Stable numerical schemes for modelling hemodynamic flows in time-dependent domains

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Mathematical models and numerical methods in biology and medicine

INM RAS

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Three ideas

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- Moving domain $\Rightarrow$ moving mesh $\Rightarrow$ $\frac{\partial \mathbf{v}}{\partial t}$? $\Rightarrow$ reformulate the problem in a steady reference domain at the cost of unsteady coefficients

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- Linearization of the fully implicit scheme by extrapolation in time $\Rightarrow$ stability (estimate) for large Δt and solution of one linear system per time step

Three ideas

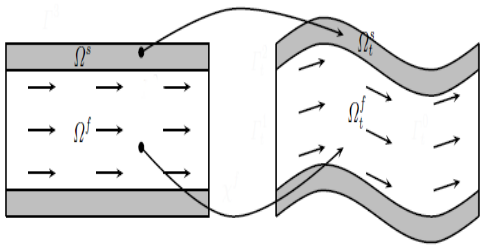
- Moving domain $\Rightarrow$ moving mesh $\Rightarrow$ $\frac{\partial \mathbf{v}}{\partial t}$? $\Rightarrow$ reformulate the problem in a steady reference domain at the cost of unsteady coefficients
- Linearization of the fully implicit scheme by extrapolation in time $\Rightarrow$ stability (estimate) for large Δt and solution of one linear system per time step
- P_2/P_1 (Taylor-Hood FE) is a feasible compromise between accuracy and coarse unstructured tetrahedrizations

Computational methodology

- Consistent unstructured tetrahedral mesh
- LBB-stable pair for velocity and pressure P_2/P_1, P_2 for displacements
- Open source software `Ani3D` (Advanced numerical instruments 3D, K.Lipnikov, Yu.Vassilevski et al.)
  `http://sf.net/p/ani3d/`
- MUMPS as the parallel solver of linear systems
- $\sim 10^2$ cores, $\sim$ 10 hours (3D FSI)

Navier-Stokes equations in moving domains

Prerequisites



- reference domain Ω_f
- transformation $\boldsymbol{\xi}$ maps Ω_f to $\Omega_f(t)$
- $\mathbf{v}$ and $\mathbf{u}$ denote velocities and displacements in Ω_f
- $\boldsymbol{\xi}(\mathbf{x}) := \mathbf{x} + \mathbf{u}(\mathbf{x})$, $\mathbf{F} := \nabla\boldsymbol{\xi} = \mathbf{I} + \nabla\mathbf{u}$, $J := \det(\mathbf{F})$
- Cauchy stress tensor $\boldsymbol{\sigma}_f$
- pressures p_f
- density ρ_f is constant

Incompressible fluid flow in a moving domain

Navier-Stokes equations in reference domain Ω_f

Let $\boldsymbol{\xi}$ mapping Ω_f to $\Omega_f(t)$, $\mathsf{F} = \nabla\boldsymbol{\xi} = \mathsf{I} + \nabla\mathsf{u}$, $J = \det(\mathsf{F})$ be given

Dynamic equations

$$\frac{\partial \mathsf{v}}{\partial t} = (J\rho_f)^{-1}\mathrm{div}\,(J\boldsymbol{\sigma}_f\mathsf{F}^{-T}) - \nabla\mathsf{v}\left(\mathsf{F}^{-1}\left(\mathsf{v} - \frac{\partial \mathsf{u}}{\partial t}\right)\right) \quad \text{in } \Omega_f$$

Fluid incompressibility

$$\mathrm{div}\,(J\mathsf{F}^{-1}\mathsf{v}) = 0 \quad \text{in } \Omega_f$$

Constitutive relation for the fluid stress tensor

$$\boldsymbol{\sigma}_f = -p_f\mathsf{I} + \mu_f((\nabla\mathsf{v})\mathsf{F}^{-1} + \mathsf{F}^{-T}(\nabla\mathsf{v})^T) \quad \text{in } \Omega_f$$

Finite element scheme

Let $\mathbb{V}_h, \mathbb{Q}_h$ be Taylor-Hood P_2/P_1 finite element spaces.

