# Delay epidemic models based on disease duration

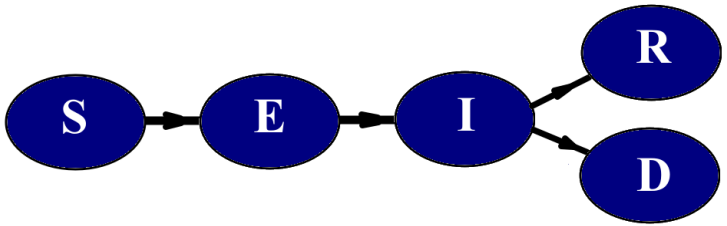
Masoud Saade

S.M. Nikol'skii Mathematical Institute, Peoples' Friendship University of Russia (RUDN University)

# Contents

# 1. An epidemic delay model determined by the infectivity and disease durations

## 1.1. Model with distributed parameters




$$\frac{dS(t)}{dt}=-J(t)=-\frac{\beta}{N}S(t)I(t), \quad (1a)$$

$$\frac{dE(t)}{dt}=J(t)-J(t-\tau_0), \quad (1b)$$

$$S(t)+E(t)+I(t)+R(t)+D(t)=N\,(=\text{constant}). \quad (0)$$

$\beta$ is disease transmisson rate, $\tau_0$ is the infectivity period.

$$\frac{dI(t)}{dt}=J(t-\tau_0)-\int_0^t \rho(t-\eta)J(\eta-\tau_0)d\eta-\int_0^t \mu(t-\eta)J(\eta-\tau_0)d\eta, \quad (1c)$$

$$\frac{dR(t)}{dt}=\int_0^t \rho(t-\eta)J(\eta-\tau_0)d\eta, \quad (1d)$$

$$\frac{dD(t)}{dt}=\int_0^t \mu(t-\eta)J(\eta-\tau_0)d\eta, \quad (1e)$$

where $\rho(t-\eta)$ and $\mu(t-\eta)$ are the recovery and death rates at time $t$ of the individuals became infectious at time $\eta$. $J(t)$ is the size of newly exposed at time $t$.

The model is completed with the initial conditions:

$$S(0)=N-I(0), I(0)>0, E(0)=R(0)=D(0)=0,$$

$$S(t)=N-I(t),\ \ I(t)\geq 0,\ \ E(t)=R(t)=D(t)=0,\ \ \forall t\in[-\tau_0,0). \quad (2)$$

**Theorem 1.** Solution of system (1) with conditions (2) exists and it is positive and unique.

## 1.2. Reduction to the ODE model

Assuming that recovery and death rates are uniformly distributed in time during disease duration

$$\rho(t-\eta)=\begin{cases}\rho_0 & :t-\tau_1\le\eta\le t\\ 0 & :\eta<t-\tau_1\end{cases},$$

$$\mu(t-\eta)=\begin{cases}\mu_0 & :t-\tau_1\le\eta\le t\\ 0 & :\eta<t-\tau_1\end{cases}, \quad (3)$$

where $\tau_1>0$ is the disease duration.

Assuming that $\psi=\frac{1}{\tau_0}$ , then system (1) can be reduced to the ODE model:

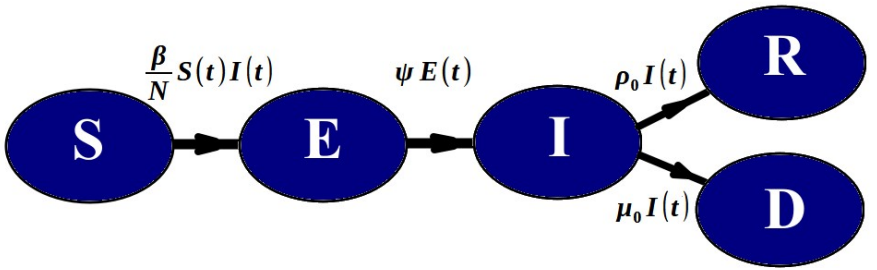
$$\frac{dS(t)}{dt} = -\frac{\beta}{N} S(t) I(t), \quad (4a)$$

