Investigation of a multi-group epidemiological model

Abdou Raouf Cheha Mohamed

S.M. Nikol'skii Mathematical Institute

Peoples' Friendship University of Russia - RUDN University

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Problem Statement

1. Analysis of the model.
2. Reduction to the integral equations.
3. Stationary solutions of the integral equations.
4. Existence of positive stationary solution.
5. Stability of the stationary solutions.
6. Positivity of the Solutions $J_1(t)$ and $J_2(t)$.
7. Existence and uniqueness of Solutions
8. Numerical simulations.

Multi-group epidemic models

This type of models recognize that different people contribute to the spread of infection and are susceptible to its consequences in different ways.

It takes into account[2]:

- Age (children vs adults vs elderly),
- Social activity (number of contacts per day),
- Immune status (vaccinated vs unvaccinated),
- Geography (urban vs rural population),
- Presence of comorbidities.

[2] Britton T, Ball F, Trapman P. A mathematical model reveals the influence of population heterogeneity on herd immunity to SARS-CoV-2, Science, 2020

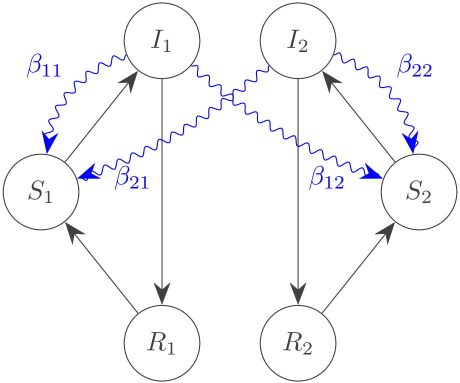
Multi-group epidemic model with time delay

$$\begin{cases} \dfrac{dS_1(t)}{dt} = -J_1(t) + J_1(t-\tau_1-\tau_2), \\ \dfrac{dS_2(t)}{dt} = -J_2(t) + J_2(t-\tau_1-\tau_2), \\ \dfrac{dI_1(t)}{dt} = J_1(t) - J_1(t-\tau_1), \\ \dfrac{dI_2(t)}{dt} = J_2(t) - J_2(t-\tau_1), \\ \dfrac{dR_1(t)}{dt} = J_1(t-\tau_1) - J_1(t-\tau_1-\tau_2), \\ \dfrac{dR_2(t)}{dt} = J_2(t-\tau_1) - J_2(t-\tau_1-\tau_2). \end{cases} \tag{1}$$

where τ_1 is disease duration, τ_2 is immunity duration, $J_1(t)$ and $J_2(t)$ are the numbers of newly infected in each group

$$J_1(t) = \frac{S_1(t)}{N}(\beta_{11}I_1(t) + \beta_{12}I_2(t)) \tag{1.1}$$

$$J_2(t) = \frac{S_2(t)}{N}(\beta_{21}I_1(t) + \beta_{22}I_2(t)). \tag{1.2}$$



Here β_{11}, β_{12}, β_{21}, β_{22} are infection transmission rates,

$N = N_1 + N_2$,

$N_1 = S_1(t) + I_1(t) + R_1(t)$,

$N_2 = S_2(t) + I_2(t) + R_2(t)$.

Integral Equations

$$\begin{cases} \dfrac{dS_1(t)}{dt} = -J_1(t) + J_1(t-\tau_1-\tau_2), \\ \dfrac{dS_2(t)}{dt} = -J_2(t) + J_2(t-\tau_1-\tau_2), \\ \dfrac{dI_1(t)}{dt} = J_1(t) - J_1(t-\tau_1), \\ \dfrac{dI_2(t)}{dt} = J_2(t) - J_2(t-\tau_1), \\ \dfrac{dR_1(t)}{dt} = J_1(t-\tau_1) - J_1(t-\tau_1-\tau_2), \\ \dfrac{dR_2(t)}{dt} = J_2(t-\tau_1) - J_2(t-\tau_1-\tau_2). \end{cases}$$