Find $\{\mathbf{v}^k, p^k\} \in \mathbb{V}_h \times \mathbb{Q}_h$ satisfying b.c.

("do nothing"$\boldsymbol{\sigma}\mathbf{F}^{-T}\mathbf{n} = 0$ or no-penetration no-slip $\mathbf{v}^k = (\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1})/\Delta t$)

$$\int_{\Omega_f} J_k \frac{\mathbf{v}^k - \mathbf{v}^{k-1}}{\Delta t} \cdot \boldsymbol{\psi}\, \mathrm{d}\mathbf{x} + \int_{\Omega_f} J_k \nabla \mathbf{v}^k \mathbf{F}_k^{-1} \left(\mathbf{v}^{k-1} - \frac{\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1}}{\Delta t} \right) \cdot \boldsymbol{\psi}\, \mathrm{d}\mathbf{x} -$$
$$\int_{\Omega_f} J_k p^k \mathbf{F}_k^{-T} : \nabla \boldsymbol{\psi}\, \mathrm{d}\mathbf{x} + \int_{\Omega_f} J_k q \mathbf{F}_k^{-T} : \nabla \mathbf{v}^k\, \mathrm{d}\mathbf{x} +$$
$$\int_{\Omega_f} \nu J_k (\nabla \mathbf{v}^k \mathbf{F}_k^{-1} \mathbf{F}_k^{-T} + \mathbf{F}_k^{-T} (\nabla \mathbf{v}^k)^T \mathbf{F}_k^{-T}) : \nabla \boldsymbol{\psi}\, \mathrm{d}\mathbf{x} = 0$$

$$\int_{\Omega_f} J_k \nabla \mathbf{v}^k : \mathbf{F}_k^{-T} q\, \mathrm{d}\Omega = 0$$

for all $\boldsymbol{\psi}$ and q from the appropriate FE spaces

Finite element scheme

The scheme

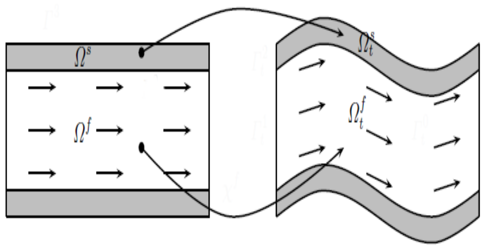
- semi-implicit
- produces one linear system per time step
- first order in time (may be generalized to the second order)
- unconditionally stable (stability estimate without CFL restriction) and 2nd order accurate, proved with assumptions:
  - $\inf_Q J \geq c_J > 0, \quad \sup_Q(\|\mathbf{F}\|_F + \|\mathbf{F}^{-1}\|_F) \leq C_F$
  - LBB-stable pairs (e.g. P_2/P_1)
  - Δt is not large

A.Danilov, A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A finite element method for the Navier-Stokes equations in moving domain with application to hemodynamics of the left ventricle. *Russian J. Numer. Anal. Math. Modelling, 32, 2017*

A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A quasi-Lagrangian finite element method for the Navier-Stokes equations in a time-dependent domain. Comput.Methods Appl.Mech. Engrg.333,2018

Fluid-Structure Interaction problem

Prerequisites



- reference subdomains Ω_f, Ω_s
- transformation $\boldsymbol{\xi}$ maps Ω_f, Ω_s to $\Omega_f(t)$, $\Omega_s(t)$
- $\mathbf{v}$ and $\mathbf{u}$ denote velocity and displacement in $\Omega := \Omega_f \cup \Omega_s$
- $\boldsymbol{\xi}(\mathbf{x}) := \mathbf{x} + \mathbf{u}(\mathbf{x})$, $\mathbf{F} := \nabla\boldsymbol{\xi} = \mathbf{I} + \nabla\mathbf{u}$, $J := \det(\mathbf{F})$
- Cauchy stress tensors $\boldsymbol{\sigma}_f$, $\boldsymbol{\sigma}_s$
- pressure p_f,p_s
- density ρ_f is constant