$$\frac{dE(t)}{dt} = \frac{\beta}{N} S(t) I(t) - \psi E(t), \quad (4b)$$

$$\frac{dI(t)}{dt} = \psi E(t) - (\rho_0 + \mu_0) I(t), \quad (4c)$$

$$\frac{dR(t)}{dt} = \rho_0 I(t), \quad (4d)$$

$$\frac{dD(t)}{dt} = \mu_0 I(t), \quad (4e)$$




$$S(t) + E(t) + I(t) + R(t) + D(t) = N \,(= \text{constant}).$$

with the initial conditions

$$S(0) = N - I(0), I(0) > 0, E(0) = R(0) = D(0) = 0. \quad (5)$$

## 1.3. Reduction to the delay model

Assuming that the individuals $J(t-\tau_0-\tau_1)$ exposed at time $t-\tau_0-\tau_1$, recover or die at time $t$ with certain probabilities.

This assumption is consistent with the following choice of the functions $\rho$ and $\mu$ :

$$\rho(t-\eta)=\rho_1\delta(t-\tau_1-\eta)\,,\quad \mu(t-\eta)=\mu_1\delta(t-\tau_1-\eta),$$

where $\rho_1+\mu_1=1$ , and $\delta$ is the Dirac delta-function.

Hence, we obtain our delay model.

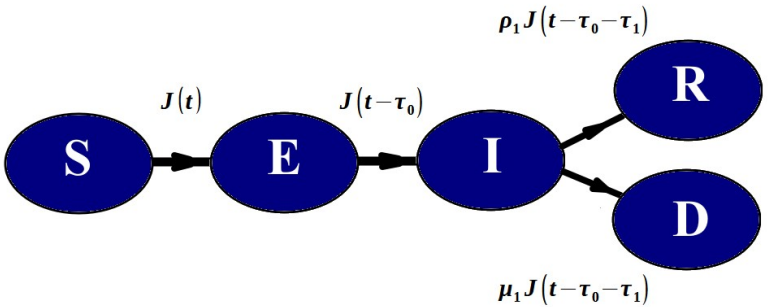
$$\frac{dS(t)}{dt} = -J(t) = -\frac{\beta}{N} S(t) I(t), \quad (6a)$$

$$\frac{dE(t)}{dt} = J(t) - J(t-\tau_0), \quad (6b)$$

$$\frac{dI(t)}{dt} = J(t-\tau_0) - J(t-\tau_0-\tau_1), \quad (6c)$$

$$\frac{dR(t)}{dt} = \rho_1 J(t-\tau_0-\tau_1), \quad (6d)$$

$$\frac{dD(t)}{dt} = \mu_1 J(t-\tau_0-\tau_1). \quad (6e)$$




$$S(t)+E(t)+I(t)+R(t)+D(t)=N\,(=\text{constant}).$$

The system is completed with the initial conditions

$$S(0)=N-I(0), I(0)>0, E(0)=R(0)=D(0)=0,$$

$$S(t)=N-I(t),\ I(t)\geq 0,\ E(t)=R(t)=D(t)=0,\ \forall t\in[-(\tau_0+\tau_1),0). \quad (7)$$

**Theorem 2.** Solution of system (6) with conditions (7) exists and it is positive and unique.

## 1.4. Epidemic characteristics for the delay model

### 1.4.1. Basic reproduction number

Describes the average number of secondary infections produced by a single infected individual in a completely susceptible populations.

