Inverse problem of recovering parameters of the medium in acoustic tomography

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We investigate the mathematical model of the 2D acoustic waves propagation in homogeneous and heterogeneous areas. The hyperbolic first order system of partial differential equations is considered and solved by the Godunov method of the first order of approximation. This is direct problem with appropriate boundary conditions.

As the main aim of the work we solve coefficient inverse problem of recovering density of the medium. Inverse problem is reduced to an optimization problem which is solved by gradient descent method.

Direct problem

Two-dimensional system of equations of acoustic waves propagation

Equations (conservations laws)

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial t} + \rho c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \Theta_\Omega(x,y) I(t)$$

$(\mathrm{x},\mathrm{y}) \in \Omega$
$0 < t < T$

Boundary and initial conditions

$$u, v, p\Big|_{(x,y)\in\partial\Omega} = 0$$

$$u, v, p\Big|_{t=0} = 0$$

u – velocity in direction x
v – velocity in direction y
p – pressure

ρ – density
c – sound speed

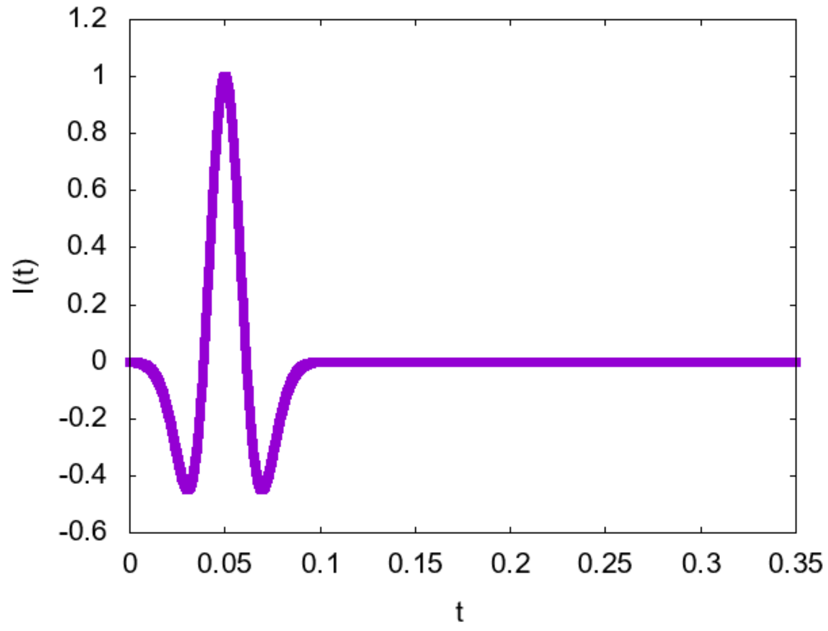
Direct problem: find u, v, p inside Ω

Domain

$$\Omega = (\mathrm{x},\mathrm{y}) \in [0{:}L] \times [0{:}L]$$

Ricker wavelet

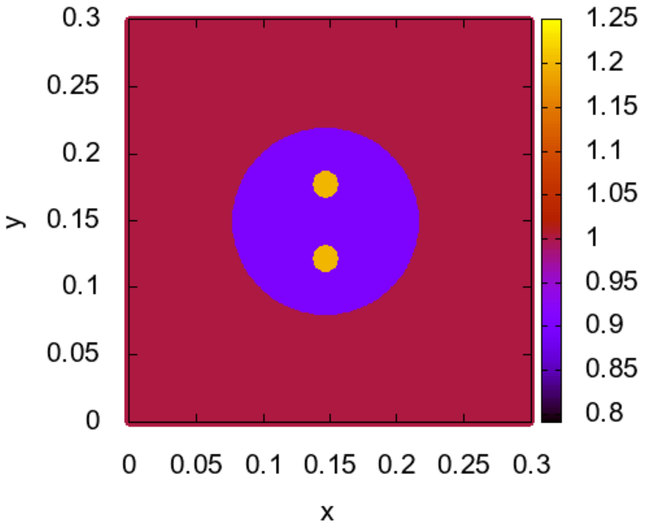
$$I(t) = \left(1 - 2\left(\pi\,\nu_0\left(t - \frac{1}{\nu_0}\right)\right)^2\right) e^{-\pi\,\nu_0\left(t-\frac{1}{\nu_0}\right)}$$