Universal equations in reference subdomains

Dynamic equations

$$\frac{\partial \mathsf{v}}{\partial t} = \begin{cases} \rho_s^{-1} \mathrm{div}\left(J \boldsymbol{\sigma}_s \mathsf{F}^{-T}\right) & \text{in } \Omega_s \\ (J\rho_f)^{-1} \mathrm{div}\left(J \boldsymbol{\sigma}_f \mathsf{F}^{-T}\right) - \nabla \mathsf{v} \left(\mathsf{F}^{-1} \left(\mathsf{v} - \frac{\partial \mathsf{u}}{\partial t} \right) \right) & \text{in } \Omega_f \end{cases}$$

Kinematic equation

$$\frac{\partial \mathsf{u}}{\partial t} = \mathsf{v} \quad \text{in } \Omega_s$$

Fluid incompressibility

$$\mathrm{div}\left(J \mathsf{F}^{-1} \mathsf{v}\right) = 0 \quad \text{in } \Omega_f$$

Constitutive relation for the fluid stress tensor

$$\boldsymbol{\sigma}_f = -p_f \mathsf{I} + \mu_f((\nabla \mathsf{v})\mathsf{F}^{-1} + \mathsf{F}^{-T}(\nabla \mathsf{v})^T) \quad \text{in } \Omega_f$$

FSI problem

User-dependent equations in reference subdomains

Constitutive relation for the solid stress tensor

$$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s(J, \mathsf{F}, p_s, \lambda_s, \mu_s, \dots) \quad \text{in } \Omega_s$$

Monolithic approach [1] : Extension of the displacement field to the fluid domain

$$\begin{aligned} G(\mathsf{u}) &= 0 \quad \text{in } \Omega_f, \\ \mathsf{u} &= \mathsf{u}^* \quad \text{on } \partial\Omega_f \end{aligned}$$

for example, vector Laplace equation or elasticity equation

$+$ Initial, boundary, interface conditions ($\boldsymbol{\sigma}_f \mathsf{F}^{-T}\mathsf{n} = \boldsymbol{\sigma}_s \mathsf{F}^{-T}\mathsf{n}$)

[1] Michler et al (2004), Hubner et al (2004), Hron&Turek (2006),...

Numerical scheme

Find $\{\mathbf{u}^{k+1},\mathbf{v}^{k+1},p^{k+1}\}\in\mathbb{V}_h^0\times\mathbb{V}_h\times\mathbb{Q}_h$ satisfying b.c. and $\left[\frac{\partial\mathbf{u}}{\partial t}\right]_{k+1}=\mathbf{v}^{k+1}$ on Γ_{fs}