Integrating these equations and using

$$J_1(t) = \frac{S_1(t)}{N}(\beta_{11}I_1(t) + \beta_{12}I_2(t)),$$

$$J_2(t) = \frac{S_2(t)}{N}(\beta_{21}I_1(t) + \beta_{22}I_2(t)),$$

we get equaivalent integral equations

$$\begin{cases} J_1(t) = \frac{1}{N}\left(S_{10} - \int\limits_{t-\tau_1-\tau_2}^{t} J_1(s)\,ds\right)\times \\ \times\left[\beta_{11}\left(I_{10} + \int\limits_{t-\tau_1}^{t} J_1(s)\,ds\right) + \beta_{12}\left(I_{20} + \int\limits_{t-\tau_1}^{t} J_2(s)\,ds\right)\right], \\ J_2(t) = \frac{1}{N}\left(S_{20} - \int\limits_{t-\tau_1-\tau_2}^{t} J_2(s)\,ds\right)\times \\ \times\left[\beta_{21}\left(I_{10} + \int\limits_{t-\tau_1}^{t} J_1(s)\,ds\right) + \beta_{22}\left(I_{20} + \int\limits_{t-\tau_1}^{t} J_2(s)\,\mathrm{d}s\right)\right]. \end{cases} \tag{1.3}$$

Here $S_{10} = S_1(0)$, $I_{10} = I_1(0)$, $I_{20} = I_2(0)$, $S_{20} = S_2(0)$.

Stationary solutions of integral equations can be found much easier!

Stationary Solutions

For stationary solutions J_{1s} and J_{2s} integral equations have the following form :

$$J_{1s} = \frac{1}{N}[S_{10} - J_{1s}(\tau_1 + \tau_2)][\beta_{11}(I_{10} + J_{1s}\tau_1) + \beta_{12}(I_{20} + J_{2s}\tau_1)],$$

$$J_{2s} = \frac{1}{N}[S_{20} - J_{2s}(\tau_1 + \tau_2)][\beta_{21}(I_{10} + J_{1s}\tau_1) + \beta_{22}(I_{20} + J_{2s}\tau_1)].$$

Using assumptions $I_{10} \ll J_{1s}\tau_1$, $I_{20} \ll J_{2s}\tau_2$ and $S_{10} \approx N_1, S_{20} \approx N_2$, we can rewrite the previous equations:

$$J_{1s} = \frac{1}{N}(N_1 - J_{1s}(\tau_1 + \tau_2))(\beta_{11}J_{1s}\tau_1 + \beta_{12}J_{2s}\tau_1), \tag{2}$$

$$J_{2s} = \frac{1}{N}(N_2 - J_{2s}(\tau_1 + \tau_2))(\beta_{21}J_{1s}\tau_1 + \beta_{22}J_{2s}\tau_1). \tag{3}$$

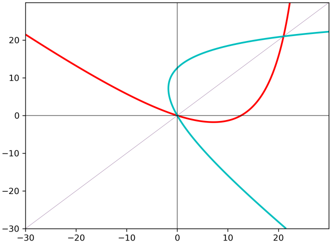
The system of equations (2)–(3) has a trivial solution $(0, 0)$.

Are there any other solutions?

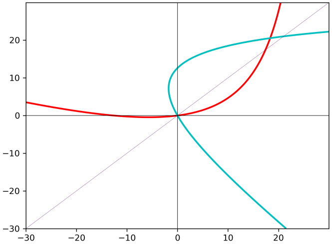
Existence of positive stationary solution

We can express J_{1s} from (3) and J_{2s} from (2), then we have

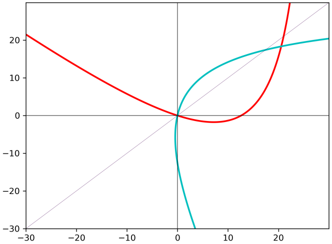
$$J_{2s} = \underbrace{\left(\frac{J_{1s}N}{N_1 - J_{1s}(\tau_1+\tau_2)} - \beta_{11}J_{1s}\tau_1\right)\frac{1}{\beta_{12}\tau_1}}_{F(J_{1s})}, \quad J_{1s} = \underbrace{\left(\frac{J_{2s}N}{N_2 - J_{2s}(\tau_1+\tau_2)} - \beta_{22}J_{2s}\tau_1\right)\frac{1}{\beta_{21}\tau_1}}_{G(J_{2s})}.$$