$$\Re_0 = \beta \tau_1 \frac{S_0}{N}. \quad (8)$$

$$(S_0 \approx N \rightarrow \Re_0 = \beta\tau_1)$$

### 1.4.2. Final size of epidemic

The final size of susceptible compartment can be calculated from the formula

$$\ln(\alpha) = \Re_0(\alpha - 1), \quad (9)$$

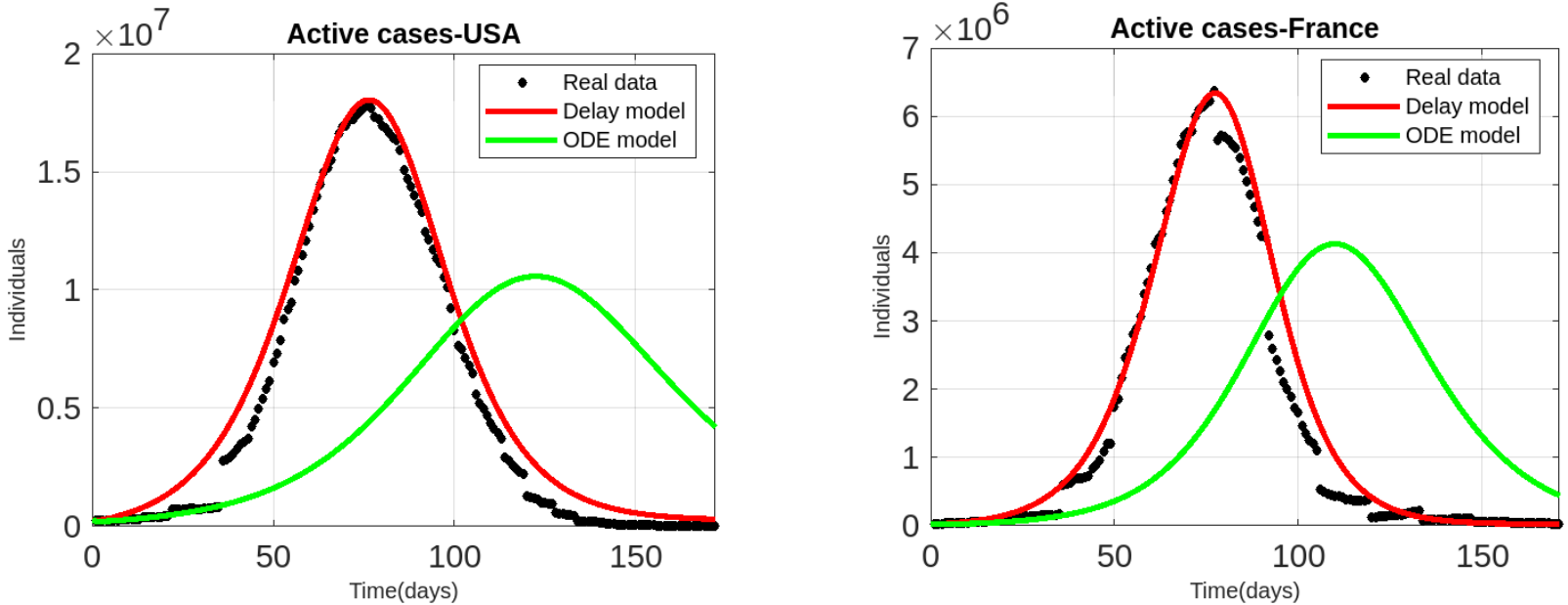
where $\alpha = \frac{S_f}{S_0}$.

Integrating (6d) and (6e) and taking the limits as $t \rightarrow \infty$ , we obtain the final size of recovered and dead populations:

$$R_f = \rho_1(S_0 - S_f), \quad D_f = \mu_1(S_0 - S_f). \quad (10)$$

## 1.5. Numerical simulation and model comparison

**Figure 1.**




Numerical simulations of the delay model and ODE model for different countries and for the period of time from November 15, 2021 to May 15, 2022.

Left:

$\beta=0.23, \tau_0=2, \tau_1=6\,(\Re_0=1.32), \rho_1=0.97, \mu_1=0.03,$

$N=331.9\times10^6, I(t<0)=130000, I(0)=257000.$

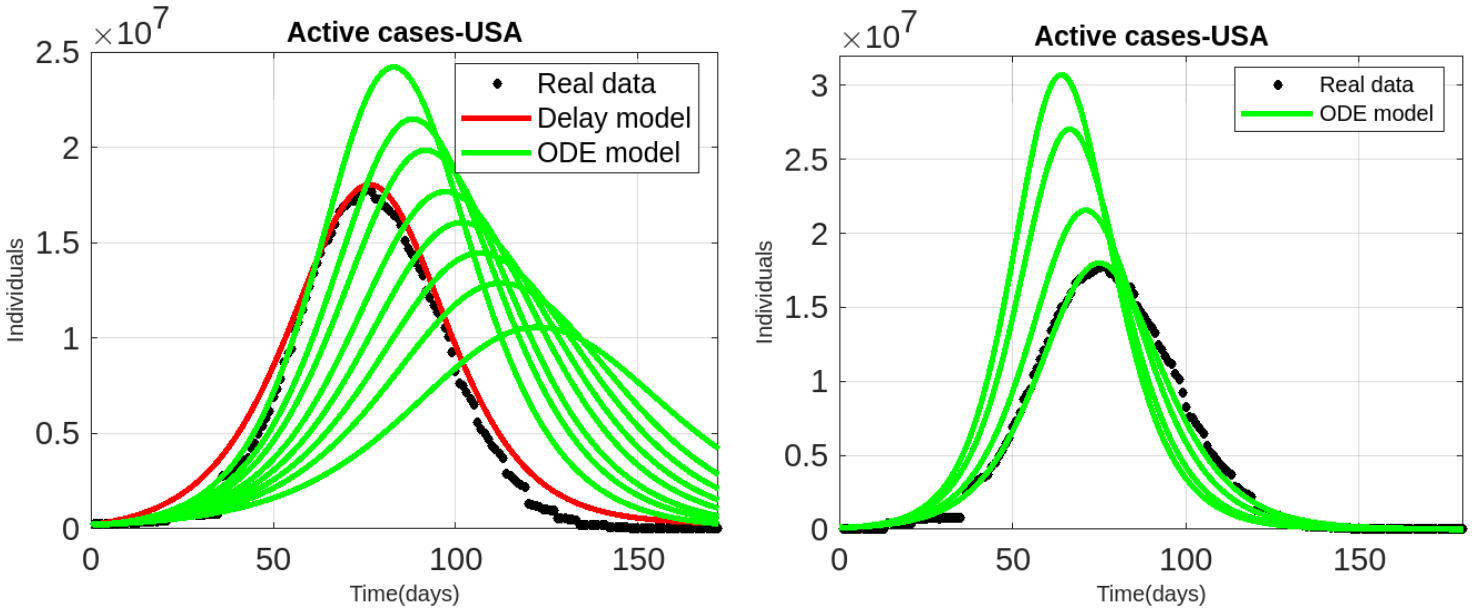
Right:

$\beta=0.276, \tau_0=3, \tau_1=6, \rho_1=0.97, \mu_1=0.03,$

$N=67.75\times10^6, I(t<0)=9000, I(0)=23550.$

$$\psi=1/\tau_0,\ \ \rho_0=1/\rho_1,\ \ \mu_0=1/\mu_1.$$

**Figure 2.**




Numerical simulations of the ODE model for USA, for the period of time from November 15, 2021 to May 15, 2022.

Left:

$\beta \in [0.23, 0.281], \tau_0 = 2, \tau_1 = 6, \rho_1 = 0.97, \mu_1 = 0.03$

$N = 331.9 \times 10^6, I(t<0) = 130000, I(0) = 257000$

Right:

$\beta = 0.4, \psi = 0.5, I(0) = 150000$

$\rho_0 + \mu_0 \in \{0.25 (\Re_0 = 1.6),\ 0.24,\ 0.22, 0.21\}$

# 2. An epidemic delay model determined by the disease and immunity durations

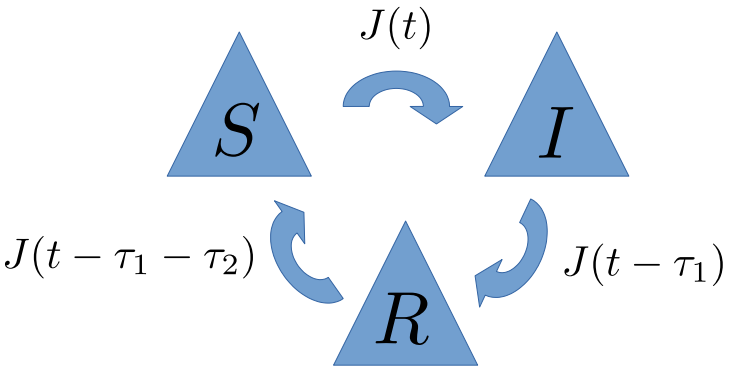
## 2.1. An Epidemic Delay Model with immunity waning

$$\frac{dS(t)}{dt} = -J(t) + J(t-\tau_1-\tau_2), \quad (11a)$$

$$\frac{dI(t)}{dt} = J(t) - J(t-\tau_1), \quad (11b)$$

$$\frac{dR(t)}{dt} = J(t-\tau_1) - J(t-\tau_1-\tau_2), \quad (11c)$$

$$J(t) = \frac{\beta}{N} S(t) I(t), \quad (11e)$$




where

$\tau_1$ is disease duration, $\tau_2$ is duration of natural immunity, $\beta$ is disease transmission rate.