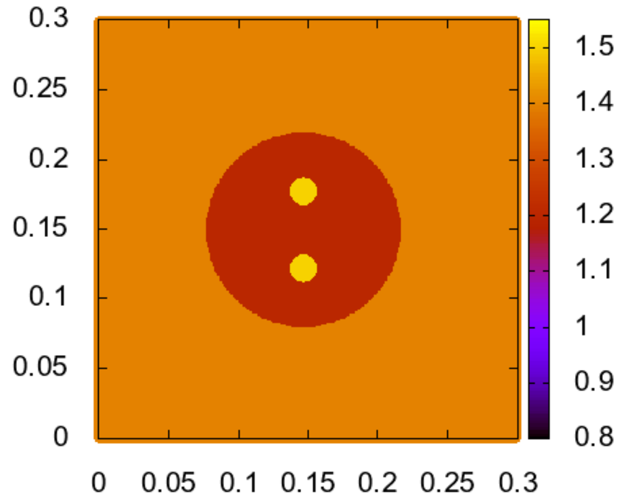


Types of medium

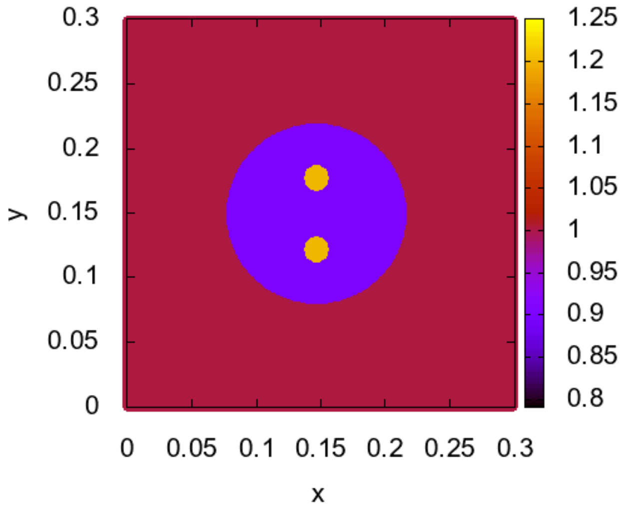
$\rho(x, y)$ $c(x, y)$

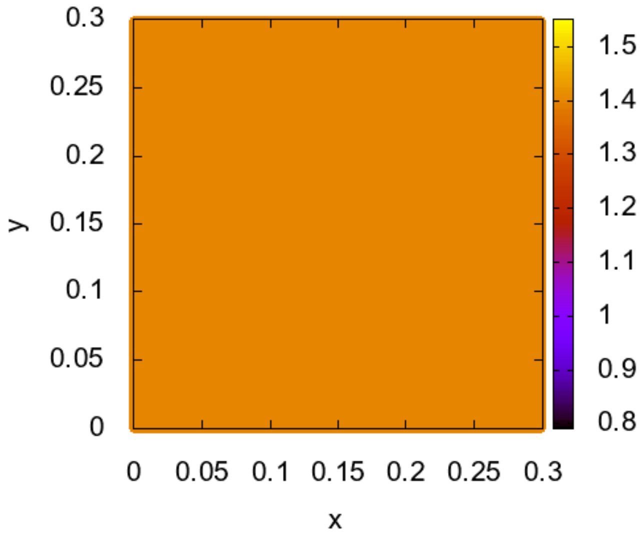
Heterogeneous density and heterogeneous velocity