J2=F(J1) (red) и J1=G(J2) (blue).
Parameters: N1=500, N2=500, tau1=7, tau2=10,
b11=0.5, b12=0.5, b21=0.5, b22=0.5



J2=F(J1) (red) и J1=G(J2) (blue).
Parameters: N1=500, N2=500, tau1=7, tau2=10,
b11=0.2, b12=0.5, b21=0.5, b22=0.5



J2=F(J1) (red) и J1=G(J2) (blue).
Parameters: N1=500, N2=500, tau1=7, tau2=10,
b11=0.5, b12=0.5, b21=0.5, b22=0.2

Basic reproduction number

Thus, we can introduce basic reproduction number for our model:

$$\mathcal{R}_0 = \max_{i=1,2,3} \{R_i\},$$

where

$$R_1 = \beta_{11}\tau_1 \frac{N_1}{N}, \qquad R_2 = \beta_{22}\tau_1 \frac{N_2}{N}, \qquad R_3 = \frac{\frac{N_1}{N}\beta_{11}\tau_1 + \frac{N_2}{N}\beta_{22}\tau_1 + \beta_{12}\beta_{21}\tau_1^2 \frac{N_1 N_2}{N^2}}{1 + \beta_{12}\beta_{21}\tau_1^2 \frac{N_1 N_2}{N^2}}.$$

Theorem

The system (2)–(3) has a positive solution if $\mathcal{R}_0 > 1$.
If $\mathcal{R}_0 < 1$ The system (2)–(3) does not have positive solutions.

Stability of the stationary solutions

We linearize integral equations with respect to the stationary solution (J_{1s}, J_{2s}).
Substituting instead of $J_1(t)$ and $J_2(t)$ the expressions $J_{1s}+\nu(t)$ $J_{2s}+\omega(t)$:

$$\nu(t) = a_1(J_{1s}) \int\limits_{t-\tau_1}^{t} \nu(s)\,\mathrm{d}s \; + a_2(J_{1s}) \int\limits_{t-\tau_1}^{t} \omega(s)\,\mathrm{d}s - a_3(J_{1s}, J_{2s}) \int\limits_{t-\tau_1-\tau_2}^{t} \nu(s)\,\mathrm{d}s. \tag{4}$$

$$\omega(t) = b_1(J_{2s}) \int\limits_{t-\tau_1}^{t} \nu(s)\,\mathrm{d}s \; + b_2(J_{2s}) \int\limits_{t-\tau_1}^{t} \omega(s)\,\mathrm{d}s - b_3(J_{1s}, J_{2s}) \int\limits_{t-\tau_1-\tau_2}^{t} \omega(s)\,\mathrm{d}s \tag{5}$$

$$a_1(J_{1s}) = \frac{\beta_{11}}{N}(N_1 - J_{1s}(\tau_1+\tau_2)), \qquad b_1(J_{2s}) = \frac{\beta_{21}}{N}(N_2 - J_{2s}(\tau_1+\tau_2)),$$

$$a_2(J_{1s}) = \frac{\beta_{12}}{N}(N_1 - J_{1s}(\tau_1+\tau_2)), \qquad b_2(J_{2s}) = \frac{\beta_{22}}{N}(N_2 - J_{2s}(\tau_1+\tau_2)),$$

$$a_3(J_{1s}, J_{2s}) = \frac{[\beta_{11}J_{1s}\tau_1 + \beta_{12}J_{2s}\tau_1]}{N}, \quad b_3(J_{1s}, J_{2s}) = \frac{[\beta_{21}J_{1s}\tau_1 + \beta_{22}J_{2s}\tau_1]}{N}.$$

