Modelling the influence of lockdown on epidemic progression

By

Anastasia Mozokhina, Ivan Popravka, Masoud Saade, and Vitaly Volpert

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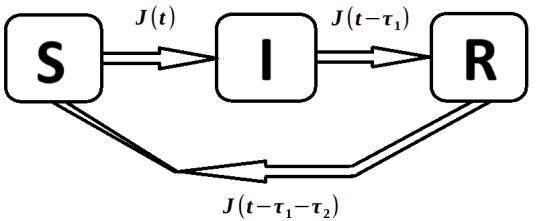
1. Epidemic delay model

$$\frac{dS(t)}{dt} = -J(t) + J(t-\tau_1-\tau_2), \quad (1a)$$

$$\frac{dI(t)}{dt} = J(t) - J(t-\tau_1), \quad (1b)$$

$$\frac{dR(t)}{dt} = J(t-\tau_1) - J(t-\tau_1-\tau_2), \quad (1c)$$

$$J(t) = \frac{\beta}{N} S(t) I(t), \quad (1d)$$



where τ_1 is the disease duration, τ_2 is the duration of immunity, and

β is the disease transmission rate.

Total size of population $N = S(t) + I(t) + R(t).$ (2)

System (1) is completed with the initial conditions

$$S(\theta) = N, \; I(\theta) = R(\theta) = 0 \quad \forall \theta \in [-\tau_1-\tau_2, 0), S(0) = N - I(0), \; I(0) > 0, \; R(0) = 0. \quad (3)$$

2. Existence and stability of stationary solutions

2.1 Reduction to an integral equation

Integrating Eq (1a) and Eq (1b) from 0 to t, then substituting in Eq (1d), we obtain

$$J(t)=\frac{\beta}{N}\left(S_0-\int_{t-\tau_1-\tau_2}^{t} J(\eta)d\eta\right)\left(I_0+\int_{t-\tau_1}^{t} J(\eta)d\eta\right). \qquad (4)$$

Thus, we have reduced system (1) to a single integral equation.

2.2 Stationary solutions

Stationary solutions of equation (4) can be found from the following algebraic equation:

$$J_s = \frac{\beta}{N}(S_0 - (\tau_1 + \tau_2) J_s)(I_0 + \tau_1 J_s). \qquad (5)$$

The positive solution of previous equation is given by the formula

$$J_s = \frac{-(\frac{\beta}{N} + (\tau_1 + \tau_2) I_0 - \tau_1 S_0) + \sqrt{\Delta}}{2\tau_1(\tau_1 + \tau_2)}, \qquad (6)$$

$$\Delta = (\frac{\beta}{N} + (\tau_1 + \tau_2) I_0 - \tau_1 S_0)^2 + 4 S_0 I_0 \tau_1 (\tau_1 + \tau_2) > 0.$$

If $I_0 \approx 0$ and $S_0 \approx N$, then we find two approximate solutions of the previous equation:

$$J_s = 0, \quad J_s = \frac{N(\beta\tau_1 - 1)}{\beta\tau_1(\tau_1 + \tau_2)}. \qquad (7)$$

In this case, a positive stationary solution exists if the basic reproduction number $\Re_0 = \beta\tau_1$ is larger than 1, allowing us to determine the stationary values of susceptible, infected, recovered and productive as:

$$S_s = \frac{N}{\beta\tau_1}, \quad I_s = \frac{N(\beta\tau_1 - 1)}{\beta(\tau_1 + \tau_2)}, \quad R_s = N - S_s - I_s. \qquad (8)$$

2.3. Stability of the stationary solution

Equation (4), linearized about the stationary solution, by setting $J(t)=J_s+\epsilon e^{\lambda t}$ and keeping the first-order terms with respect to ϵ, has the following form

$$v(t)=-a_1\int_{t-\tau_1-\tau_2}^{t} v(x)\,dx+a_2\int_{t-\tau_1}^{t} v(x)\,dx, \quad (9)$$

where

$$a_1=\frac{\beta}{N}\left(I(0)+\frac{N}{\beta}\frac{\beta\tau_1-1}{\tau_1+\tau_2}\right),\ \ a_2=\frac{\beta}{N}\left(S(0)-\frac{N}{\beta\tau_1}(\beta\tau_1-1)\right). \quad (10)$$

