Time series analysis using persistent homology methods

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British–Russian Workshop

Mathematical and Computational Modelling in Cardiovascular Problems

Moscow, April 15–17, 2014

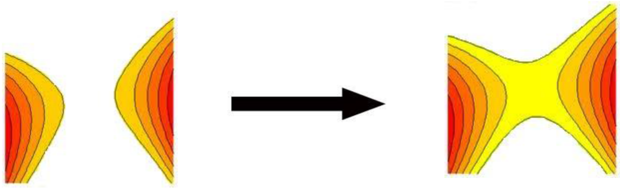
Processing and analysis of digital information

- How to analyse and compare digital data?

Definitions

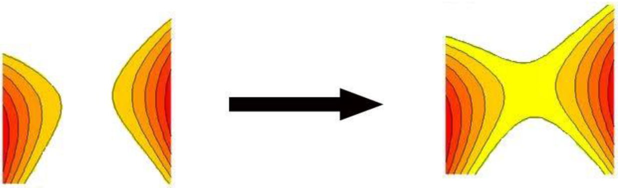
Introduce the following notation:

- X — topological space,
- $f : X \mapsto \mathbb{R}$ — continuous mapping
- $X_a = \{p \in X \mid f(p) \leqslant a\}\ \forall a \in \mathbb{R}$
- $C_a = C(X_a)\ \forall a \in \mathbb{R}$ — set of components of connectivity $X_a \subset X$
- $f_a^b : C_a \to C_b$ — mapping of the sets, induced by embedding $X_a \subset X_b$ $\forall a < b$.



Critical values

Value $a \in \mathbb{R}$ is called *critical* for function f, if for any sufficiently small $\varepsilon > 0$ the mapping $f_{a-\varepsilon}^{a+\varepsilon} : C_{a-\varepsilon} \to C_{a+\varepsilon}$ is not bijection.
Let $a_1, \ldots, a_n$ be all critical values of function f.
Denote by $C_i = C_{a_i}$ a set of connectivity components, $i = 1, \ldots, n$

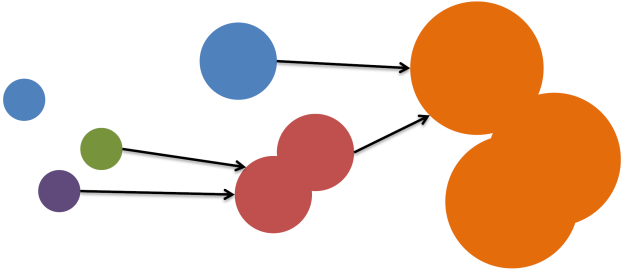


Merge tree

Construct graph Γ in the following way.

1. Set of vertices is $\cup_{i=1}^{n} C_i$.
2. Connect every vertex $c \in C_i$ with an edge to vertex $d = f_i(c) = f_{a_i}^{a_{i+1}}(c)$.

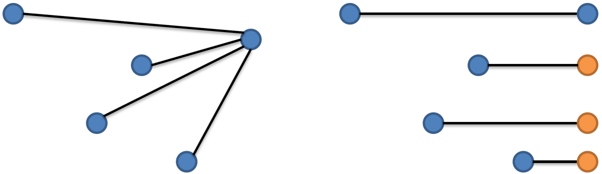
Graph Γ is a tree, which we call a *merge tree*.



Barcode. Persistent diagram

For every $Y \in C_i$ we define $f(Y) = a_i$ and $w(Y) = inf_{p \in Y} f(p)$.
Construct a graph Γ' from graph Γ in the following way.

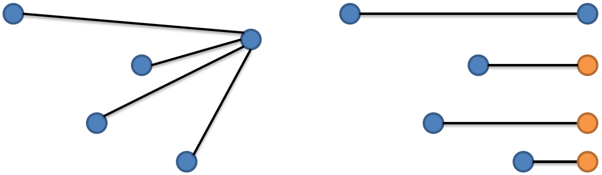
1. Let $d \in C_{i+1}$ be a vertex of Γ and $f_i^{-1}(d) = \{c_1, c_2, \ldots, c_k\} \subset C_i$.
2. Renumber vertices $c_j, j = 1, \ldots, k$ such that $w(c_1) \geq w(c_2) \geq \ldots \geq w(c_k)$.
3. Disjoin every edge $c_j d$, $j = 2, \ldots, k$ from graph Γ, and add a new vertex d_j, which will be the end of that edge.