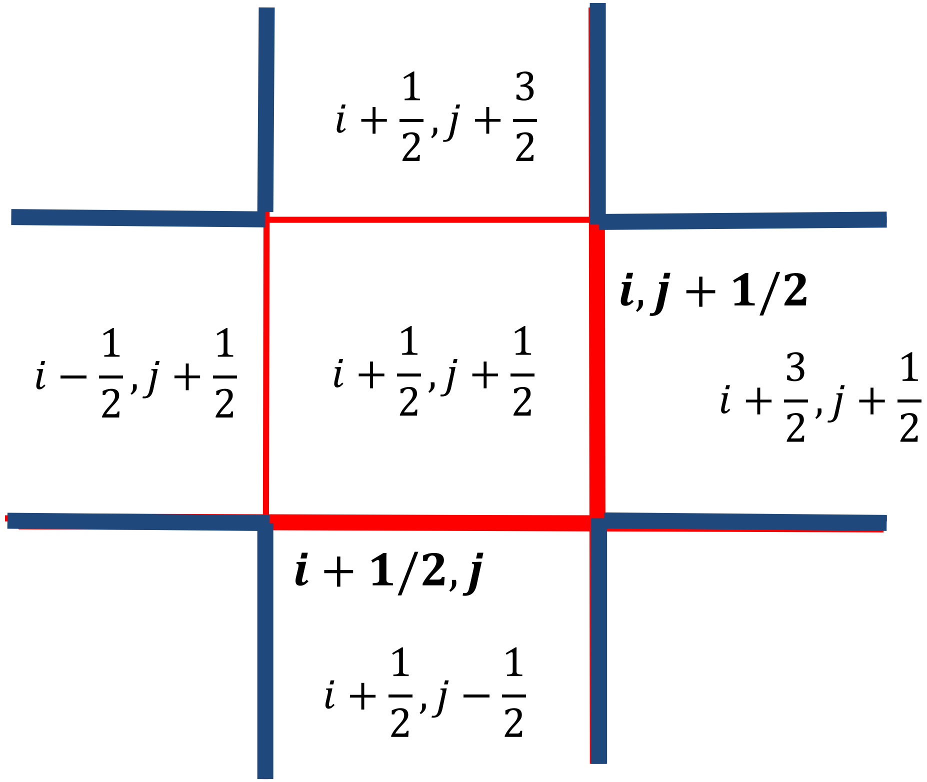


Heterogeneous density and homogeneous velocity





Godunov method for 1D acoustic equations.



Part 1. Finding solution of decay of discontinuity on each boundary

$$U_{i,j+1/2} = \frac{u_{i-1/2,j+1/2} + u_{i+1/2,j+1/2}}{2} - \frac{1}{\rho c}\frac{p_{i+1/2,j+1/2} - p_{i-1/2,j+1/2}}{2}$$

$$P_{i,j+1/2} = \frac{p_{i-1/2,j+1/2} + p_{i+1/2,j+1/2}}{2} - \rho c\frac{u_{i+1/2,j+1/2} - u_{i-1/2,j+1/2}}{2}$$

$$V_{i+1/2,j} = \frac{v_{i+1/2,j-1/2} + v_{i+1/2,j+1/2}}{2} - \frac{1}{\rho c}\frac{p_{i+1/2,j+1/2} - p_{i+1/2,j-1/2}}{2}$$

$$P_{i+1/2,j} = \frac{p_{i+1/2,j-1/2} + p_{i+1/2,j+1/2}}{2} - \rho c\frac{v_{i+1/2,j+1/2} - v_{i+1/2,j-1/2}}{2}$$