$$
\begin{aligned}
&\int_{\Omega_s}\rho_s\left[\frac{\partial\mathbf{v}}{\partial t}\right]_{k+1}\boldsymbol{\psi}\,\mathrm{d}\Omega+\int_{\Omega_s}J_k\mathbf{F}(\widetilde{\mathbf{u}}^k)(\lambda_s\mathtt{tr}(\mathbf{E}(\mathbf{u}^{k+1},\widetilde{\mathbf{u}}^k))\mathbf{I}+2\mu_s\mathbf{E}(\mathbf{u}^{k+1},\widetilde{\mathbf{u}}^k)):\nabla\boldsymbol{\psi}\,\mathrm{d}\Omega+\\
&\int_{\Omega_f}\rho_f J_k\left[\frac{\partial\mathbf{v}}{\partial t}\right]_{k+1}\boldsymbol{\psi}\,\mathrm{d}\Omega+\int_{\Omega_f}\rho_f J_k\nabla\mathbf{v}^{k+1}\mathbf{F}^{-1}(\widetilde{\mathbf{u}}^k)\left(\widetilde{\mathbf{v}}^k-\widetilde{\left[\frac{\partial\mathbf{u}}{\partial t}\right]}_k\right)\boldsymbol{\psi}\,\mathrm{d}\Omega+\\
&\int_{\Omega_f}2\mu_f J_k\mathbf{D}_{\widetilde{\mathbf{u}}^k}\mathbf{v}^{k+1}:\mathbf{D}_{\widetilde{\mathbf{u}}^k}\boldsymbol{\psi}\,\mathrm{d}\Omega-\int_{\Omega_f}p^{k+1}J_k\mathbf{F}^{-T}(\widetilde{\mathbf{u}}^k):\nabla\boldsymbol{\psi}\,\mathrm{d}\Omega=0\quad\forall\boldsymbol{\psi}\in\mathbb{V}_h^0
\end{aligned}
$$

$$
\int_{\Omega_s}\left[\frac{\partial\mathbf{u}}{\partial t}\right]_{k+1}\boldsymbol{\phi}\,\mathrm{d}\Omega-\int_{\Omega_s}\mathbf{v}^{k+1}\boldsymbol{\phi}\,\mathrm{d}\Omega+\int_{\Omega_f}G(\mathbf{u}^{k+1})\boldsymbol{\phi}\,\mathrm{d}\Omega=0\quad\forall\boldsymbol{\phi}\in\mathbb{V}_h^{00}
$$

$$
\int_{\Omega_f}J_k\nabla\mathbf{v}^{k+1}:\mathbf{F}^{-T}(\widetilde{\mathbf{u}}^k)q\,\mathrm{d}\Omega=0\quad\forall\, q\in\mathbb{Q}_h
$$

$J_k:=J(\widetilde{\mathbf{u}}^k)$, $\widetilde{\mathbf{f}}^k:=2\mathbf{f}^k-\mathbf{f}^{k-1}$, $\mathbf{D}_\mathbf{u}\mathbf{v}:=\{\nabla\mathbf{v}\mathbf{F}^{-1}(\mathbf{u})\}_s$, $\{\mathbf{A}\}_s:=\frac{1}{2}(\mathbf{A}+\mathbf{A}^T)$ $\mathbf{E}(\mathbf{u}_1,\mathbf{u}_2):=\{\mathbf{F}(\mathbf{u}_1)^T\mathbf{F}(\mathbf{u}_2)-\mathbf{I}\}_s$

Numerical scheme

The scheme

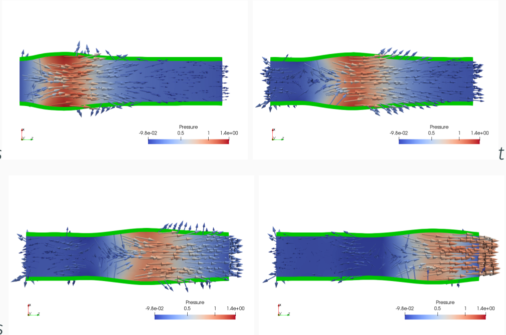
- provides strong coupling on interface
- semi-implicit
- produces one linear system per time step
- may be first or second order in time
- unconditionally stable (stability estimate without CFL restriction), proved with assumptions:
  - 1st order in time
  - St. Venant–Kirchhoff inc./comp. (experiment: Neo-Hookean inc./comp.)
  - extension of $\mathbf{u}$ to Ω_f guarantees $J_k > 0$
  - Δt is not large

A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. Analysis and assessment of a monolithic FSI finite element method. *Computers and Fluids, 179, 2019*

A.Lozovskiy, M.Olshanskii, V.Salamatova, Yu.Vassilevski. An unconditionally stable semi-implicit FSI finite element method. *Comput.Methods Appl.Mech.Engrg., 297, 2015*