Definition

A set B of all such intervals $[c, d)$ is called *barcode* of function f on X.

Persistent diagram of function f is a set $D(f)$ of points $(c, d) \in \mathbb{R}^2, [c, d) \in B$, united with the set of diagonal points $\Delta = \{(x, x) | x \in \mathbb{R}\}$.



Stability

Define a distance between two sets D_1 and D_2:

$$d_B(D_1, D_2) = \inf_{\gamma} \sup_{p \in D_1} \|p - \gamma(p)\|_\infty, \quad \gamma : D_1 \to D_2 \text{ — bijection.}$$

Theorem

Let X be a topological space, $f, g : X \to \mathbb{R}$. Then

$$d_B(D(f), D(g)) \leqslant \|f - g\|_\infty.$$

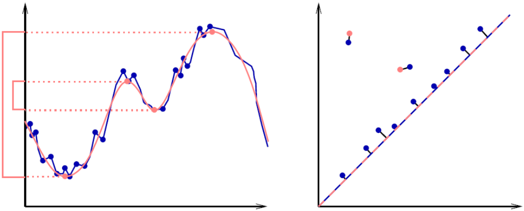
Thus, persistent diagram $D(f)$ is stable with respect to pertubations of the function f.

[D. Cohen-Steiner, H. Edelsbrunner, J. Harer, 2007]

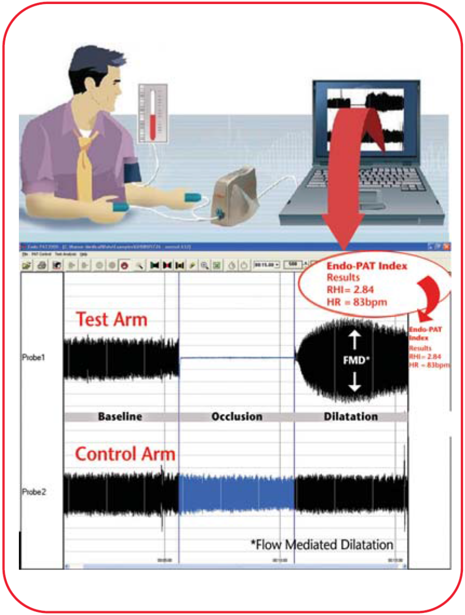
Stability

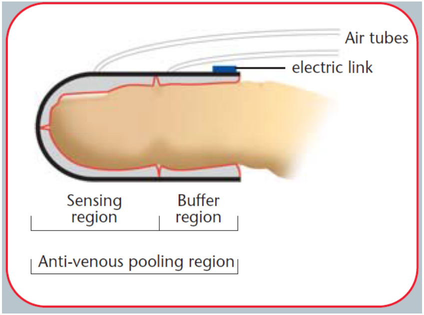
When the function f with diagram $D(f)$ is pertubed:

1. some points move over a short distance;
2. a number of points close to diagonal, moves onto the diagonal;
3. a part of points comes out from the diagonal.