Part 2. Applying finite-difference conservation laws

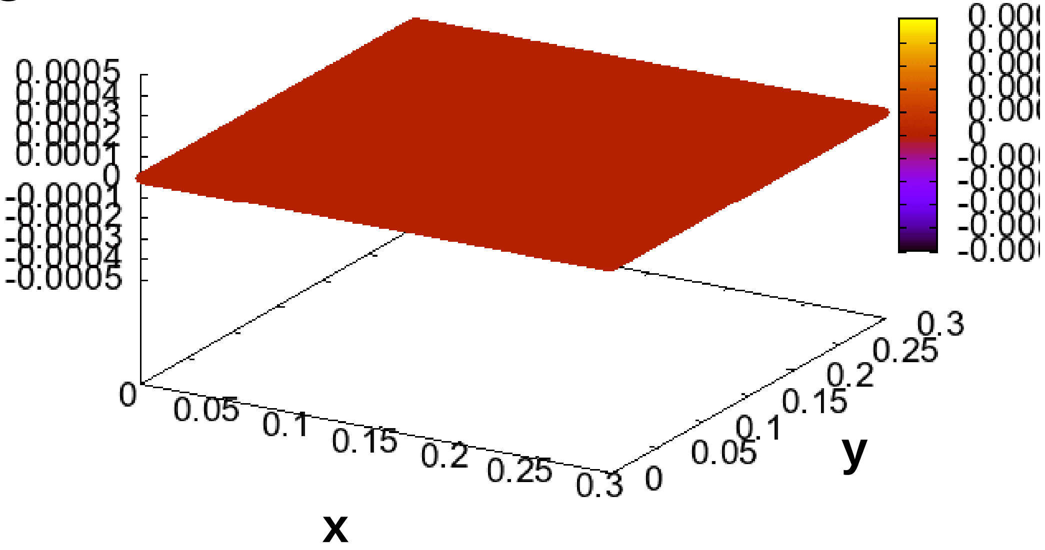
$$u^{i+1/2,j+1/2} = u_{i+1/2,j+1/2} - \frac{\tau}{\rho h_x}\left(P_{i+1,j+1/2} - P_{i,j+1/2}\right)$$

$$v^{i+1/2,j+1/2} = v_{i+1/2,j+1/2} - \frac{\tau}{\rho h_y}\left(P_{i+1/2,j+1} - P_{i+1/2,j}\right)$$

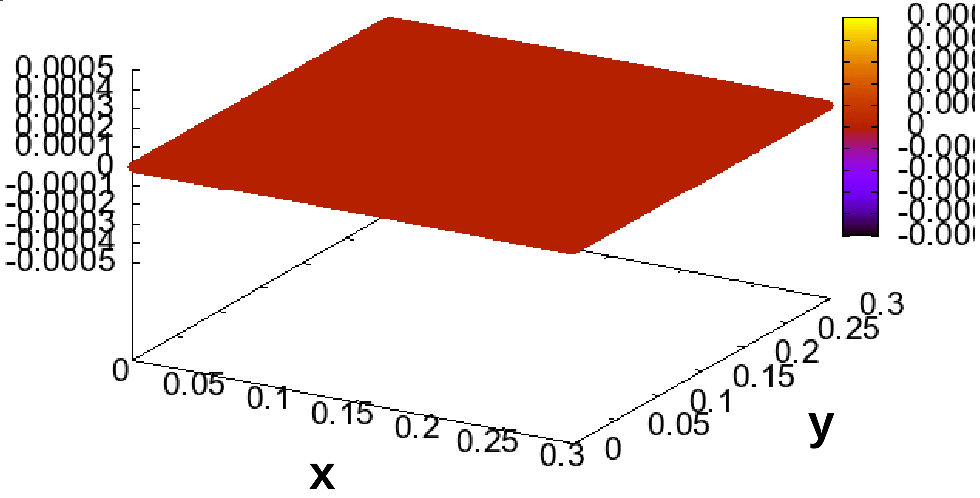
$$p^{i+1/2,j+1/2} = p_{i+1/2,j+1/2} - \frac{\tau}{h_x}\rho c^2\left(U_{i+1,j+1/2} - U_{i,j+1/2}\right) - \frac{\tau}{h_y}\rho c^2\left(V_{i+1/2,j+1} - V_{i+1/2,j}\right) - \tau\theta_{\Omega_{i+1/2,j+1/2}} I_k$$