Set $v(t)=e^{\lambda t}$. Then, from (9) we obtain the equation for the principal eigenvalue which determines infection growth rate as follows

$$\lambda=-a_1\left(1-e^{-(\tau_1+\tau_2)\lambda}\right)+a_2\left(1-e^{-\tau_1\lambda}\right). \quad (11)$$

Clearly, $\lambda = 0$ is a solution of Equation (11). We will study the existence of solutions of this equation with a positive real part, which determines the loss of stability of the stationary solution.

In order to simplify this analysis, we set $I(0)=0$, $S(0)=N$ in (10), so we get

$$a_1 = \frac{\beta \tau_1 - 1}{\tau_1 + \tau_2}, \quad a_2 = \frac{1}{\tau_1}.$$

Theorem 1. The following properties hold:

- If $\Re_0 > 1$ and $J_s > 0$, then equation (11) does not have nontrivial positive real solutions.
- Let $J_s = 0$. If $\Re_0 > 1$, then equation (11) has exactly one positive real solution. If $\Re_0 < 1$, then this equation has only negative real solutions.
- There exists some value $\Re_c > 1$, for which equation (11) has a pure imaginary solution .

3. Influence of isolation on epidemic progression

Isolation of a part of the population can influence epidemic progression and decrease the number of infected individuals.

3.1 Model without immunity waning (single outbreak)

We model partial lockdown in which a part of the population is isolated and cannot be infected.

Isolation begins before epidemic outbreak and stops after the outbreak.

We will determine the optimal proportion of isolated people which minimizes the number of infected individuals.

We consider the following system of equations:

$$\frac{dS(t)}{dt} = -J(t), \quad (12a)$$

$$\frac{dI(t)}{dt} = J(t) - J(t-\tau_1), \quad (12b)$$

$$\frac{dR(t)}{dt} = J(t-\tau_1), \quad (12c)$$

$$J(t) = \frac{\beta}{N(t)} S(t) I(t). \quad (12d)$$

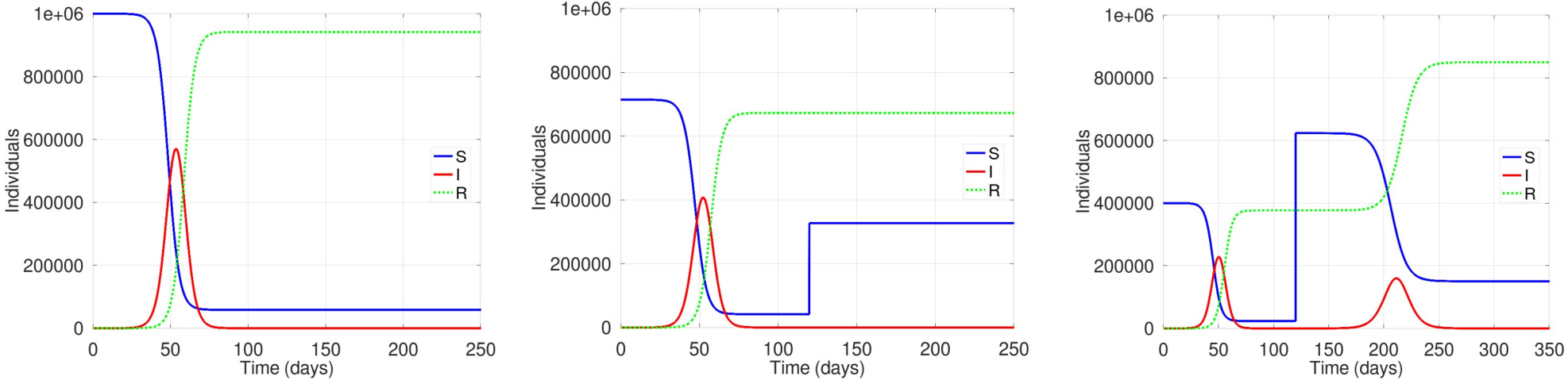
We impose a partial lockdown at time t_1 with duration T_1 and proportion of isolated population $k_1 \in (0, 1)$ of the total population N_0:

$$N(t_1) = (1-k_1)N_0, \; S(t_1) = S(t_1) - k_1 N_0, \quad N(t_1+T_1) = N_0, \; S(t_1+T_1) = S(t_1+T_1) + k_1 N_0. \quad (13)$$

System (12) is considered with the following initial conditions

$$S(\theta) = N_0, \; I(\theta) = 0 : \forall \theta \in [-\tau_1, 0), \quad S(0) = S_0 > 0, \; I(0) = I_0 > 0 \; (S_0 + I_0 = N_0), \; R(0) = 0. \quad (14)$$

Figure 1.



Susceptible, infected and recovered populations in numerical simulations of System (12) for initial conditions $N_0 = 10^6$, $S(t) = N_0$, $I(t) = R(t) = 0$ $\forall t \in [-\tau_1, 0)$, $S(0) = N_0 - 1$, $I(0)=1$, $R(0) = 0$ and parameters $\beta = 0.3$, $\tau_1 = 10$. Panel (a): without isolation. Panel (b): a part of the population is isolated before epidemic outbreak and returns afterwards ($t_1 = 0$, $T_1 = 120$, $k_1 \approx 0.29$). Panel (c): $t_1 = 0$, $T_1 = 120$, $k_1 = 0.6$.

Analytical estimate:

Let us determine the value of k_1 which provides the minimal total number of infected individuals. The equation which find the number of susceptible individuals S_f at the end of the outbreak, taking into account that a part of the population is isolated is given by

where

$$\ln(w) = \Re_0 (w-1), \qquad (15)$$

$$w = \frac{S_f}{(1-k_1) N_0} \quad \text{and} \quad \Re_0 = \beta \tau_1 .$$

Eq (15) has a solution w ∈ (0, 1) if $\Re_0 > 1$. When isolated people are returned after lockdown is finished, the total number of susceptible becomes $S_f + k_1 N_0$, and the new value of the basic reproduction number is $\Re_0' = \Re_0((1-k_1)w + k_1)$.

Epidemic does not restart after the end of isolation when $\Re_0' = 1$ and we obtain

