Class of Chemotactic System of Keller-Segel based on Einstein method of Brownian motion Modeling

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Goal of the project

We study the movement of the living organism in a band form towards the presence of chemical substrates based on a system of partial differential evolution equations. We incorporate Einstein's method of Brownian motion to deduce the chemotactic model exhibiting a traveling band. We have shown that in the presence of limited and unlimited substrate, traveling bands are achievable and it has been explained accordingly. We also study the stability of the constant steady states for the system. The linearized system about a constant steady state is obtained under the mixed Dirichlet and Neumann boundary conditions. We are able to find explicit conditions for linear instability. The linear stability is established with respect to the L^2-norm, H^1-norm, and L^∞-norm under certain

Observed Experiment

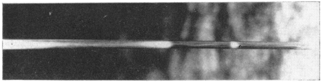

Figure 1: Chemotactic Bacteria Flow in the Tube

Model of the Flow

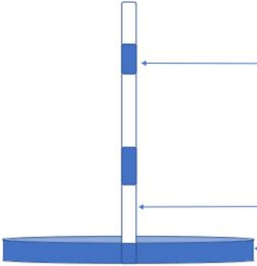

Figure 2: Petrie Dish Bacteria Flow

Models with consumption or reaction term

Let $X = (x, t)$ is an observable point in space $x \in \mathbb{R}$ at time $t \in (0, \infty)$. Late space of observation for $t > 0$ bounded by two planes x and $x + dx$ perpendicular to the x-axis. τ time interval between the collision of two particles , the interval τ is “sufficiently small” compared to the time scale t .Δ be the distance each particle makes during the time interval $(t, t + \tau)$ and $\varphi_\tau(\Delta)$ be the probability density function of non-collision.
$w(x, t)$ is the number of the particles in the volume $[x, x + dx]$.

Definition 1

(Expected value of the length of free jump)

$$\Delta_e = \int \Delta \varphi_\tau(\Delta) d\Delta.$$

Definition 2

(Standard variance of free jump)

$$\sigma^2 = \int (\Delta - \Delta_e)^2 \varphi_\tau(\Delta) d\Delta.$$

Then the number of particles found at time $t+\tau$ between two planes perpendicular to the x-axis, with abscissas x and $x+dx$, is given by

$$(w(x,t+\tau))\cdot dx = \left(\underbrace{\mathbb{E}[w(x+\Delta,t)]}_{I_1} + \underbrace{w\frac{\partial \Delta_e}{\partial x}}_{I_2} + \underbrace{\frac{1}{\tau}\int_t^{t+\tau} f(x,\xi)d\xi}_{I_3} \right)\cdot dx \tag{1.1}$$

Here, $\mathbb{E}[w(x+\Delta,t)] = \int_{-\infty}^{\infty} w(x+\Delta,t)\varphi_\tau(\Delta)d\Delta$

In the right-hand side of Eq. (1.1), the first term, I_1, describes the particle distribution due to random walk. The second term, I_2, explains the adjective flux of particles dependent on the gradient of the expected length. And the last term, I_3, represents the birth or death of particles during $[t, t+\tau]$.

τ, Δ, and $\varphi_\tau(\Delta)$ can be functions of spatial distance x and the time variable t and of any other physical quantity such as density or the number of particles, etc. In our case, we will assume, for now, τ to be independent of the concentration of particles $w(x,t)$. And $\varphi_\tau(\Delta)$ is fixed with respect to $w(x,t)$.