3D: pressure wave in flexible tube

L.Formaggia et al., *CMAME* 191: 561–582, 2001



$t = 0.004s$ $t = 0.006s$

$t = 0.008s$ $t = 0.01s$

Pressure wave: middle cross-section velocity field, pressure distribution, velocity vectors and 10× enlarged structure displacement for several time instances

- The tube (fixed at both ends) is 50mm long, it has inner diameter of 10mm and the wall (SVK) is 1mm thick.
- Left end: external pressure p_{ext} is set to $1.333 \cdot 10^3$Pa for $t \in (0, 3 \cdot 10^{-3})$s and zero afterwards, $\boldsymbol{\sigma}_f \mathbf{F}^{-T}\mathbf{n} = p_{ext}\mathbf{n}$. Right end: open boundary
- Simulation was run with $\Delta t = 10^{-4}$ s
- $\#Tets(\Omega_s) = 6336/11904/38016$, $\#Tets(\Omega_f) = 13200/29202/89232$

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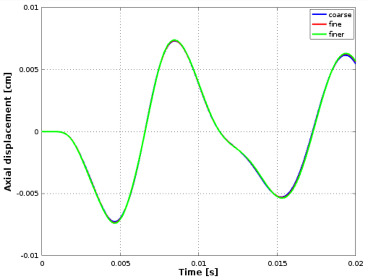
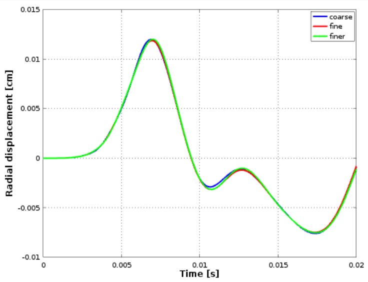


axial component

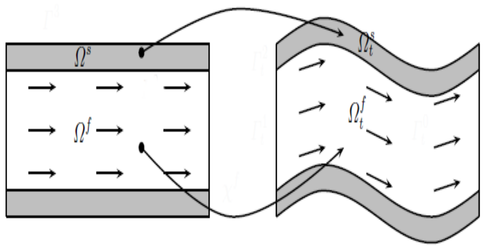


radial component

Pressure wave: The radial and axial components of displacement of the inner tube wall at half the length of the pipe. Solutions are shown for three sequentially refined meshes. The plots are almost indistinguishable.

Fluid-Porous Structure Interaction problem

Prerequisites



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- $\boldsymbol{\xi}(\mathbf{x}) := \mathbf{x} + \mathbf{u}(\mathbf{x})$, $\mathbf{F} := \nabla \boldsymbol{\xi} = \mathbf{I} + \nabla \mathbf{u}$, $J := \det(\mathbf{F})$
- pressures p_f,p_d
- Cauchy stress tensors $\boldsymbol{\sigma}_f$, $\boldsymbol{\sigma}_s$, poroelastic stress $\boldsymbol{\sigma}_p = \boldsymbol{\sigma}_s - \alpha p_d \mathbf{I}$
- porosity ϕ, density of saturated porous medium $\rho_p = \rho_s(1 - \phi) + \rho_f \phi$
- filtration flux $\mathbf{q} = \phi(\mathbf{v} - \mathbf{v}_s)$, permeability tensor K