Direct problem solution

U



V



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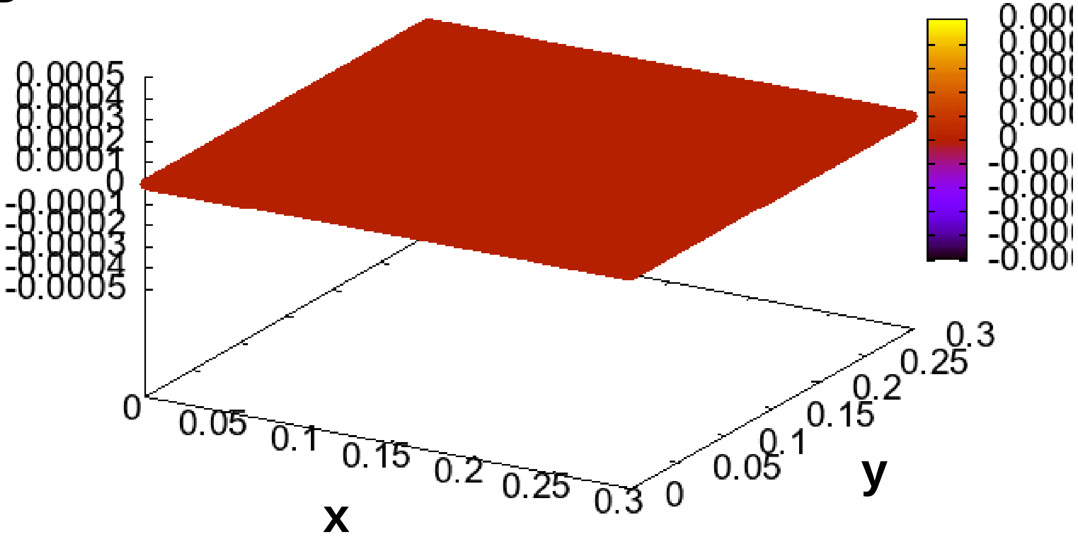
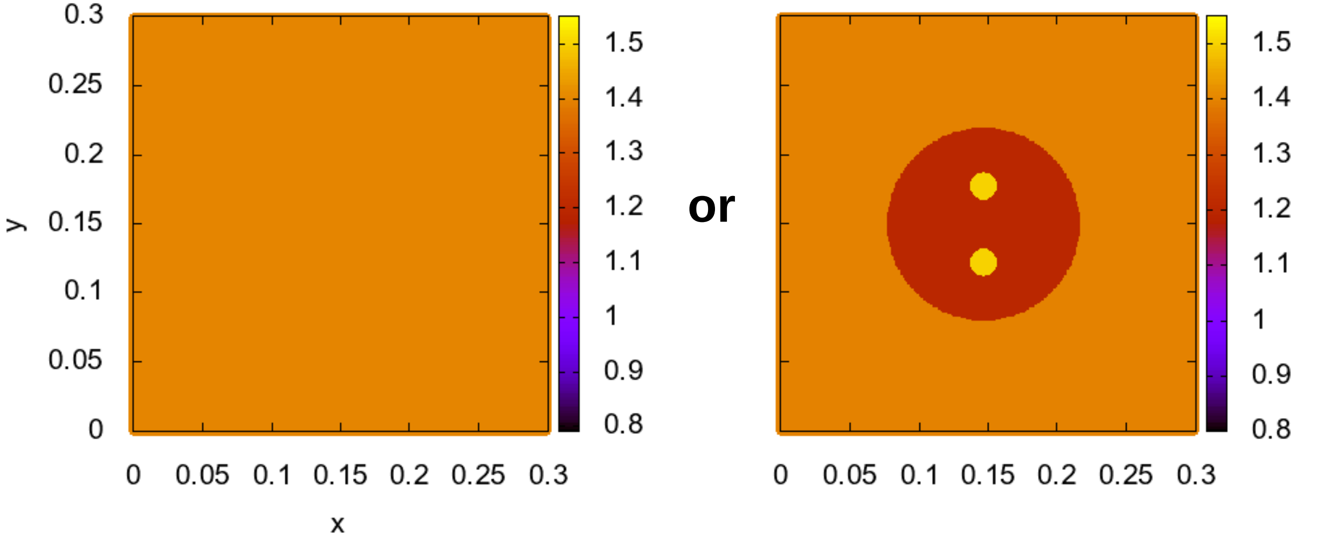


