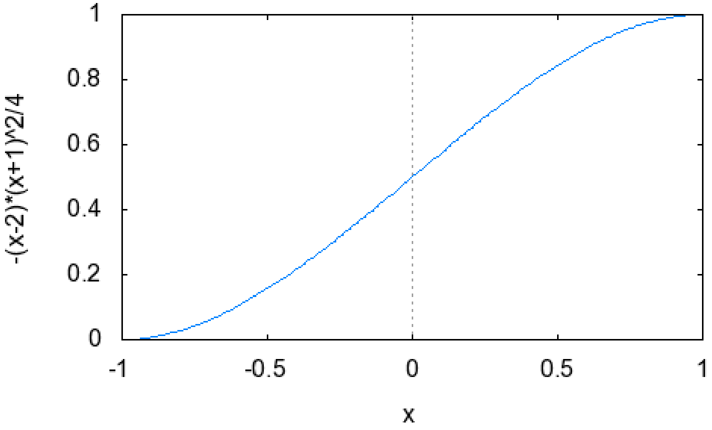
and its derivative: $\phi'_{-1}(x) = (3x^2 - 3)/4$.


1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
x
(x-1)^2*(x+2)/4

Similarly we construct the basis function ϕ_1:
$\phi_1(-1) = 0$, $\phi_1'(-1) = 0$, $\phi_1(1) = 1$, $\phi_{-1}(1) = 0$.
Therefore ϕ_1 has the form

$$\phi_1(x) = -\frac{(x-2)(x+1)^2}{4}.$$

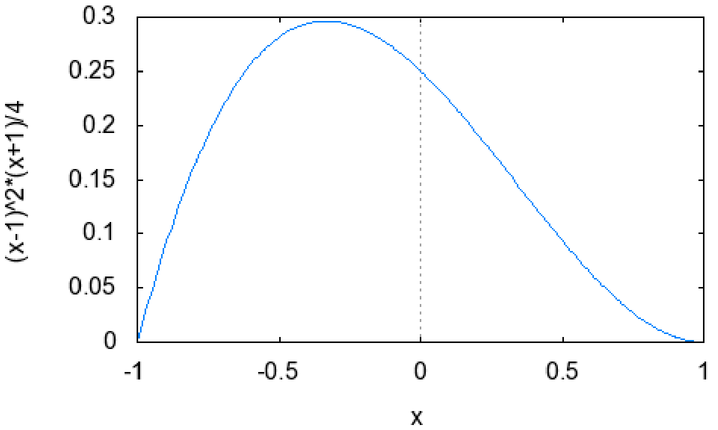
Its derivative:$\phi_1' = -(3 * x^2 - 3)/4$.


1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
x
-(x-2)*(x+1)^2/4

One more basis function ψ_{-1} for point -1:
$\psi_{-1}(-1) = 0$, $\psi'_{-1}(-1) = 1$, $\psi_{-1}(1) = 0$, $\psi'_{-1}(1) = 0$.
It has the form:

$$\psi_{-1} = (x) = \frac{(x-1)^2\,(x+1)}{4}$$

Its derivative: $\psi'_{-1} = ((x-1) * (3 * x + 1))/4$

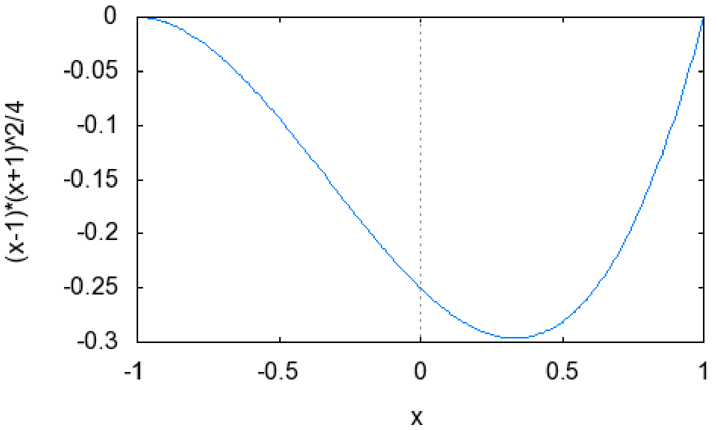


0.3
0.25
0.2
0.15
0.1
0.05
0
-1
-0.5
0
0.5
1
x
(x-1)^2*(x+1)/4

One more basis function ψ_1 for point 1:
$\psi_1(-1) = 0$, $\psi_1'(-1) =$, $\psi_1(1) = 0$, $\psi_1'(1) = 1$.
Therefore