System (11) is completed with the initial conditions:

$$S(t) = I(t) = R(t) = 0 : \forall t \in [-(\tau_1+\tau_2), 0), \; S(0) = N - I(0), I(0) > 0, R(0) = 0. \quad (12)$$

Note that

$$S(t) + I(t) + R(t) = N \quad : \forall t \in \mathbb{R}^+. \quad (13)$$

## 2.2. Integral Equation and Stationary Solutions

Integrating equations (11a) and (11b) from 0 to t, and substituting in (11e) we reduce system (11) to the following single integral equation

$$J(t)=\frac{\beta}{N}\left(S(0)-\int_{t-\tau_1-\tau_2}^{t} J(x)dx\right)\left(I(0)+\int_{t-\tau_1}^{t} J(x)dx\right). \quad (14)$$

Stationary solutions of this equation can be found from the following algebraic equation:

$$J_s=\frac{\beta}{N}\left(S(0)-(\tau_1+\tau_2)J_s\right)\left(I(0)+\tau_1 J_s\right). \quad (15)$$

The positive solution of this equation is given by the formula

$$J_s=\frac{-\left(\frac{N}{\beta}+(\tau_1+\tau_2)I(0)-\tau_1 S(0)\right)+\sqrt{\Delta}}{2\tau_1(\tau_1+\tau_2)}, \quad (16)$$

where

$$\Delta=\left(\frac{N}{\beta}+(\tau_1+\tau_2)I(0)-\tau_1 S(0)\right)^2+4S(0)I(0)\tau_1(\tau_1+\tau_2)>0. \quad (17)$$

If we consider $I(0)\approx 0$ and $S(0)\approx N$, then we find two approximate solutions

$$J_s=0 \quad , \quad J_s=\frac{N}{\beta\tau_1}\frac{\beta\tau_1-1}{\tau_1+\tau_2}. \quad (18)$$

Hence, there exists a positive stationary solution if $\Re_0=\beta\tau_1>1.$

In this case, we can determine the stationary values of susceptible, infected, and recovered as:

$$S_s=\frac{N}{\beta\tau_1}, \quad I_s=\frac{N}{\beta}\frac{\beta\tau_1-1}{\tau_1+\tau_2}, \quad R_s=N-S_s-I_s. \quad (19)$$

## 2.3. Stability of the Stationary Solution

Equation (14), linearized about the stationary solution, by setting $J(t)=J_s+\epsilon e^{\lambda t}$ and keeping the first-order terms with respect to $\epsilon$, has the following form

$$v(t)=-a_1\int_{t-\tau_1-\tau_2}^{t} v(x)dx+a_2\int_{t-\tau_1}^{t} v(x)dx, \quad (20)$$

where

$$a_1=\frac{\beta}{N}\left(I(0)+\frac{N}{\beta}\frac{\beta\tau_1-1}{\tau_1+\tau_2}\right), \quad a_2=\frac{\beta}{N}\left(S(0)-\frac{N}{\beta\tau_1}(\beta\tau_1-1)\right). \quad (21)$$

Set $v(t)=e^{\lambda t}$. Then, from (20) we obtain

$$\lambda=-a_1\left(1-e^{-(\tau_1+\tau_2)\lambda}\right)+a_2\left(1-e^{-\tau_1\lambda}\right). \quad (22)$$

Clearly, $\lambda=0$ is a solution of Equation (22). We will study the existence of solutions of this equation with a positive real part, which determines the loss of stability of the stationary solution.