$$k_1 = \frac{1/\Re_0 - w}{1-w}. \qquad (16)$$

The minimal total number of infected individuals is given by the formula

$$I_{total} = (1-k_1)(1-w) N_0. \qquad (17)$$

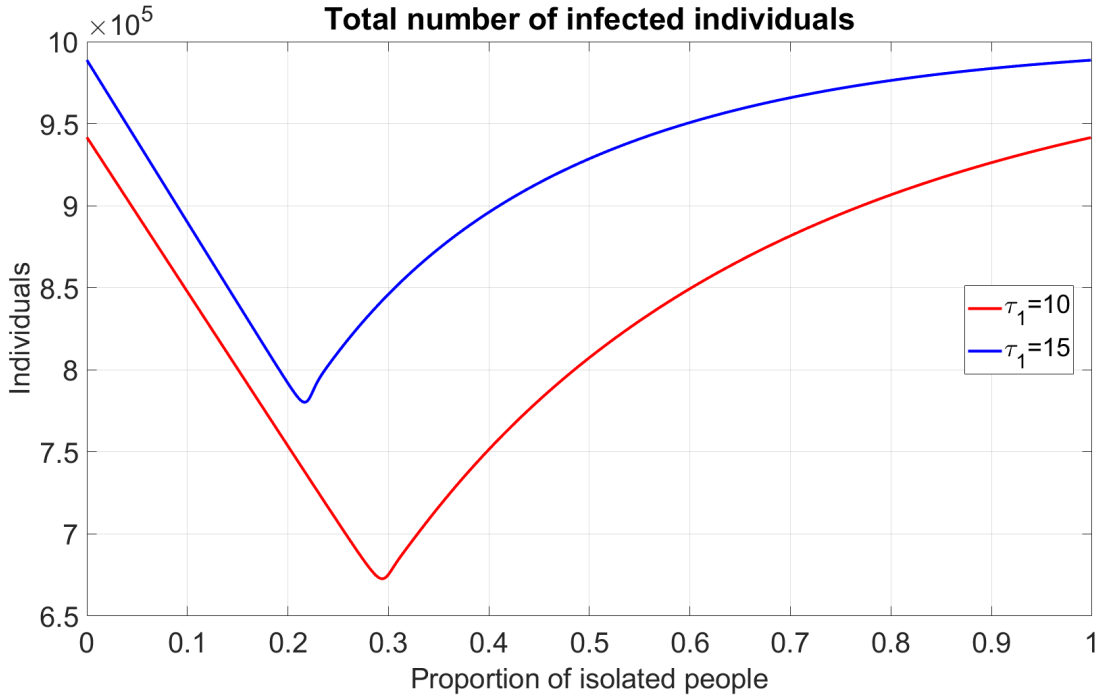
Example: If $\Re_0 = 3$, then from equation (15) we obtain w≈0.06, and from (16) and (17), $k_1 \approx 0.29$, $I_{total}/N_{total} \approx 0.67$.

On the other hand, without lockdown ($k_1 = 0$), we obtain $I_{total}/N_{total} \approx 0.94$.

Thus, isolation allows about 30% reduction of the proportion of infected individuals.

This analytical result is confirmed numerically by Fig 2.

Figure 2.



Dependence of the total number of infected individuals I_{total} on the proportion of isolated population k_1 in the model (12) without immunity waning.

The values of parameters: $N_0 = 10^6$, $\beta = 0.3$, $t_1 = 0$, $T_1 = 120$, $I(0) = 1$.

3.2 Model with immunity waning (periodic outbreaks)

$$\frac{dS(t)}{dt} = -J(t) + J(t-\tau_1-\tau_2), \quad (18a)$$

$$\frac{dI(t)}{dt} = J(t) - J(t-\tau_1), \quad (18b)$$

$$\frac{dR(t)}{dt} = J(t-\tau_1) - J(t-\tau_1-\tau_2), \quad (18c)$$

$$J(t) = \frac{\beta}{N(t)} S(t) I(t), \quad (18d)$$

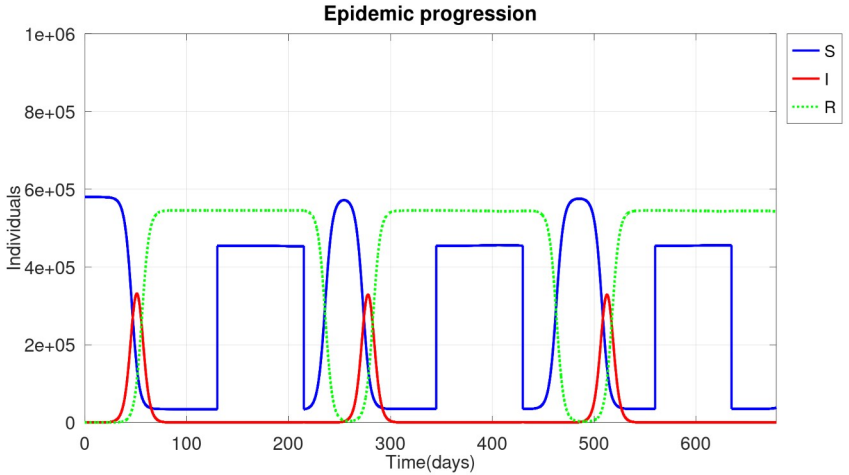
and we impose consecutive lockdowns (before each epidemic outbreak) characterized by the moment t_j when they begin, duration T_j , and the proportion $k_j \in (0, 1)$ of isolated population:

$$N(t_j) = (1-k_j) N_0, \; S(t_j) = S(t_j) - k_j N_0, \quad N(t_j+T_j) = N_0, \; S(t_j+T_j) = S(t_j+T_j) + k_j N_0. \quad (19)$$