$$\psi_1(x) = \frac{(x-1)(x+1)^2}{4}$$

Its derivative: $\psi_1'(x) = ((x+1) * (3 * x - 1))/4$


0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-1
-0.5
0
0.5
1
x
(x-1)*(x+1)^2/4

Let u be a function and at the nodes x_i fulfilled the conditions

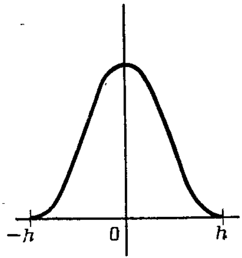
$$s(x_i) = u(x_i), s'(x_i) = u'(x_i). \quad (3)$$

then the following theorem holds.

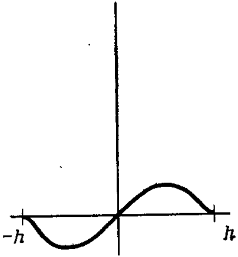
Theorem

Estimates are valid

$$||s^{\nu} - u^{\nu}||_{\infty} \leq K_{\nu} h^{4-\nu} ||u^4||_{\infty}, \quad \nu = 0, 1, 2, 3.$$



Φ_{2i}



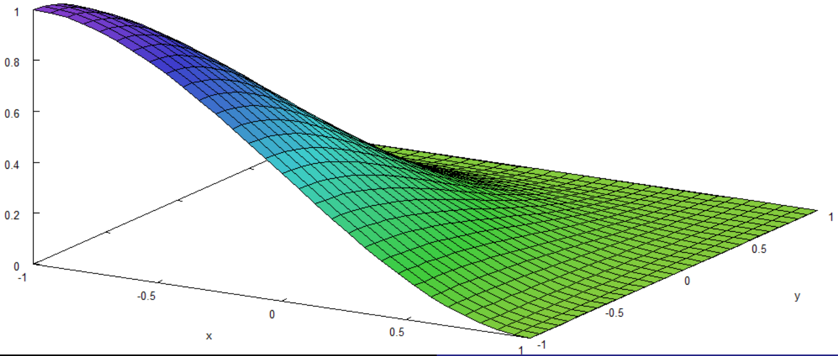
Φ_{2i+1}

Bogner-Fox-Schmit element

Bogner-Fox-Schmit element has 16 degrees of freedom and is defined at each vertex z_i of square $\Omega = [-1,1] \times [-1,1]$ by four basis functions φ_i, φ_{xi}, φ_{yi}, φ_{xyi}.
Basis functions φ_i take value 1 at node z_i: $\varphi_i(z_i) = 1$ and 0 at rest nodes: $\varphi_i(z_j) = 0$, $i \neq j$. The values of first derivatives and mixed derivative are equal to 0 at all nodes.
First derivatives along x of basis functions φ_{xi} take value 1 at node z_i, and 0 at rest nodes: $\partial\varphi_{xi}(z_j)/\partial x = 0$, if $i \neq j$:
$\partial\varphi_{xi}(z_j)/\partial y = 0$,
$\partial^2\varphi_{xi}(z_j)/\partial x\partial y = 0$, $i,j = 1,2,3,4$.
It is the same is valid for φ_{yi}, φ_{xyi}:
$\partial^2\varphi_{xyi}(z_i)/\partial x\partial y = 1$,
$\partial^2\varphi_{xyi}(z_j)/\partial x\partial y = 0$, $i \neq j$;
$\partial\varphi_{xyi}(z_j)/\partial x = 0$, $\partial\varphi_{xi}(z_i)/\partial y = 0$, $i,j = 1,2,3,4$.