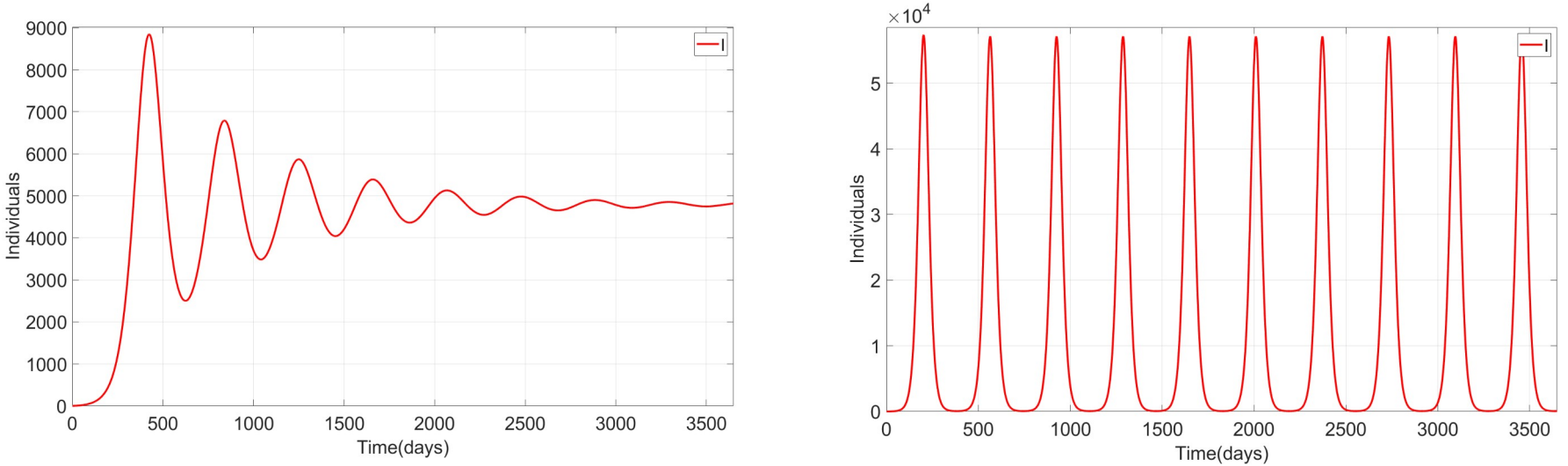
In order to simplify this analysis, we set $I(0)=0$, $S(0)=N$ in (21), so we get

$$a_1=\frac{\beta\tau_1-1}{\tau_1+\tau_2}, \quad a_2=\frac{1}{\tau_1}.$$

**Theorem 3.** The following properties hold:

- If $\Re_0 > 1$ and $J_s > 0$, then equation (22) does not have nontrivial positive real solutions.
- If $\Re_0 > 1$ and $J_s = 0$, then equation (22) has exactly one positive real solution. If $\Re_0 < 1$, then this equation has only negative real solutions.
- There exists some value $\Re_c > 1$, for which equation (22) has a pure imaginary solution .