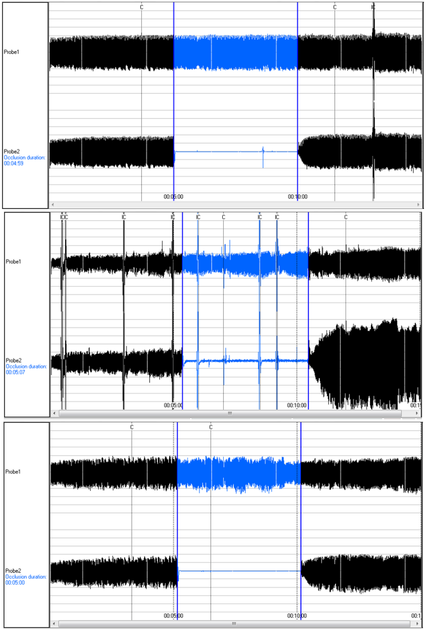


Endo-PAT2000

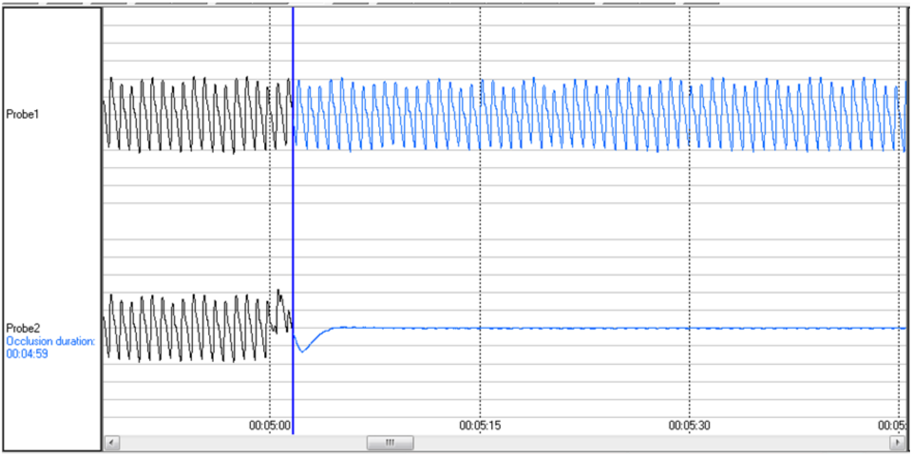




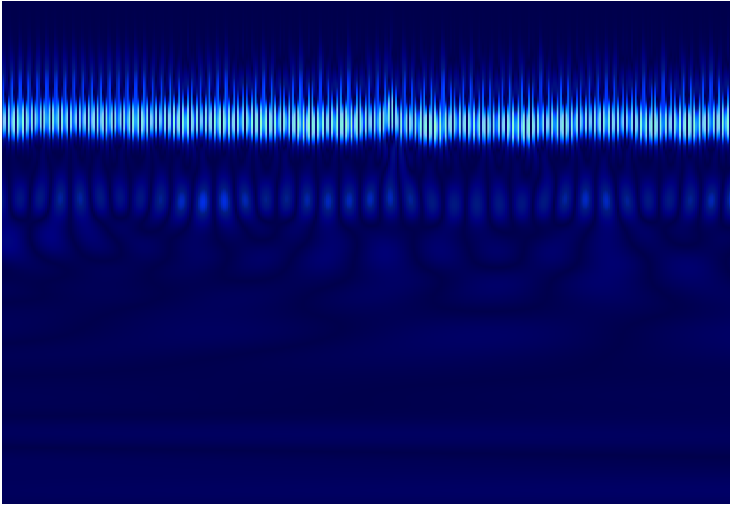
Endo-PAT2000 data



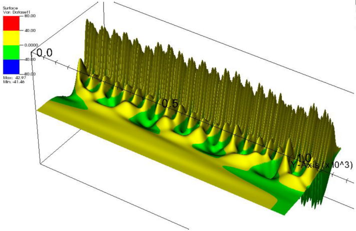
Endo-PAT2000 data



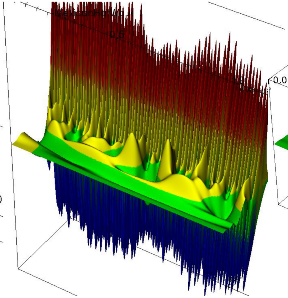
Wavelet scalograms



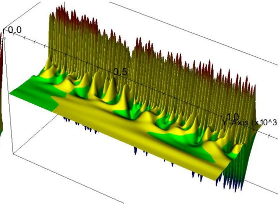
Wavelet scalograms



A

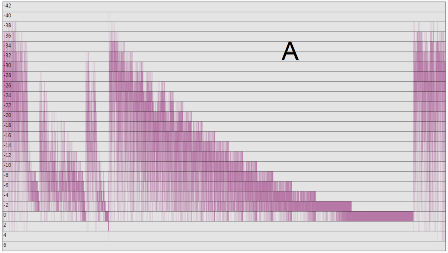


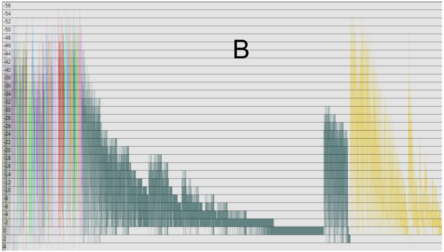
B

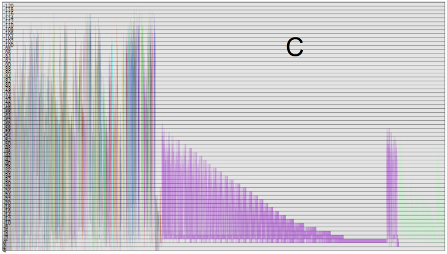


C

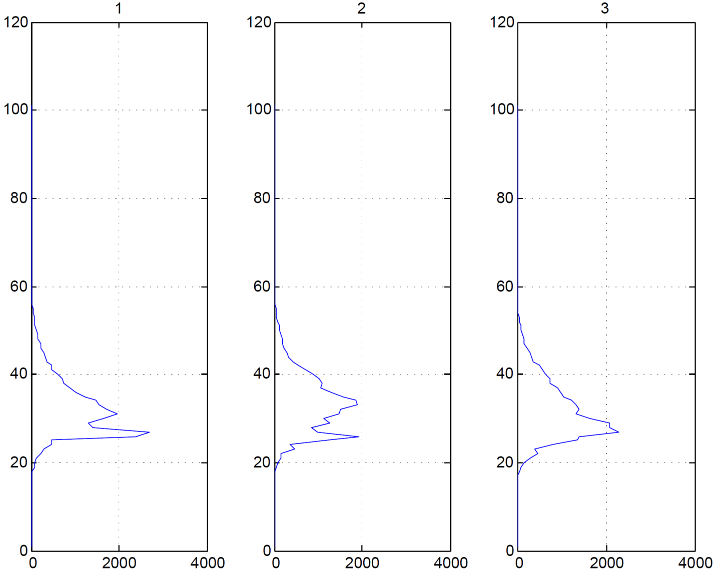
Barcodes


A
-42
-40
-38
-36
-34
-32
-30