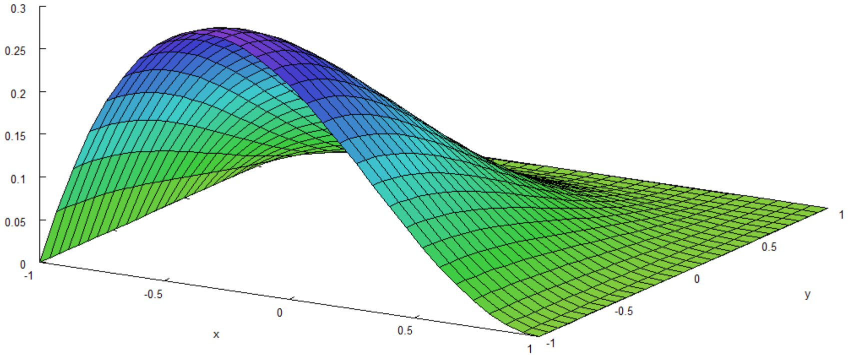
Bogner-Fox-Schmit element

Basis functions for Bogner-Fox-Schmit element are constructed with the help of one-dimensional basis functions for Hermitian element.
Enumerate vertices of $\Omega = [-1,1] \times [-1,1]$: $z_1 = (-1,-1)$, $z_2 = (1,-1)$, $z_3 = (1,1)$, $z_4 = (-1,1)$.
Then for node z_1 we have the following.
$\varphi_1(x,y) = \phi_{-1}(x)\phi_{-1}(y)$:



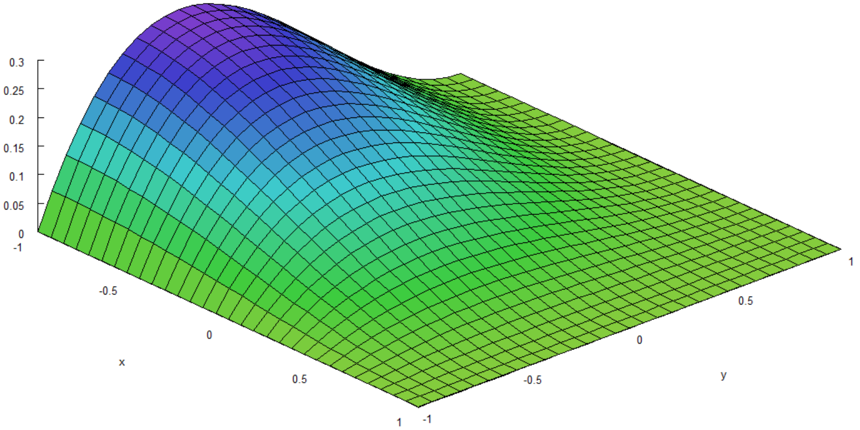
Bogner-Fox-Schmit element

$\varphi_{x1}(x, y) = \psi_{-1}(x)\phi_{-1}(y)$:



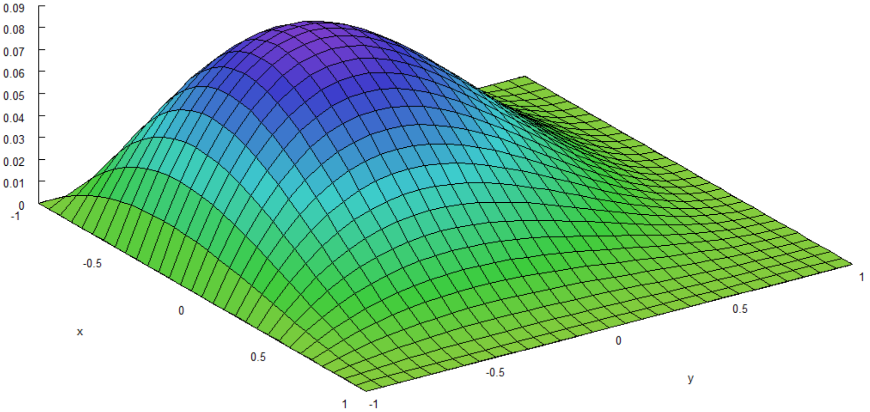
Bogner-Fox-Schmit element

$\varphi_{y1}(x, y) = \phi_{-1}(x)\psi_{-1}(y)$:



Bogner-Fox-Schmit element

$\varphi_{xy1}(x, y) = \psi_{-1}(x)\psi_{-1}(y)$:



Enumerate all 16 basis functions of Bogner-Fox-Schmit element by following way: $\Phi_1(x,y) = \varphi_1(x,y)$, $\Phi_2(x,y) = \varphi_{x1}(x,y)$, $\Phi_3(x,y) = \varphi_{y1}(x,y)$, $\Phi_4(x,y) = \varphi_{xy1}(x,y)$, $\Phi_5(x,y) = \varphi_2(x,y)$, $\Phi_6(x,y) = \varphi_{x2}(x,y)$, $\Phi_7(x,y) = \varphi_{y2}(x,y)$, $\Phi_8(x,y) = \varphi_{xy2}(x,y)$, $\Phi_9(x,y) = \varphi_3(x,y)$, $\Phi_{10}(x,y) = \varphi_{x3}(x,y)$, $\Phi_{11}(x,y) = \varphi_{y3}(x,y)$, $\Phi_{12}(x,y) = \varphi_{xy3}(x,y)$, $\Phi_{13}(x,y) = \varphi_4(x,y)$, $\Phi_{14}(x,y) = \varphi_{x4}(x,y)$, $\Phi_{15}(x,y) = \varphi_{y4}(x,y)$, $\Phi_{16}(x,y) = \varphi_{xy4}(x,y)$.

Hermit splines 5 order

We want to construct a function $s \in C^2$ and In each node of the mesh $\omega_h = \{x_i\}$ values $s(x_i), s'(x_i), s''(x_i)$ are known.
At each interval $[x_i, x_{i+1}]$ spline is polynomial 5 order
$s = a_0 + a_1 x + a_2 x^2 + ... + a_5 x^5$ and can be written as

$$s(x) = \varphi_1(t)s(x_i) + \varphi_2(t)s(x_{i+1}) + h\varphi_1^1(t)s'(x_i) + h\varphi_2^1(t)s'(x_{i+1}) +$$

$$+h^2\varphi_1^2(t)s''(x_i) + h^2\varphi_2^2(t)s''(x_{i+1}),$$

h is mesh size.

Hermit splines 5 order

Let u be a function and at the nodes x_i fulfilled the conditions

$$s(x_i) = u(x_i), s'(x_i) = u'(x_i), s''(x_i) = u''(x_i). \tag{4}$$

then the following theorem holds.

Theorem

Estimates are valid

$$||s^\nu - u^\nu||_\infty \leq K_\nu h^{6-\nu} ||u^6||_\infty, \quad \nu = 0, 1, 2, 3.\Box$$

Galerkin formulation

For Laplace operator, Galerkin formulation is written as follows:

$$\mathcal{L}(u,v) = \int_{\Omega} u'_x v'_x + u'_y v'_y d\Omega = (f,v), \forall v \in H_0^1.$$

System of linear algebraic equations to find vector u^h has the form:

$$Au^h = F,$$

$A = (a_{ij}),\ F = (F_i), i,j = 1,...,N.$

$$a_{ij} = \mathcal{L}(\Psi_i, \Psi_j), F_i = (f, \Psi_i), i,j = 1,...,N.$$

Theorem [Streng, Fix 1973]

$$||u - u^h||_s \leq Ch^{4-s}||u||_4, \quad s \geq -2,$$

$$||u - u^h||_s \leq Ch^6||u||_4, \quad s \leq -2.$$

Negative norm

$s \leq 0$

$$||u||_s = \max_g \frac{|(g,u)|}{||g||_{-s}}.$$

So, if $s = -1$ and $v \equiv 1$

$$||u - u^h||_{-1} = \max_v \frac{|\int_\Omega (u - u^h) v dx|}{||v||_1} \geq \frac{|\int_\Omega (u - u^h) dx|}{(mes \ \ \Omega)^{1/2}}.$$

Remark

The residual is orthogonal to any function in the space of Hermitian elements S^h

$$(L(u-u^h),\phi)=0,\forall\phi\in S^h.$$

$$((u-u^h),L^*\phi)=0,\forall\phi\in S^h,$$

where L^* is the adjoint operator of L.
So

$$((u-u^h),w)=0,\forall w\in LS^h.$$

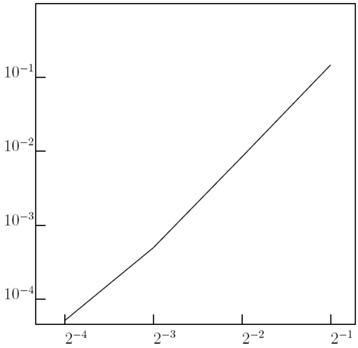
where $LS^h=\{\psi|\psi=L\phi,\phi\in S^h\}$.