**Figure 3.**



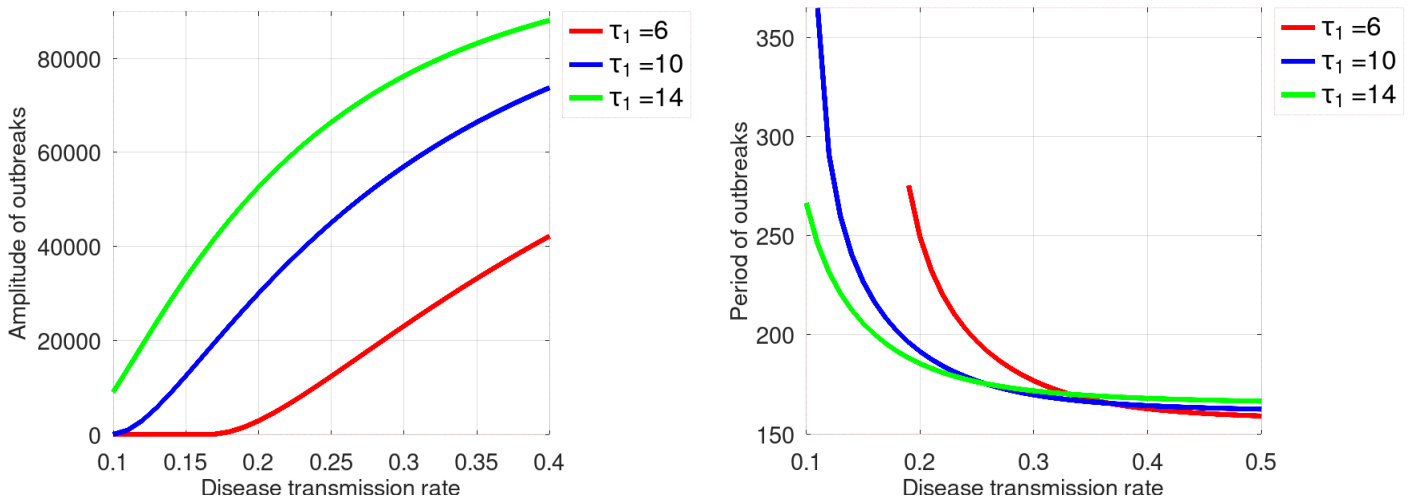

$(a)$ $(b)$

Simulation of System (11) for the following initial conditions and parameters:

$$N = 10^6,\ S(t) = N,\ I(t) = 0,\ R(t) = 0\ \forall t < 0,\ S(0) = N - 1,\ I(0) = 1,\ R(0) = 0,$$

$$\tau_1 = 10,\ \tau_2 = 180,\ \text{and}\ (a) : \beta = 0.11\ (\Re_0 = 1.1);\ (b) : \beta = 0.13\ (\Re_0 = 1.3).$$

**Figure 4.**




Dependence of the amplitude and the period of outbreaks for model (11) on the disease transmission rate $\beta$ for the initial conditions:

$$N=10^5,\ \ S(0)=N-I(0), I(0)=10^{-4}, R(0)=V(0)=0,\ \ \tau_2=150.$$

From Fig. 4, we conclude that $\lim\limits_{\beta\to\infty} T(\beta,\tau_1,\tau_2)=\tau_1+\tau_2,\quad \lim\limits_{\beta\to\infty} A(\beta,\tau_1,\tau_2)=N,$

where $T$ is the period of outbreaks and $A$ is their amplitude.

## 3. Conclusions

- In the first section of this talk, we develop epidemiological model with distributed recovery and death rates.

- A disadvantage of this integro-differential model is that it is relatively complex and it requires the knowledge of distributed recovery and death rates which may not be available in the literature.

- This model can be reduced to the conventional ODE model using the uniform distribution of recovery and death rates in time during disease duration, and to the DDE model using the Delta-Dirac distribution.

- The point-wise delay model is quite simple, it has a clear biological meaning, and it is determined by main parameters (time delays: infectivity and disease durations) which can be easily estimated from the clinical data for each particular viral infection (or virus variant).

- Data of epidemic progression is better described in the delay model than in the ODE model.

- In the second section of this talk, we propose an epidemiological model based on delay differential equations with two time delays, representing the disease duration and the period of natural immunity.

- The reduction in the delay model to an integral equation allows us to study stationary solutions of this model and their stability. A positive stationary solution appears for the basic reproduction number larger than 1. It loses its stability and leads to periodic oscillations if the basic reproduction number exceeds some critical value. We determine this critical value and the period of emerging oscillations.

- An increase in the disease transmission rate increases the amplitude and decreases the period of the outbreaks. For a large value of it, the period of outbreaks approaches the sum of disease duration and the period of natural immunity.

## References

**[1]** Ghosh, S.; Volpert, V.; Banerjee, M. An epidemic model with time delay determined by the disease duration. Mathematics 2022,10, 2561.

**[2]** Saade, M.; Ghosh, S.; Banerjee, M.; Volpert, V. An epidemic model with time delays determined by the infectivity and disease durations. Math. Biosci. Eng. 2023, 20, 12864–12888.

**[3]** Saade M.; Aniţa S.; Volpert V., Dynamics of persistent epidemic and optimal control of vaccination, Mathematics 11 (17) (2023) 3770.

# Thank you for attention !