Inverse problem

What is the goal of solving inverse problem?

$\rho(x, y)$ $c(x, y)$

Density reconstruction **?**

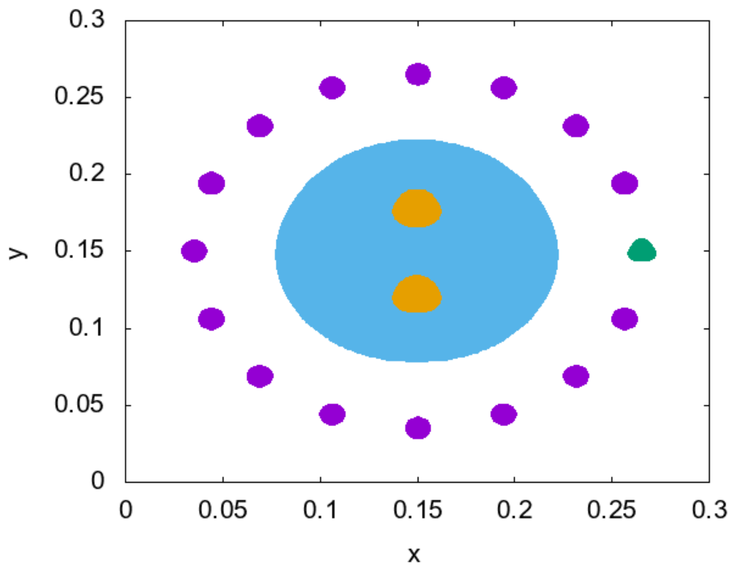


We have additional information about solution

There is system of receivers where data is registered

$$f_i(t) = p(x_i, y_i, t),\quad i = 1 \dots N$$

Here N – is a number of all points (x_i, y_i) that represent any part of receiver



System of
1 source (green)
15 receivers (violet)
1 object (blue)
2 inclusions (orange)

The statement of inverse problem

Equations (conservations laws)

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial t} + \rho c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \Theta_\Omega(x,y)I(t)$$

$(\mathrm{x},\mathrm{y}) \in \Omega$
$0 < t < T$

Boundary and initial conditions

$$u, v, p\Big|_{(x,y)\in\partial\Omega} = 0$$

$$u, v, p\Big|_{t=0} = 0$$

Additional information (data)