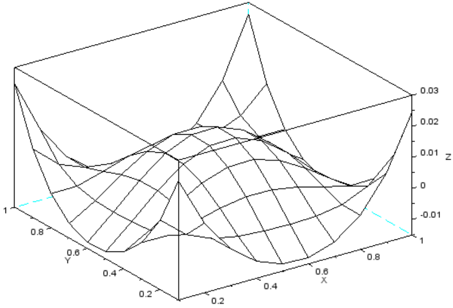
$$(u - u^h, f) = (u - u^h, f - f_{LS}),$$

$\forall f_{LS} \in LS^h$

Approximation properties of LS^h

Numerical example with $f = \sin(\pi x)\sin(\pi y)$ on $[0,1]^2$. Chosen f_{LS} as the approximation the least squares method.





$$||f - f_{LS}|| \leq Kh^{3.5}.$$

Postprocessing

Let s be a polynomial $s = \sum_{l=0}^{n} a_l \psi_l (x - x_0)$ such that

$$||s - u^h||^2_{L_2(\Omega')} + \alpha ||Ls - f||^2_{L_2(\Omega')} \to \min, \quad (5)$$

where ψ_i are given linearly independent functions, Ω' is the cell of grid.
The problem (5) is reduced to the solution of the system of linear algebraic equations

$$Ba = d.$$

So, s is postprocessing solution.

Error estimate $u - s$

Let u_l be solution of problem

$$||u - u_l||^2_{L_2(\Omega')} + \alpha||Lu_l - f||^2_{L_2(\Omega')} \to \min, \tag{6}$$

where u_l similar to the function s and

$$v = s - u_l.$$

Rewrite

$$||u^h - u - v||^2_{L_2(\Omega')} + \alpha||Lv||^2_{L_2(\Omega')} + ||u - u_I||^2_{L_2(\Omega')} + \alpha||Lu_I - f||^2_{L_2(\Omega')} \to \min,$$

Hence

$$||u^h - u - v||^2_{L_2(\Omega')} + \alpha||Lv||^2_{L_2(\Omega')} \to \min .$$

Represent $v = v_1 + v_2$ where

$$Lv_1 \equiv 0.$$

Then

$$||u^h - u - v_1||^2_{L_2(\Omega')} \to \min .$$

$$||u^h - u - v_1 - v_2||^2_{L_2(\Omega')} + \alpha||Lv_2||^2_{L_2(\Omega')} \to \min .$$

$$||v_1|| \gg ||v_2||.$$

Represent v_1 in the form $v_1 = a_0 + a_1 x + a_2 y + a_3 xy$. Estimate a_0. It is easy to see that

$$a_0 \sim (u^h - u, 1)/ \text{ mes } (\Omega') \leq ||u^h - u||||1 - f_{LS}|| \text{ mes } (\Omega')^{-1/2}$$

Finally

Error estimate

$$||u - s|| \sim ||u - u^h||||f - f_{SL}||.$$

Model Problem

$$-u'' = 12x^2 - 6x, x \in (0, 1),$$

$$u(0) = 0, u(1) = 0.$$

Exact solution

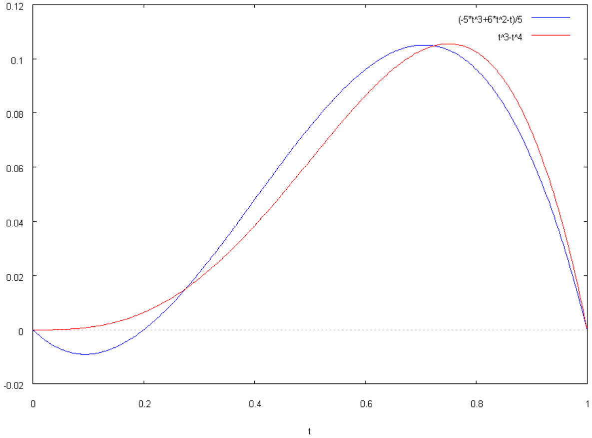
$$u = (1 - x)x^3.$$

In simplest case without dividing interval (0,1) we shall find solution in the form