-28
-26
-24
-22
-20
-18
-16
-14
-12
-10
-8
-6
-4
-2
0
2
4
6

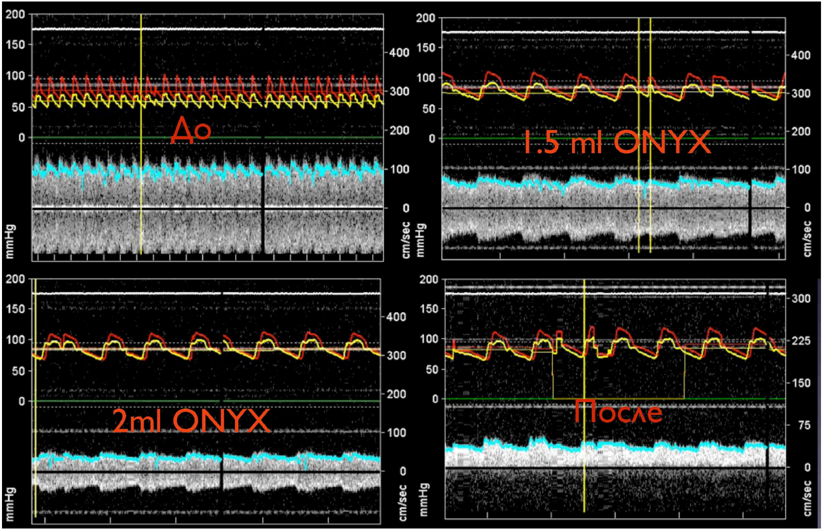

B
-56
-54
-52
-50
-48
-46
-44
-42
-40
-38
-36
-34
-32
-30
-28
-26
-24
-22
-20
-18
-16
-14
-12
-10
-8
-6
-4
-2
0
2
4


C

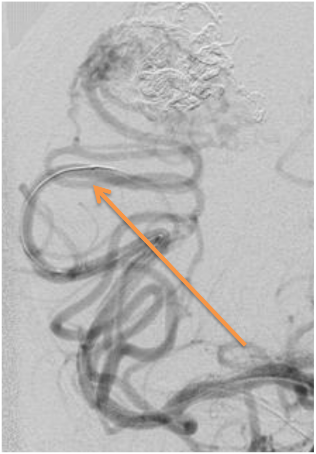
Barcodes. Distribution

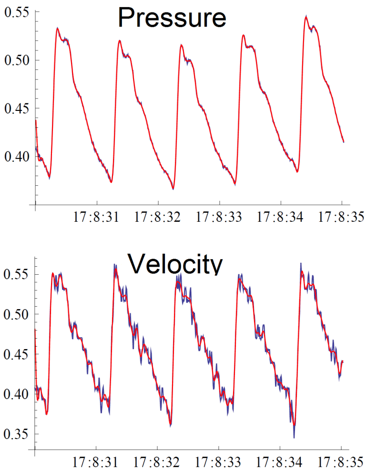


Endovascular measurements

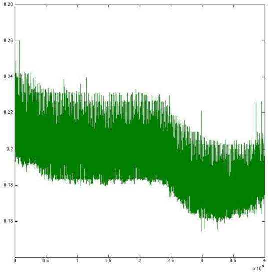


Measurement data

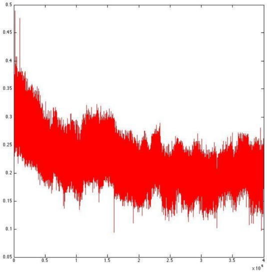




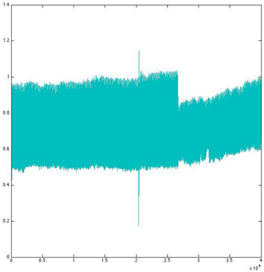
Data analysis



Pressure

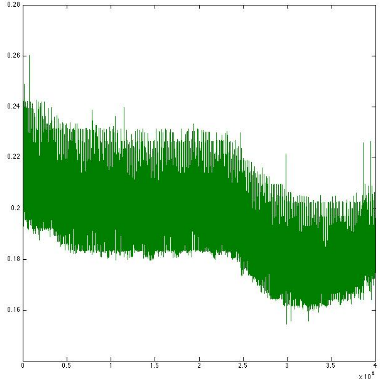