Stability of the stationary solutions

Let $\nu(t) = pe^{\lambda t}$ and $\omega(t) = qe^{\lambda t}$. Substituting these expressions into (4)–(5), calculating the integrals and reducing by $e^{\lambda t}$, we obtain:

$$\begin{cases} p\lambda = pa_1(1 - e^{-\tau_1 \lambda}) + qa_2(1 - e^{-\tau_1 \lambda}) - pa_3(1 - e^{-(\tau_1+\tau_2)\lambda}), \\ \\ q\lambda = pb_1(1 - e^{-\tau_1 \lambda}) + qb_2(1 - e^{-\tau_1 \lambda}) - qb_3(1 - e^{-(\tau_1+\tau_2)\lambda}). \end{cases} \tag{6}$$

Clearly, $\lambda = 0$ is a solution of the system (6).

Are there any positive solutions?

Stability of the stationary solutions

System (6) can be rewritten in the following form:

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$c_{11} = a_1(1 - e^{-\tau_1 \lambda}) - a_3(1 - e^{-(\tau_1+\tau_2)\lambda}) - \lambda, \qquad c_{12} = a_2(1 - e^{-\tau_1 \lambda}),$$
$$c_{21} = b_1(1 - e^{-\tau_1 \lambda}), \qquad c_{22} = b_2(1 - e^{-\tau_1 \lambda}) - b_3(1 - e^{-(\tau_1+\tau_2)\lambda}) - \lambda.$$

The system has a nontrivial solution if $\Delta = c_{11}c_{22} - c_{21}c_{12} = 0$. In other words,

$$\left[a_1(1 - e^{-\tau_1 \lambda}) - a_3(1 - e^{-(\tau_1+\tau_2)\lambda}) - \lambda\right] \left[b_2(1 - e^{-\tau_1 \lambda}) - b_3(1 - e^{-(\tau_1+\tau_2)\lambda}) - \lambda\right] - \\ - \left[a_2(1 - e^{-\tau_1 \lambda})\right] \left[b_1(1 - e^{-\tau_1 \lambda})\right] = 0$$

Stability of the trivial stationary solution

Theorem

If $\mathcal{R}_0 > 1$, then there exists a positive real solution λ of

$$\left[a_1(1-e^{-\tau_1\lambda}) - a_3(1-e^{-(\tau_1+\tau_2)\lambda}) - \lambda\right]\left[b_2(1-e^{-\tau_1\lambda}) - b_3(1-e^{-(\tau_1+\tau_2)\lambda}) - \lambda\right] - \\ - \left[a_2(1-e^{-\tau_1\lambda})\right]\left[b_1(1-e^{-\tau_1\lambda})\right] = 0$$

If $\mathcal{R}_0 < 1$ then previous equation does not have positive solutions.

Positivity of the Solutions $J_1(t)$ and $J_2(t)$

Consider the system of integral equations (1.3).

Remark

If $I_{10} = I_{20} \equiv 0$, then from (1.1), (1.2) it is easy to see that $J_1(t) = J_2(t) \equiv 0$ is the trivial solution of the system (1.3).

Hereafter, we will assume that the following holds:

(i) $S_{10}, S_{20}, I_{10}, I_{20} > 0$

(ii) $I_1(t) = I_2(t) = 0$ for $t < 0$, this means that $J_1(t) = J_2(t) = 0$ for $t < 0$

(iii) $\beta_{ij} > 0$

(iv) $\tau_1, \tau_2 > 0$

(v) $N = S_{10} + S_{20} + I_{10} + I_{20} + R_{10} + R_{20}$

(vi) The functions $J_1(t)$ and $J_2(t)$ are continuous for positive t. — Is this true?