We add and subtract $w(x,t)$ on the right-hand side of the Eq. (1.1) and then we compute as follows

$$\begin{aligned}&\big(w(x,t+\tau)-w(x,t)\big)\cdot dx\\&=\left(\mathbb{E}[w(x+\Delta,t)]-\mathbb{E}w[(x,t)]+w\cdot\frac{\partial\Delta_e}{\partial x}+\frac{1}{\tau}\int_t^{t+\tau}f(x,\xi)d\xi\right)\cdot dx.\end{aligned}\tag{1.2}$$

Assume that $w(x,t)$ is four time differentiable function on $\mathbb{R}$ and bounded, then $\big(\mathbb{E}[w(x+\Delta,t)]-w(x,t)\big)$ can be well approximated by formulae

$$\big(\mathbb{E}[w(x+\Delta,t)]-w(x,t)\big)=\frac{1}{2}\sigma^2\frac{\partial^2 w(x,t)}{\partial x^2}+\Delta_e\frac{\partial w(x,t)}{\partial x}.\tag{1.3}$$

Using properties of the function φ and applying Eq. (1.3) on (1.2), we get

$$\tau\frac{\partial w}{\partial t}=\Delta_e\frac{\partial w}{\partial x}+w\cdot\frac{\partial\Delta_e}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 w}{\partial x^2}+\frac{1}{\tau}\int_t^{t+\tau}f(x,\xi)d\xi,\tag{1.4}$$

or, equivalently,

$$\tau\frac{\partial w}{\partial t}=\frac{\partial\left(w\cdot\Delta_e\right)}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 w}{\partial x^2}+\frac{1}{\tau}\int_t^{t+\tau}f(x,\xi)d\xi.\tag{1.5}$$

Equation for bacteria in the presents of chemotactic force

The corresponding expression of the Eq. (1.5) for bacteria is

$$\tau_u \frac{\partial u}{\partial t} - \frac{\partial \left(u \cdot \Delta_{e,u} \right)}{\partial x} - \frac{1}{2} \sigma_u^2 \frac{\partial^2 u}{\partial x^2} - \frac{1}{\tau_u} \int_t^{t+\tau_u} f_u(x, \xi) d\xi = 0. \qquad (1.6)$$

The chemotactic response of the bacteria $u(x,t)$ in the medium can be influenced by the concentration of chemical substrate $v(x,t)$ and/or ∇v. We assume that the chemotactic response, which causes the event of movement of the organism towards food (or any attractor), is proportional to relative changes of v in space with respect to the amount of food. We assume that $\Delta_{e,u}$ and σ_u also depend on $v(x,t)$. Keller-Segel suggested a density-dependent sensitivity with a singularity at $v = 0$. Following the Keller-Segel assumption, we suggested the dynamics of directed movement characterized by the expected value of free jump $\Delta_{e,u}$ as follows,

$$\Delta_{e,u}(v) = -\beta(v) \frac{\partial v}{\partial x} = -\frac{\beta}{v} \frac{\partial v}{\partial x} = -\beta \frac{\partial \ln v}{\partial x}. \qquad (1.7)$$

β is a positive chemotactic coefficient having dimension $[L^2]$.

Final equation for bacteria under no reaction force

Although in real life, the standard deviation is a composite parameter depending on v, ∇v, $u, \nabla u$, x, t, *etc*, in this we consider the dynamics of processes with constant standard deviation. Namely,

$$\sigma_u^2(v) = \mu, \tag{1.8}$$

where μ is the motility parameter or diffusion coefficient of the organism with dimension $[L^2]$. Both μ and β can be obtained from analyses of the dynamics of the process, using image processing.
f_u is the number of organisms that are born or die per unit volume. We will assume that

$$\int_t^{t+\tau_u} f_u(x,\xi)d\xi = \tau_u g(u,v). \tag{1.9}$$

Here $g(u,v)$ is the rate of born or death of the organism with dimension $[\frac{1}{T}]$. Since the growth or reproduction of an organism happens on a large time scale and chemotaxis occurs on a very small time scale, we can ignore the growth term. Then $g(u,v) = 0$ in Eq. (1.10) gives

$$\tau_u \frac{\partial u}{\partial t} + \beta \frac{\partial}{\partial x}\left(u \frac{\partial \ln v}{\partial x}\right) - \frac{\mu}{2}\frac{\partial^2 u}{\partial x^2} = 0. \tag{1.10}$$

The first term on the right-hand side of Eq. (1.10) is the chemotactic response of the organism. The second term is the change in the density of an organism due to random motion.

