On Modifications of the Navier–Stokes Equations

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Let Ω be a bounded domain with Lipshitz boundary in R^3.

We use the following notations: independent variables are denoted as $x = (x_1, x_2, x_3)$, $x' = (x_1, x_2)$; the variable x_3 is denoted as z also. In the space of vector functions, introduce the norms:

$$\|\mathbf{f}\|^2 = \sum_{i=1}^{3} \int_{\Omega} f_i^2(x)dx, \qquad Q_T = \Omega \times [0,T],$$

$$\|\mathbf{f}_x\|^2 = \sum_{i,j=1}^{3} \int_{\Omega} \left(\frac{\partial f_i}{\partial x_j}\right)^2 dx,$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \qquad \partial_{x_i} = \frac{\partial}{\partial x_i}, \qquad \|\cdot\|_q = \|\cdot\|_{L_q}$$

The system of Navier–Stokes equations describing dynamics of incompressible viscous flow is of the form

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + \nu\partial_3^2\mathbf{u} + \nabla p + u_k\mathbf{u}_{x_k} = \mathbf{0}, \qquad \operatorname{div}\mathbf{u} = 0,$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \ \operatorname{div}\mathbf{u}_0 = 0, \qquad \mathbf{u}|_{\partial\Omega\times[0,T]} = \mathbf{0}. \tag{1}$$

The Leray hypothesis is formulated as follows: *For any $\nu > 0$, arbitrary smooth initial condition $\mathbf{u}_0$ and arbitrary time interval to prove existence and uniqueness of a solution $\mathbf{u} \in \mathbf{H}^1(Q_T)$ to (1).*

In 1966 Ladyzhenskaya suggested modified equations

$$\mathbf{u}_t - \nu \mathrm{div}\,((1+\varepsilon \mathbf{u}_x^2)\nabla \mathbf{u}) + \nabla p + u_k \mathbf{u}_{x_k} = \mathbf{0},$$

$$\mathrm{div}\,\mathbf{u} = 0,\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{u}|_{\partial\Omega\times[0,T]} = \mathbf{0}. \tag{1'}$$

For this problem the Leray hypothesis is valid.

In practice, there are problems when a viscosity coefficient is different in various directions. In particular, we consider the case when the viscosity coefficient in horizontal direction equals ν, while in the vertical direction z it equals

$\mu \geq \nu$. In this case equations (1) take the form

$$\mathbf{u}_t - \nu\Delta'\mathbf{u} + \mu\frac{\partial^2\mathbf{u}}{\partial z^2} + \nabla p + u_k\mathbf{u}_{x_k} = 0, \quad \operatorname{div}\mathbf{u} = 0,$$

$$\mathbf{u}(x,0) = \mathbf{u}^0(x),\ \operatorname{div}\mathbf{u}^0 = 0, \quad \mathbf{u}|_{\partial\Omega\times[0,T]} = 0. \tag{2}$$

Consider the solvability "in the large" problem for (2). The following theorem holds:

Theorem 1. *For any sufficiently smooth initial condition* $\mathbf{u}_0$*, any* $\nu > 0$ *and arbitrary time interval* $[0,T]$ *there is* $\mu > 0$ *such that there exists a solution to (2) "in the large", i.e. there exists* $\mathbf{u} \in \mathbf{H}^2(Q_T)$ *satisfying (2) in a week sense and the norm* $\|\mathbf{u}_x\|$ *is continuous in time on* $[0,T]$*. Moreover, in this case the following inequality holds*

$$\|\mathbf{u}_t(t)\| \leq \|\mathbf{u}_t(0)\| \quad \forall t > 0.$$

Proof. To prove the theorem we use the Ladyzhenskaya inequality

$$\|f\|_4^4 \leq c_1 \|f_{x_1}\| \, \|f_{x_2}\| \, \|f_{x_3}\| \, \|f\|. \tag{3}$$

being valid for any $f \in H_0^1(\Omega)$.

Take scalar product in $\mathbf{L}_2$ the first equation of (2) and $\mathbf{u}$. As a result, we have

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \nu\|\mathbf{u}_{x'}\|^2 + \mu\|\mathbf{u}_z\|^2 = 0; \tag{4}$$

integration in time gives

$$\|\mathbf{u}(t)\| \leq \|\mathbf{u}^0\| \equiv M. \tag{5}$$

From (4) and (5) one gets

$$\nu\|\mathbf{u}_{x'}\|^2 + \mu\|\mathbf{u}_z\|^2 \leq M\|\mathbf{u}_t\|. \tag{6}$$

Differentiate (2) in t:

$$\mathbf{u}_{tt} - \nu\Delta\mathbf{u}_t + \nu\partial_3^2\mathbf{u}_t + \nabla p_t + u_k\mathbf{u}_{tx_k} + u_{kt}\mathbf{u}_{x_k} = 0,$$

$$\operatorname{div}\mathbf{u}_t = 0. \tag{7}$$

Take now a scalar product of the first equation (7) and $\mathbf{u}_t$:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2 + \nu\|\mathbf{u}_{tx'}\|^2 + \mu\|\mathbf{u}_{tz}\|^2 + (u_{kt}\mathbf{u}_{x_k}, \mathbf{u}_t) = 0. \tag{8}$$