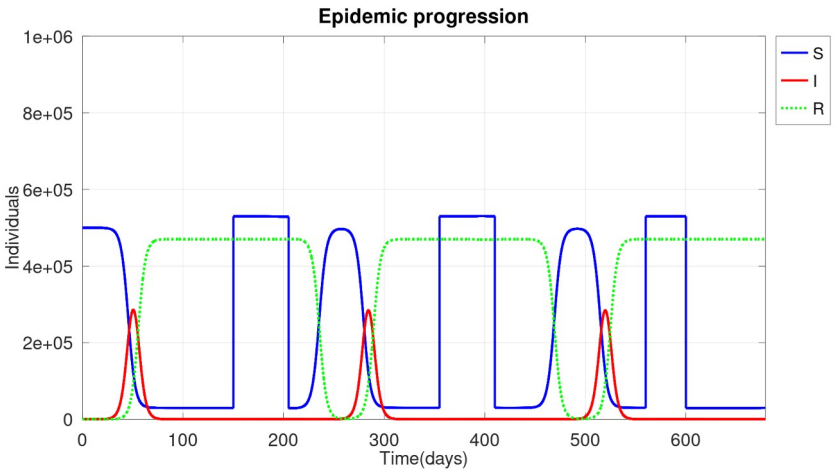
System (20) is considered with the following initial conditions

$$S(\theta) = N_0, \; I(\theta) = 0 : \forall \theta \in [-(\tau_1+\tau_2), 0), \quad S(0) = S_0 > 0, \; I(0) = I_0 > 0 \; (S_0 + I_0 = N_0), \; R(0) = 0. \quad (20)$$

Figure 3.



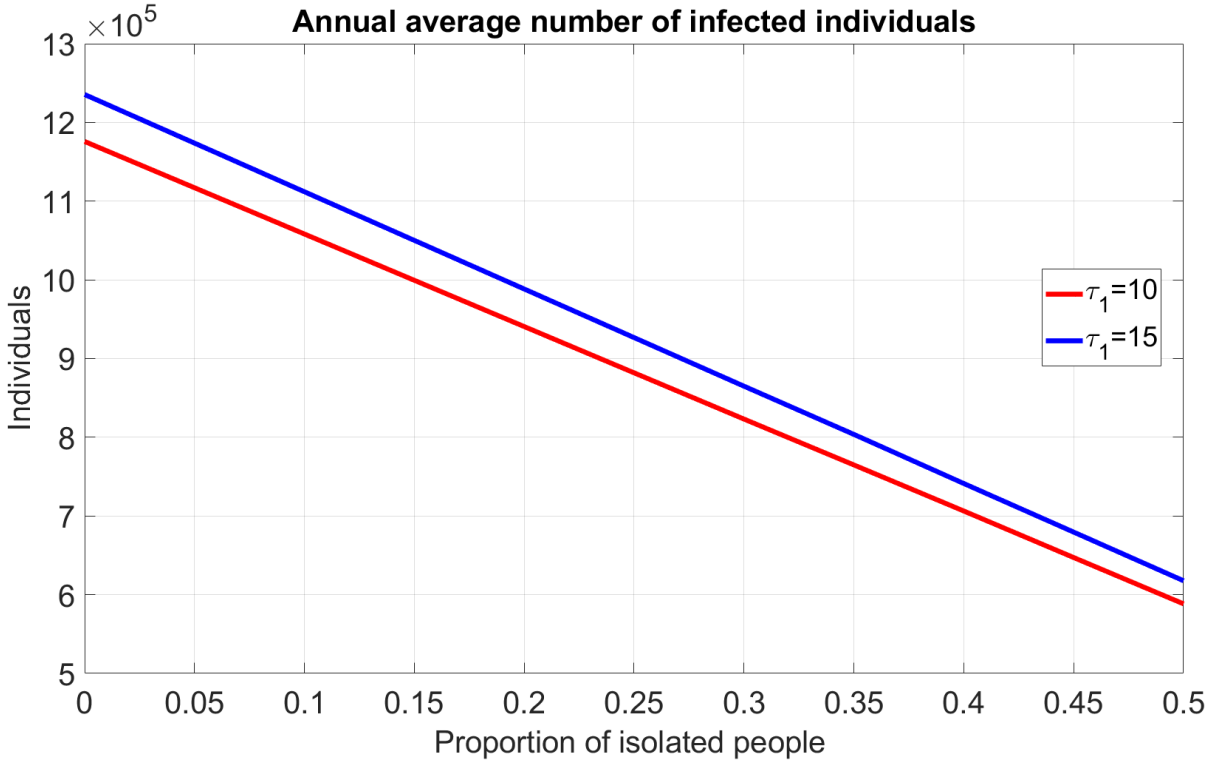
(a)