Existence and uniqueness of Solutions

Set $\theta = \tau_1 + \tau_2$ and denote by $(S_{1,n}, S_{2,n}, I_{1,n}, I_{2,n}, R_{1,n}, R_{2,n})$ the restriction of the solution $(S_1, S_2, I_1, I_2, R_1, R_2)$ to the interval $[(n-1)\theta, n\theta]$, where $n \in \mathbb{N}$. If $t \in [(n-1)\theta, n\theta]$, where $n \in \mathbb{N}$, then $t - \theta \in [(n-2)\theta, (n-1)\theta]$ and the functions $J_1(t-\theta) = \phi_n^1(t)$, $J_2(t-\theta) = \phi_n^2(t)$ are called the *history functions* for $J_1(t)$, $J_2(t)$. For $t \in [(n-1)\theta, n\theta]$, system (1) takes the form:

$$\frac{dS_1(t)}{dt} = -J_1(t) + \phi_n^1(t),$$
$$\frac{dS_2(t)}{dt} = -J_2(t) + \phi_n^2(t),$$
$$\frac{dI_1(t)}{dt} = J_1(t) - J_1(t-\tau_1),$$
$$\frac{dI_2(t)}{dt} = J_2(t) - J_2(t-\tau_1),$$
$$\frac{dR_1(t)}{dt} = J_1(t-\tau_1) - \phi_n^1(t),$$
$$\frac{dR_2(t)}{dt} = J_2(t-\tau_1) - \phi_n^2(t),$$

Existence and uniqueness of Solutions

where

$$\phi_n^1(t) = \frac{S_1(t-\theta)}{N}\left(\beta_{11} I_1(t-\theta) + \beta_{12} I_2(t-\theta)\right)$$

and

$$\phi_n^2(t) = \frac{S_2(t-\theta)}{N}\left(\beta_{21} I_1(t-\theta) + \beta_{22} I_2(t-\theta)\right).$$

$$\hat{\Sigma}_n = \left\{(S_1, S_2, I_1, I_2, R_1, R_2) \in (\Sigma_n)^6 : \phi_n^1(t) \geq 0, \phi_n^2(t) \geq 0, \forall t \in [(n-1)\theta, n\theta]\right\},$$

where Σ_n is defined as

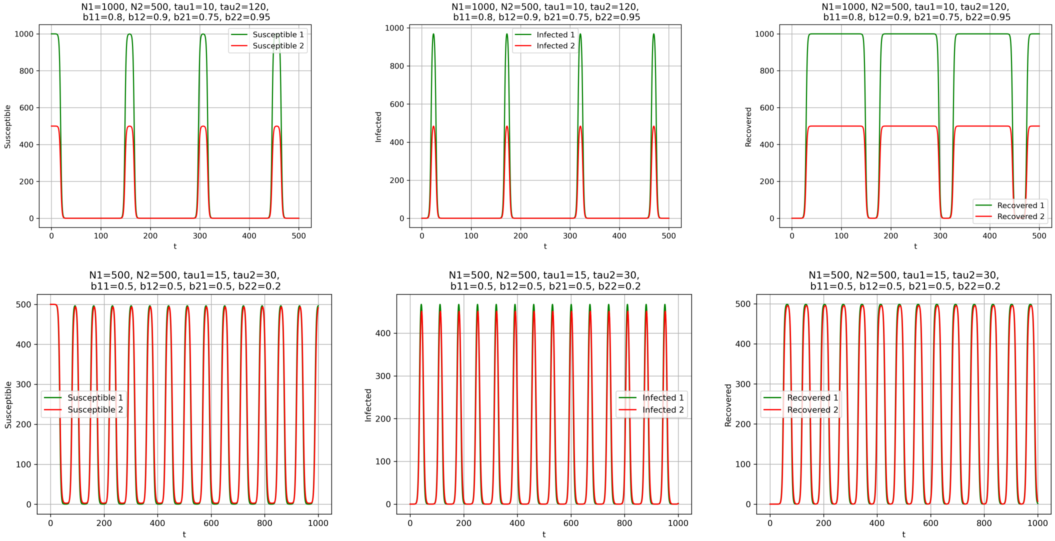
$$\Sigma_n = \{T_n \in C([(n-1)\theta, n\theta]) : 0 \leq T_n(t) \leq N, \forall t \in [(n-1)\theta, n\theta]\}.$$

We formulate the following theorem.