Assumption on birth of organisms and main chemotactic system

And the concentration $v(x,t)$ of chemical substrates can be given by the equation

$$\tau_v \frac{\partial v}{\partial t} - \frac{\partial\left(v \cdot \Delta_{e,v}\right)}{\partial x} - \frac{1}{2}\sigma_v^2 \frac{\partial^2 v}{\partial x^2} - \frac{1}{\tau_v}\int_t^{t+\tau_v} f_v(x,\xi)\,d\xi = 0. \qquad (1.11)$$

Assuming $\Delta_{e,v} = 0$ and $-\frac{1}{\tau_v}\int_t^{t+\tau_v} f_v(x,\xi)\,d\xi = \tau_v k(u,v)u$ one will get

$$\tau_u \frac{\partial u}{\partial t} + \beta \frac{\partial}{\partial x}\left(u \frac{\partial \ln v}{\partial x}\right) - \frac{\mu}{2}\frac{\partial^2 u}{\partial x^2} = 0, \qquad (1.12a)$$

$$\tau_v \frac{\partial v}{\partial t} - \frac{D}{2}\frac{\partial^2 v}{\partial x^2} + \tau_v k(u,v)u = 0. \qquad (1.12b)$$

Summary of the Biological Assumptions

Assumption 1

Food (chemical substrate) is considered to be immovable, so no chemical interaction between particles of substrates is possible under our assumption. Hence,

$$\Delta_{e,v} = 0 \text{ and } \sigma_v^2 = D,$$

with D being the diffusion constant of the chemical substrate.

Assumption 2

f_v is defined to be the consumption of substrate cells and

$$\int_t^{t+\tau_v} f_v(x,\xi)d\xi = H(u,v) = -\tau_v k(v)u,$$

where $k(v)$ is the rate of consumption of the substrate with dimension $[\frac{1}{T}]$.

Traveling bands

This section is dedicated to showing that chemotactic models in the presence of unlimited and limited substrate exhibits traveling band. First, we will define the traveling band.

Definition 3

(Traveling Band) The system of Eqs. (1.10) and (**??**) exhibits a traveling band form if there exist solutions $u(x, t)$ and $v(x, t)$ of the following form

$$u(x, t) = U(x - ct) \text{ and } v(x, t) = V(x - ct) \text{ for all } x \in \mathbb{R} \text{ and } t \geq 0 \tag{1.13}$$

where $c > 0$ is the constant band speed, and U, V are functions from $\mathbb{R}$ to $(0, \infty)$ such that $\lim\limits_{\zeta \to \pm\infty} U(\zeta)$ and $\lim\limits_{\zeta \to \pm\infty} V(\zeta)$ exist and belong to $[0, \infty)$, $, \zeta = x - ct$.

We will also assume $D = 0$ in the Eq. (**??**) since its effect is trivial in the chemotactic model. And, for the sake of simplicity, we will use $\tau = \tau_u$ and $\tau_v = 1$.

In the presence of an abundance of the substrate, the rate of consumption of the food, $k(v)$, does not depend on the concentration of food.

$$\tau \frac{\partial u}{\partial t} + \beta \frac{\partial}{\partial x}\left(u \frac{\partial \ln v}{\partial x}\right) - \frac{\mu}{2}\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.14}$$

$$\frac{\partial v}{\partial t} + ku = 0, \tag{1.15}$$

Theorem 4

If $d = \dfrac{2\beta}{\mu}$ *and* $d \geq 1$*, then for any* $\tau, \beta, \mu, k, c, V_\infty, C_0 > 0$ *and* $d \geq 1$*, there exist solutions*

$$U(\zeta) = C_0 V^d(\zeta) e^{-\frac{2\tau c \zeta}{\mu}}\ ,, \tag{1.16}$$

$$V(\zeta) = \begin{cases} [\frac{1}{2} C_0 k c^{-2} \tau^{-1} \mu (d-1) e^{-\frac{2\tau c \zeta}{\mu}} + V_\infty^{-d+1}]^{-\frac{1}{d-1}} & \text{for } d > 1, \\ V_\infty e^{-\frac{1}{2} C_0 k c^{-2} \tau^{-1} \mu e^{-\frac{2\tau c}{\mu}\zeta}} & \text{for } d = 1. \end{cases} \tag{1.17}$$