$$f_i(t) = p(x_i, y_i, t),$$
$$i = 1 \dots N$$

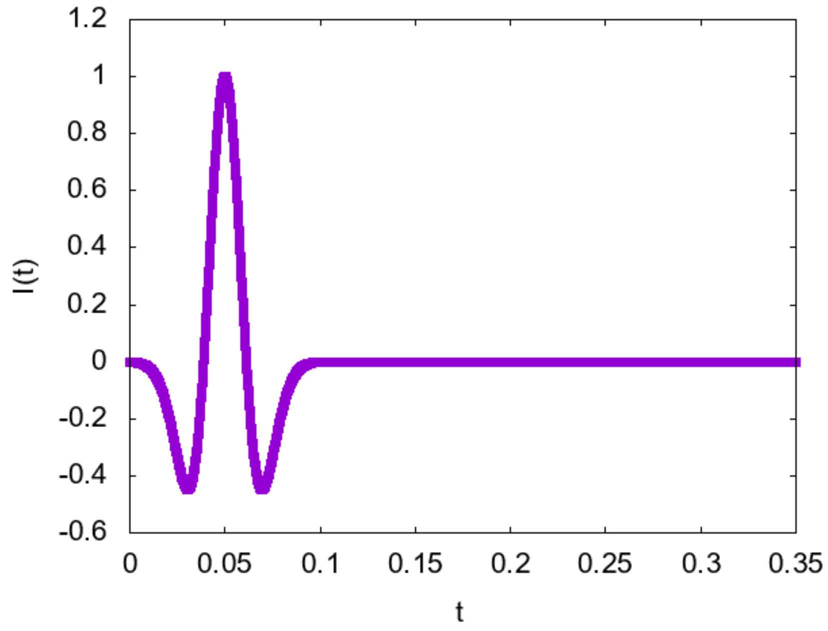
Inverse problem: find density ρ inside Ω

Domain

$$\Omega = (\mathrm{x},\mathrm{y}) \in [0:L] \times [0:L]$$

Ricker wavelet

$$I(t) = \left(1 - 2\left(\pi\,\nu_0\left(t - \frac{1}{\nu_0}\right)\right)^2\right) e^{-\pi\,\nu_0\left(t - \frac{1}{\nu_0}\right)}$$



Method for solving inverse problem

Inverse problem is reduced to optimization problem. Cost functional is

$$J(\rho) = \int_0^T \sum_{i=1}^{N} [p(x_i, y_i, t; \rho) - f_i(t)]^2 dt =$$

$$= \int_0^T \int_0^L \int_0^L \sum_{i=1}^{N} \delta(x - x_i, y - y_i)[p(x, y, t; \rho) - f_i(t)]^2 dxdydt \to \min_{\rho}$$

Idea: to reduce deviation of approximated and measured data

Method: gradient descent method

How to compute gradient from one source?

Classical method: $\boldsymbol{\rho_{n+1} = \rho_n - \alpha J'(\rho_n)}$

$$J'(\rho) = J'(\rho)(x,y) = \int_0^T -u\psi_{1_t} - v\psi_{2_t} + \frac{1}{\rho}\psi_3(u_x + v_y)dt$$

u, v, p - solution of **direct** problem

ψ_1, ψ_2, ψ_3 - solution of **adjoint** problem

$$\frac{\partial\psi_1}{\partial t} + \frac{1}{\rho}\frac{\partial\psi_3}{\partial x} = 0$$

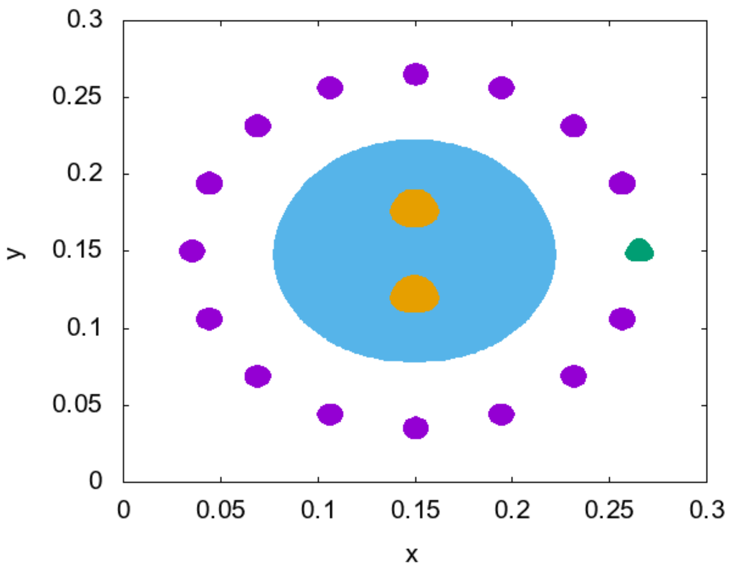
$$\frac{\partial\psi_2}{\partial t} + \frac{1}{\rho}\frac{\partial\psi_3}{\partial y} = 0$$