Estimating the scalar product in (8) with the help of Hölder inequality and (6), we have

$$|(u_{kt}\mathbf{u}_{x_k}, \mathbf{u}_t)| = |(u_{kt}\mathbf{u}, \mathbf{u}_{tx_k})|$$
$$\leq c\|\mathbf{u}_{tx}\|^{7/4}\|\mathbf{u}_t\|^{1/4}\|\mathbf{u}_{x_1}\|^{1/4}\|\mathbf{u}_{x_2}\|^{1/4}\|\mathbf{u}_z\|^{1/4}\|\mathbf{u}\|^{1/4}$$
$$\leq c\left(\frac{M^3}{\nu^2\mu}\right)^{1/8}\|\mathbf{u}_{tx}\|^{7/4}\|\mathbf{u}_t\|^{5/8}$$
$$\leq \frac{\nu}{2}\|\mathbf{u}_{tx}\|^2 + \frac{cM^3}{\nu^9\mu}\|\mathbf{u}_t\|^5. \tag{9}$$

Substitute (9) into (8):

$$\frac{d}{dt}\|\mathbf{u}_t\|^2 + \nu\|\mathbf{u}_{tx'}\|^2 + \mu\|\mathbf{u}_{tz}\|^2 - \frac{cM^3}{\nu^9\mu}\|\mathbf{u}_t\|^5 \leq 0$$

from what follows

$$\frac{d}{dt}\|\mathbf{u}_t\|^2 + \nu\|\mathbf{u}_{tx'}\|^2 + \left(\mu - \frac{cM^3}{\nu^9\mu}\|\mathbf{u}_t\|^3\right)\|\mathbf{u}_{tz}\|^2 \leq 0. \tag{10}$$

It is obvious that the norm $\|\mathbf{u}_t(0)\|$ depends on the norm $\|\mathbf{u}_0\|_{\mathbf{H}^2}$. Now from (10) it follows that for any $\nu > 0$ and arbitrary $\|\mathbf{u}_t(0)\|$ depending on the norm of initial condition $\|\mathbf{u}_0\|_{\mathbf{H}^2}$ there exists $\mu > 0$ such that $\mu - \frac{cM^3}{\nu^9\mu}\|\mathbf{u}_t(0)\|^3 > 0$. Then from (10) we conclude that the norm $\|\mathbf{u}_t(t)\|$ satisfies the inequality

$$\|\mathbf{u}_t(t)\| \leq \|\mathbf{u}_t(0)\| \quad \forall t > 0. \tag{11}$$

The proof of existence and uniqueness of a solution "in the large" with the help of estimate (11) may be obtained in the same way as in [**?**]. The proof is completed.

Consider anisotropic modification of (1):

$$u_t^i - \nu \operatorname{div}\,(1+\varepsilon|u_x^i|^2)\nabla u^i + p_{x_i} + u^k u_{x_k}^i = 0,\ i = 1,2,$$

$$u_t^3 - \nu\Delta u^3 + p_{x_3} + u^k u_{x_k}^3 = 0,$$

$$\operatorname{div}\mathbf{u} = 0. \tag{12}$$

Let us obtain a proper a priori estimate for a solution to (12).

1. Take a scalar product of (12) and $\mathbf{u}$:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \nu\|\mathbf{u}_x\|^2 + \nu\varepsilon\sum_{i=1}^{2}\|u_x^i\|^2 = 0. \tag{13}$$

Integration in t gives

$$\max_t \|\mathbf{u}(t)\| \le \|\mathbf{u}_0\| = M, \tag{14}$$

$$\nu\int_0^\infty \left(\|\mathbf{u}_x\|^2 + \varepsilon\sum_{i=1}^{2}\|u_x^i\|_4^4\right)dt \le M^2. \tag{15}$$

2. Take a scalar product of (12) and $\mathbf{u}_t$:

$$\|\mathbf{u}_t\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\mathbf{u}_x\|^2 + \frac{\nu\varepsilon}{2}\sum_{i=1}^{2}\|u_x^i\|_4^4$$

$$+\sum_{i=1}^{2}(u_k u_{x_k}^i, u_t^i) + (u_k u_{x_k}^3, u_t^3) = 0. \tag{16}$$

Estimate scalar products from (16) using the Hölder and Ladyzhenskaya inequalities:

$$|(u_k u_{x_k}^i, u_t^i)| \le \{4,4,2\} \le \|u_k\|_4 \, \|u_x^i\|_4 \|u_t^i\|$$

$$\le \frac{1}{4}\|u_t^i\|^2 + c\|u_x^i\|_4^4,$$

$$|(u_3 u_{x_3}^3, u_t^3)| \le \{4,4,2\} \le \|u_3\|_4 \, \|u_z^3\|_4 \|u_t^3\|$$

$$\le \frac{1}{4}\|u_t^i\|^2 + c\|u_x^i\|_4^4,$$

$$|(u_i u^3_{x_i}, u^3_t)| \le \max |u_i| \cdot \|u^3_x\| \, \|u^3_t\|$$

$$\le \|u^i_x\|_4 \|u^3_x\| \, \|u^3_t\| \le \frac{1}{4}\|u^3_t\|^2 + c\|u^i_x\|^2_4 \|u^3_x\|^2.$$

Then from (16) one obtains

$$\|\mathbf{u}_t\|^2 + \frac{d}{dt}\|\mathbf{u}_x\|^2 + \sum_{i=1}^{2} \frac{d}{dt}\|u^i_x\|^4_4$$

$$\le \sum_{i=1}^{2} \|u^i_x\|^4_4 + \sum_{i=1}^{2} \|u^i_x\|^2_4 \|u^3_x\|^2.$$

Using the Gronwall inequality, from the last relation we get

$$\int_0^T \|\mathbf{u}_t\|^2 dt + \max_{0\le t\le T} \|\mathbf{u}_x\|^2 \le C_T, \qquad (17)$$

where the constant C_T depends on initial data of the problem. Estimate (17) provides the proof of existence and uniqueness of a solution to modified equations (12) “in the large”.