Fluid-Porous Structure Interaction problem

Dynamic equations

$$\begin{aligned}
\rho_p \frac{\partial \mathbf{v}_s}{\partial t} + \rho_f \frac{\partial \mathbf{q}}{\partial t} &= J^{-1}\operatorname{div}\left(J\boldsymbol{\sigma}_p(\boldsymbol{\xi}_s)\mathbf{F}^{-T}\right) && \text{in } \Omega_s \\
\rho_p \frac{\partial \mathbf{v}_s}{\partial t} + \frac{\rho_f}{\phi} \frac{\partial \mathbf{q}}{\partial t} &= -K^{-1}\mathbf{q} - \mathbf{F}^{-T}\nabla p_d && \text{in } \Omega_s
\end{aligned}$$

$$\rho_f \frac{\partial \mathbf{v}_f}{\partial t} = J^{-1}\operatorname{div}\left(J\boldsymbol{\sigma}_f(\boldsymbol{\xi}_f)\mathbf{F}^{-T}\right) - \rho_f \nabla \mathbf{v}_f \left(\mathbf{F}^{-1}\left(\mathbf{v}_f - \frac{\partial \mathbf{u}}{\partial t}\right)\right) \quad \text{in } \Omega_f$$

Mass conservation

$$\operatorname{div}\left(J\mathbf{F}^{-1}\mathbf{v}\right) = 0 \quad \text{in } \Omega_f \quad \text{and} \quad \operatorname{div}\left(\mathbf{F}^{-1}(\mathbf{v}_s + \mathbf{q})\right) = -s_0 \frac{\partial p_d}{\partial t} \quad \text{in } \Omega_s$$

A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A finite element scheme for the numerical solution of the Navier-Stokes/Biot coupled problem. *Russian J. Numer. Anal. Math. Modelling, 37(3), 2022*

Fluid-Porous Structure Interaction problem

Kinematic equation

$$\frac{\partial \mathsf{u}}{\partial t} = \mathsf{v} \quad \text{in } \Omega_s$$

Constitutive relation for the fluid stress tensor

$$\boldsymbol{\sigma}_f = -p_f \mathsf{I} + \mu_f((\nabla \mathsf{v})\mathsf{F}^{-1} + \mathsf{F}^{-T}(\nabla \mathsf{v})^T) \quad \text{in } \Omega_f$$

Constitutive relation for the solid stress tensor

$$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s(J, \mathsf{F}, p_s, \lambda_s, \mu_s, \dots) \quad \text{in } \Omega_s$$

Conditions on actual (physical) interface

$\boldsymbol{\sigma}_f \mathsf{n} = \boldsymbol{\sigma}_p \mathsf{n}, \quad \mathsf{n}^T \boldsymbol{\sigma}_f \mathsf{n} = -p_d + \frac{\rho_f}{2} |\mathsf{v}_f|^2$ balance of stresses

$\mathsf{v}_f \cdot \mathsf{n} = (\mathsf{v}_s + \mathsf{q}) \cdot \mathsf{n}$ conservation of fluid

$\mathsf{P}_\Gamma \boldsymbol{\sigma}_f \mathsf{n} = -\gamma \mathsf{P}_\Gamma K^{-\frac{1}{2}} (\mathsf{v}_f - \mathsf{v}_s)$ Beavers–Joseph–Saffman

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Fluid-Porous Structure Interaction problem

FE scheme with P_2 velocities,filtration flux,displacements and P_1 pressures

- provides strong coupling on interface
- semi-implicit
- produces one linear system per time step
- may be first or second order in time
- unconditionally stable (stability estimate without CFL restriction), proved with assumptions:
  - 1st order in time
  - St. Venant–Kirchhoff inc./comp.
  - extension of $\mathbf{u}$ to Ω_f guarantees $J_k > 0$
  - Δt is not large

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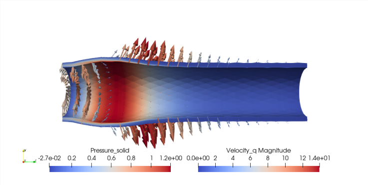
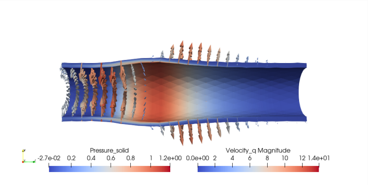
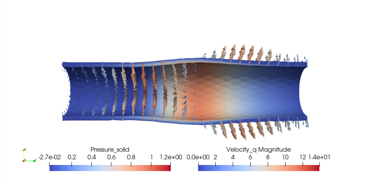
3D: pressure wave in poroelastic flexible tube