Theorem

If there exists a unique solution $(S_{1,n-1}, S_{2,n-1}, I_{1,n-1}, I_{2,n-1}, R_{1,n-1}, R_{2,n-1})$ of system (1) in the domain $\hat{\Sigma}_{n-1}$, then system (1) will have a unique solution $(S_{1,n}, S_{2,n}, I_{1,n}, I_{2,n}, R_{1,n}, R_{2,n})$ in the domain $\hat{\Sigma}_n$.

Numerical simulations



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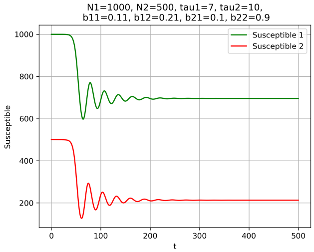

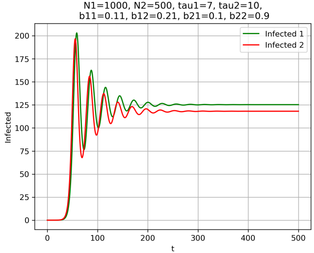

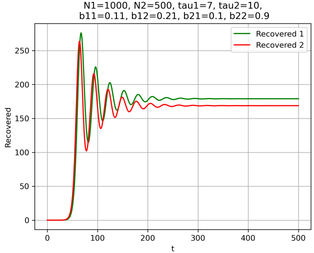

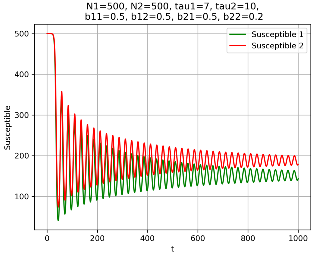

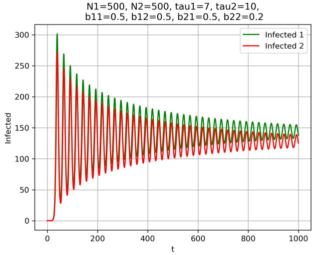

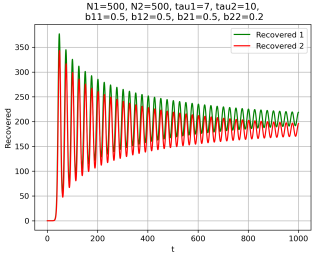

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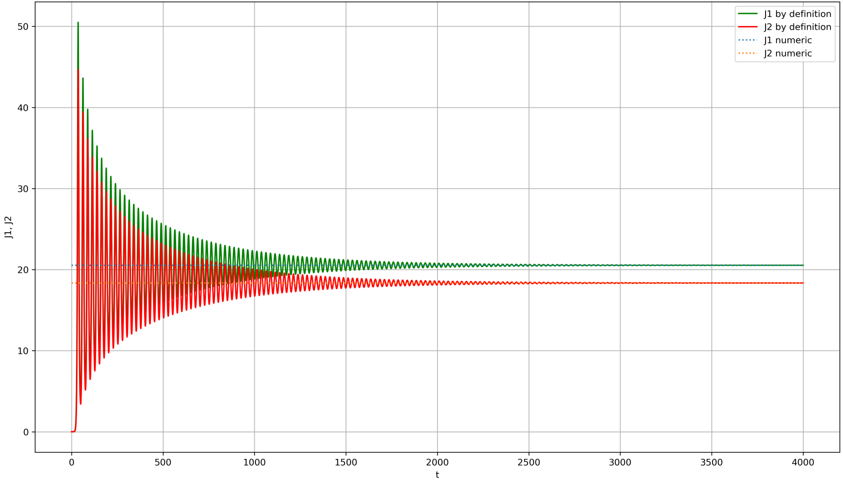

Sources and References

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THANK YOU

FOR YOUR ATTENTION!