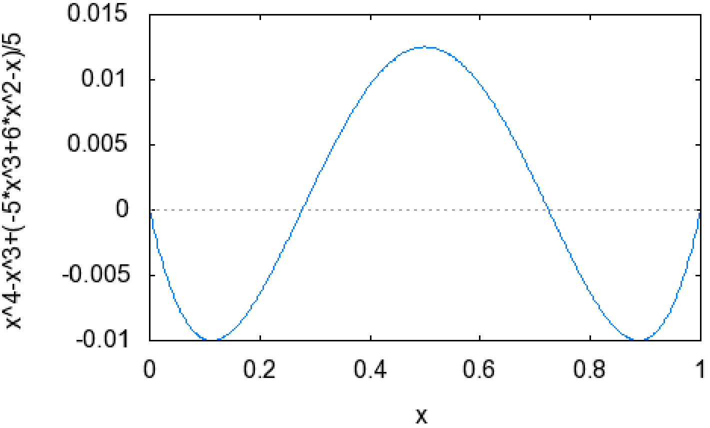
$$u^h = \sum_{i=1}^{2} u_i \Phi_i(x, y).$$

Vector of solution is $(-0.2, -0.8)$

Comparison of the exact and FEM solutions



Error



$$(u - u^h, 1) = 0, \quad (u - u^h, x) = 0.$$

In this case postprocessing solution s is equal to exact solution u.

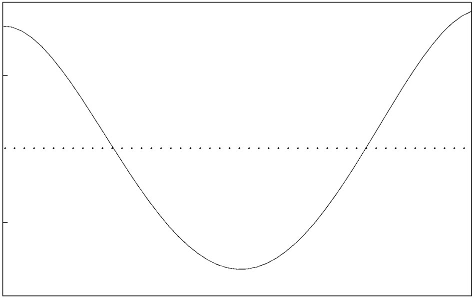
Numerical Example

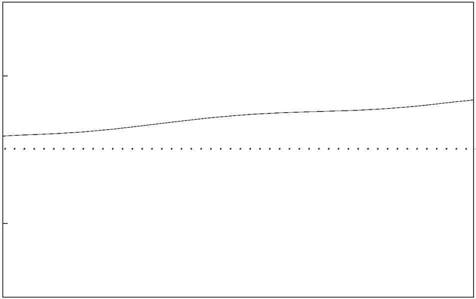
$$-u'' + u = (\pi^2 + 1)\sin(\pi x), x \in (0,1),$$
$$u(0) = 0, u(1) = 0.$$

Exact solution

$$u = \sin(\pi x).$$

h	$\|\|u-u^h\|\|_{L_2(\Omega')}$	$\|\|u-s\|\|_{L_2(\Omega')}$	$\int_{\Omega'} u-u^h dx$
1/2	0.0023703044	0.0000726709	-0.0000264585
1/4	0.0002649516	0.0000067943	-0.0000070038
1/6	0.0000599402	0.0000017142	-0.0000017577
1/8	0.0000199439	0.0000005954	-0.0000006048