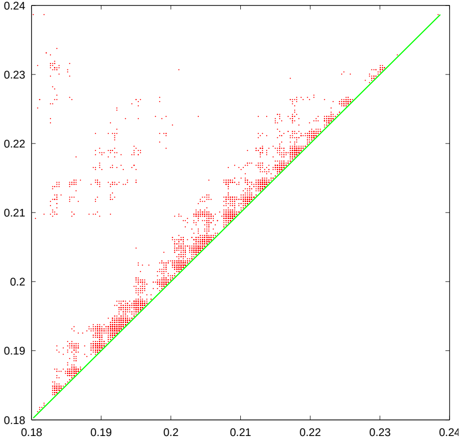
Velocity



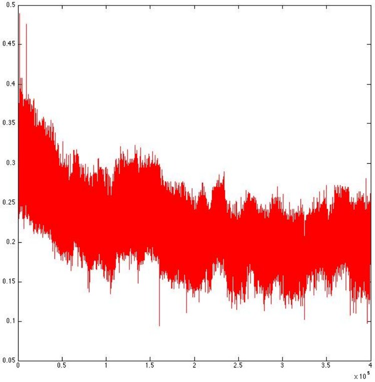
Systemic pressure

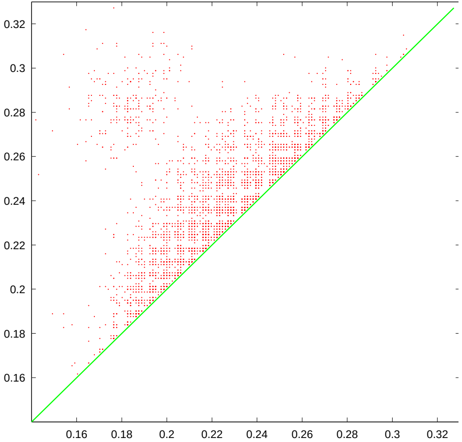
Persistent diagram: Pressure



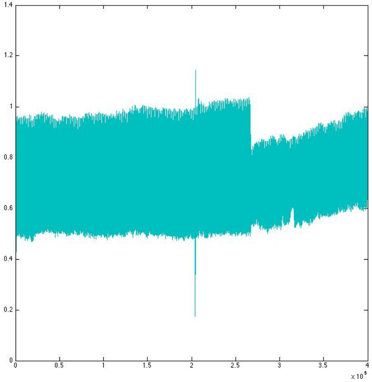


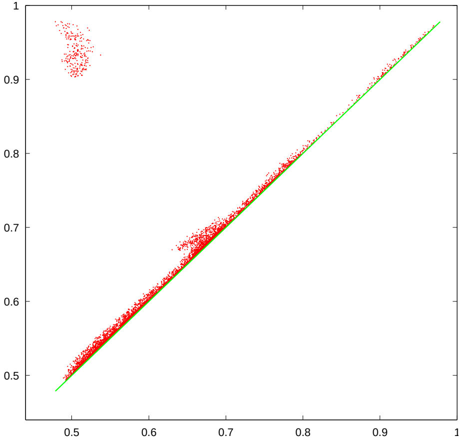
Persistent diagram: Velocity





Persistent diagram: Systemic pressure





Summary

- Methods of persistent diagrams were applied to medical data, obtained during examinations.
- Comparison of different study cases shows that barcodes and persistent diagrams can be used for data analysis.