Moreover, $(U(\zeta), V(\zeta))$ *that satisfies*

$$U(\zeta) \to 0,\ U^{'}(\zeta) \to 0,\ V(\zeta) \to V_\infty \text{ as } \zeta \to \infty, \tag{1.18}$$

$$\tau \frac{\partial u}{\partial t} + \beta \frac{\partial}{\partial x}\left(u \frac{\partial \ln v}{\partial x}\right) - \frac{\mu}{2}\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.19}$$

$$\frac{\partial v}{\partial t} + kuv = 0. \tag{1.20}$$

Theorem 5

For any $\tau, \beta, \mu, k, c > 0$ *the system has a traveling band of the form* (1.13)*. More precisely,* $U(\zeta)$ *and* $V(\zeta)$ *can be given by*

$$U(\zeta) = \frac{\tau c^2}{\beta k}\left(1 + C_0 e^{\frac{2\tau c}{\mu}\zeta}\right)^{-1}, \tag{1.21}$$

$$V(\zeta) = V_\infty \left(1 + C_0^{-1} e^{-\frac{2\tau c}{\mu}\zeta}\right)^{-\frac{\mu}{2\beta}}, \tag{1.22}$$

where $V_\infty > 0$ *and* $C_0 > 1$*. In fact,* $U(\zeta)$ *and* $V(\zeta)$ *in Eqs.* (1.21) *and* (1.22) *are unique solutions of Eqs.* (1.19) *and* (1.20) *that satisfy condition* (1.18) *and*

$$U(0) = \frac{\tau c^2 C_0}{\beta k}. \tag{1.23}$$

Illustration on pictures for U

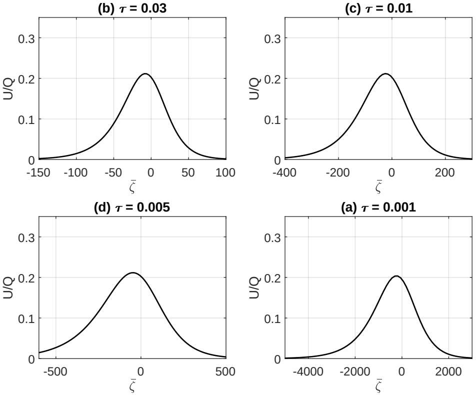

Figure 3: Concentration of bacteria $U(\zeta)$ divided by $Q = C_0 V_\infty$ for different values of τ

Limiting values of the concentrations of bacteria and the food

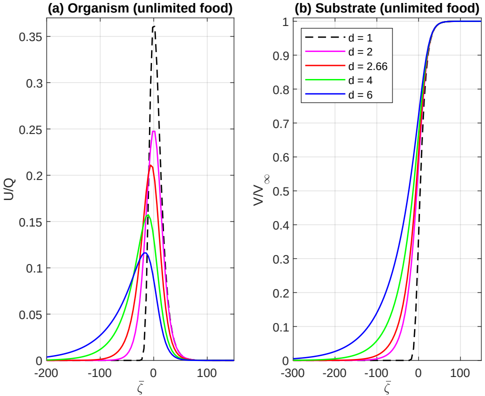

Figure 4: Showing the uniform convergence of (a) the concentration of bacteria $U(\zeta)$ divided by $Q = C_0 V_\infty$ to the solution U_1 and (b) the concentration of the substrates $V(\zeta)$ divided by V_∞ to the solution V_1 as $d \to 1$.

End

Thank you