$u-u^h$ in the cell, $u-s$ in the cell

Example

$$-\Delta u = f, x \in \Omega = [-1,1] \times [-1,1],$$

$$u = 0, x \in \partial\Omega.$$

Let

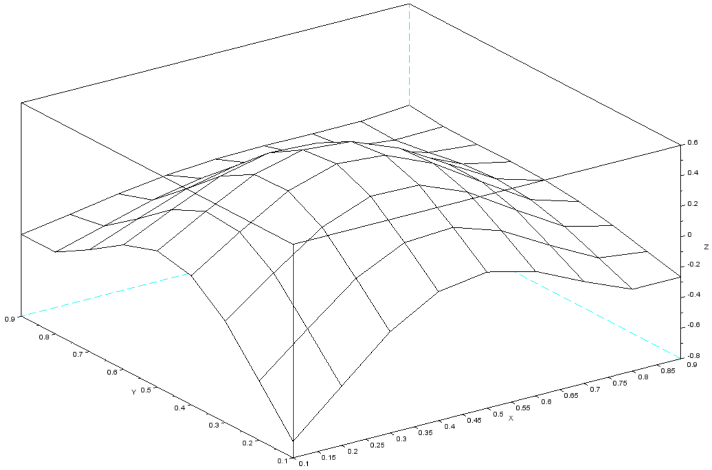
$$f(x,y) = 0.5\pi^2 \cos(\pi x/2) cos(\pi y/2),$$

Then exact solution is

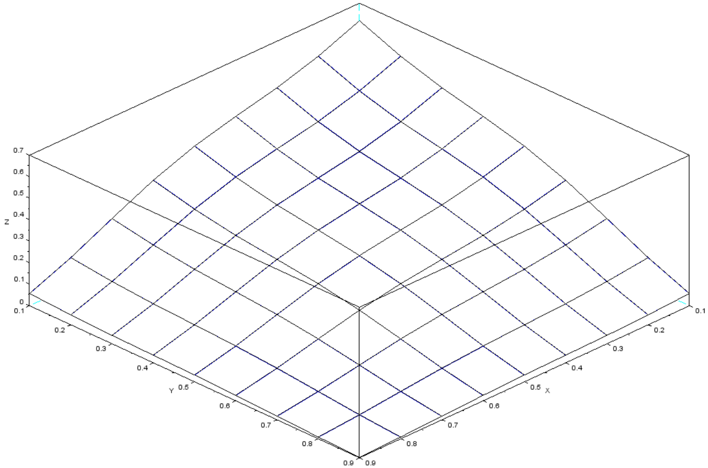
$$u(x,y) = \cos(\pi x/2)\cos(\pi y/2).$$

In two dimensions case, post-processing has been very sensitive to the form of polynomials. So using the full polynomial of fifth degree reduced the error in the L_2 norm only 2–3 times. To improve the properties of post-processing used Hermite polynomials bi fifth degree. Post-processing results are shown in Table.

h	$\|\|u - u^h\|\|_{L_2}$	$\|\|u - s\|\|_{L_2}$
0.5	0.00436993	0.00045358
0.25	0.00027572	0.00002087
0.125	0.00001746	0.00000103


0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
Z
0.9
0.8
0.7
0.6
Y 0.5
0.4
0.3
0.2
0.1
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
X
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9

The behavior of Post Processing errors in a grid cell



Defect Correction Method for FEM

Defect Correction method is the common name of special discrete Newton's method. Consider the defect correction method for linear elliptic problem. We will use the finite element method with cubic elements, but the final solution will have accuracy corresponding elements 5 order.

Further, using the finite element solution u^h and p we construct s and

$$s(x_i) = p(x_i), s'(x_i) = p'(x_i), s''(x_i) = p''(x_i). \tag{7}$$

$$\mathcal{L}(u, v) = (f, v), \forall v \in H_0^1. \tag{8}$$

Consider the identity

$$\mathcal{L}(s, v) = (f, v) - \varphi, \forall v \in H_0^1(\Omega), \tag{9}$$

where $\varphi \equiv (f, v) - \mathcal{L}(s, v)$. Subtracting (9) from (8) we have:

$$\mathcal{L}(u - s, v) = \varphi, \forall v \in H_0^1(\Omega). \tag{10}$$

Denoting $\varepsilon = u - s$ we write the equation for ε in the weak form

$$\mathcal{L}(\varepsilon, v) = \varphi, \forall v \in H_0^1(\Omega). \tag{11}$$

Corrected Solution

Further, let ε^h be an approximate solution to the problem (11) by Hermitian bicubic finite element method. Then the corrected solution is

$$s_{cor} = s + \varepsilon^h.$$

Conclusion

Demonstrated the efficiency of the post-processing to improve the accuracy of Hermitian FEM. Presented post-processing can be used for a posteriori estimates of the error of the solutions hermitian FEM.