(b)

Susceptible, infected, and recovered populations in numerical simulations of System (18) with parameter values $\beta = 0.3$, $\tau_1 = 10, \tau_2 = 180, N_0 = 10^6$, $I(0) = 1$. In panel (a), we impose the following four lockdowns: $\{(t_1 = 0, T_1 = 130, k_1 = 0.42), (t_2 = 215, T_2 = 130, k_2 = 0.42), (t_3 = 430, T_3 = 130, k_3 = 0.42), (t_4 = 635, T_4 = 130, k_4 = 0.42)\}$. The average annual number of infected individuals is $I_{avr} = 682{,}405$. In panel (b), we impose the following four lockdowns: $\{(t_1 = 0, T_1 = 150, k_1 = 0.5), (t_2 = 205, T_2 = 150, k_2 = 0.5), (t_3 = 410, T_3 = 150, k_3 = 0.5), (t_4 = 600, T_4 = 150, k_4 = 0.5)\}$. The average annual number of infected individuals is $I_{avr} = 588{,}353$.

Figure 4.



Dependence of the annual average number of infected individuals $I_{avr} = \frac{1}{n\tau_1}\int_0^{365n} I(\eta)\,d\eta$ on the proportion of isolated people in the model (18) with immunity waning and periodic outbreaks. The values of parameters: $N_0=10^6$, $\beta=0.3$, $\tau_2=180$, $I(0) = 1$.

4. Discussion

- In this talk, we proposed an epidemiological model that incorporates a system of delay differential equations, featuring two time delays correspond to the duration of the disease and the period of natural immunity.
- The reduction of the delay model to an integral equation allows us to study stationary solutions of this model and their stability. A positive stationary solution appears for the basic reproduction number larger than 1. It loses its stability and leads to periodic oscillations if the basic reproduction number exceeds some critical value.
- Imposing partial lockdowns reduces the epidemic.
- One of the main results of this work is the determination of the optimal proportion of isolated people.

- In the case without immunity waning, this optimal choice represents the maximal proportion of isolated for which the epidemic does not restart when the isolation is finished. This condition allows us to determine this optimal proportion analytically.
- In the case with immunity waning, the annual average number of infected individuals is a decreasing function with respect to the proportion of isolated people during lockdowns, and does not have a local minimum as in the model with permanent immunity.
- This study has some limitations. First, discrete delays prescribe single values of the disease duration and immunity waning instead of distributions. Furthermore, we did not consider exposed compartments, which may have some influence on the economic state of the population. These questions and some others represent interesting open questions for forthcoming works.

Thanks for attention !