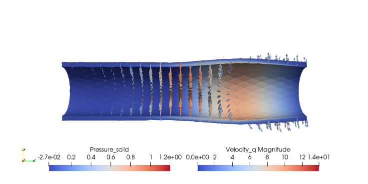
$t = 0.004s$



$t = 0.006s$



$t = 0.008s$

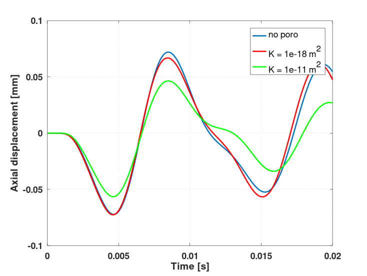


$t = 0.01s$

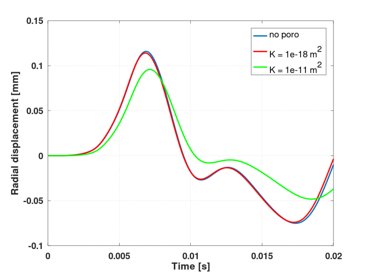
Pressure wave: porous pressure p_d and filtration velocity $\mathbf{q}$ distribution in the solid for $K = 10^{-5} mm^2$: middle cross-section view, with 10-fold enlarged structure displacement

- The tube (fixed at both ends) is 50mm long, it has inner diameter of 10mm and the wall (SVK) is 1mm thick.
- Left end: external pressure p_{ext} is set to $1.333 \cdot 10^3$Pa for $t \in (0, 3 \cdot 10^{-3})$s and zero afterwards, $\boldsymbol{\sigma}_f \mathbf{F}^{-T}\mathbf{n} = p_{ext}\mathbf{n}$. Right end: open boundary
- Simulation was run with $\Delta t = 10^{-4}$ s, $\#Tets(\Omega_S) = 7200$, $\#Tets(\Omega_f) = 13200$, $\#unknowns = 356000$

3D: pressure wave in poroelastic flexible tube



axial component



radial component

Pressure wave: The radial and axial components of displacement of the inner tube wall at half the length of the pipe. Solutions are shown for three permeabilities.

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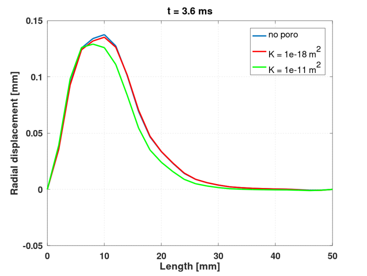
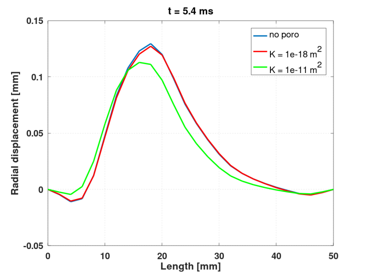
3D: pressure wave in poroelastic flexible tube



$t = 0.0036\mathrm{s}$



$t = 0.0054\mathrm{s}$

Pressure wave: Wall profile on the inner side along the tube length for two time instances.

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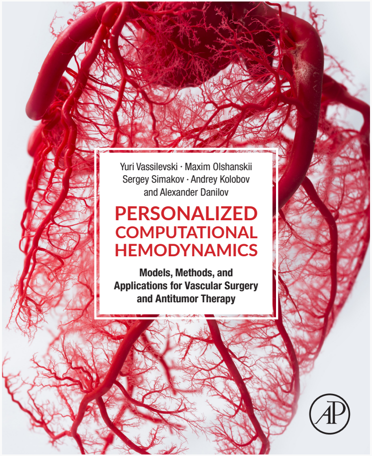
For details refer to



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