$$\frac{\partial\psi_3}{\partial t} + \rho c^2\left(\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y}\right) = 2\sum_{i=1}^{N}\delta(x - x_i, y - y_i)[p - f_i]$$

$$\psi_1, \psi_2, \psi_3\Big|_{(x,y)\in\partial\Omega} = 0$$

$$\psi_1, \psi_2, \psi_3\Big|_{t=T} = 0$$

For a current approximation of density ρ and fixed position of source !

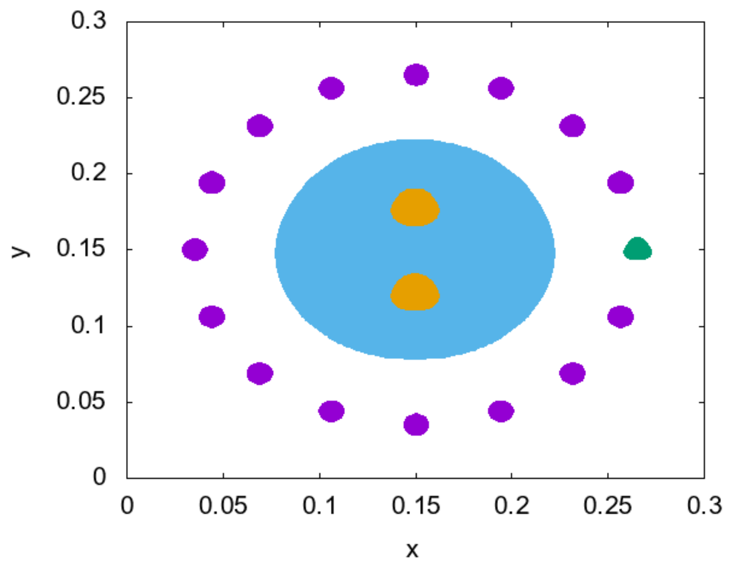


Gradient descent method with changing position of source

1) Classical version

$$\rho_{n+1} = \rho_n - \alpha J'(\rho_n)$$

Each iteration – one gradient of a source from random position in circle
Other positions - receivers



One turn – new iteration

The influence only from **one** source

Gradient descent method with changing position of source

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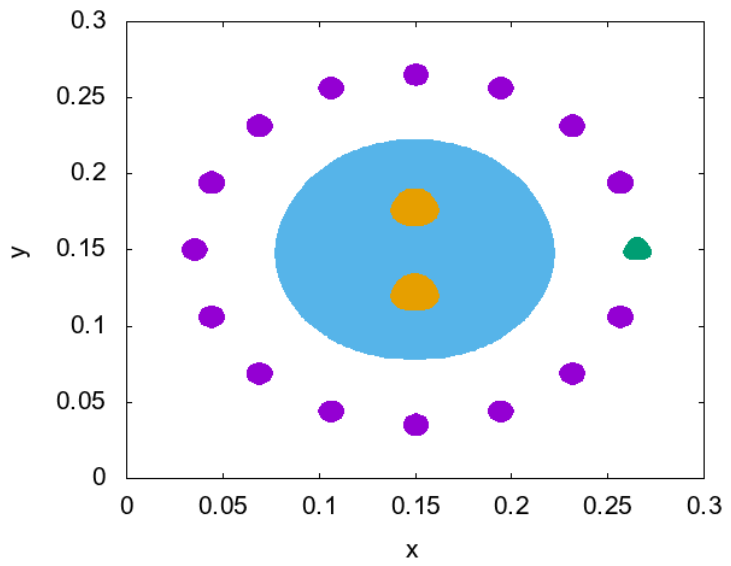
Each iteration – one gradient of a source from random position in circle
Other positions - receivers



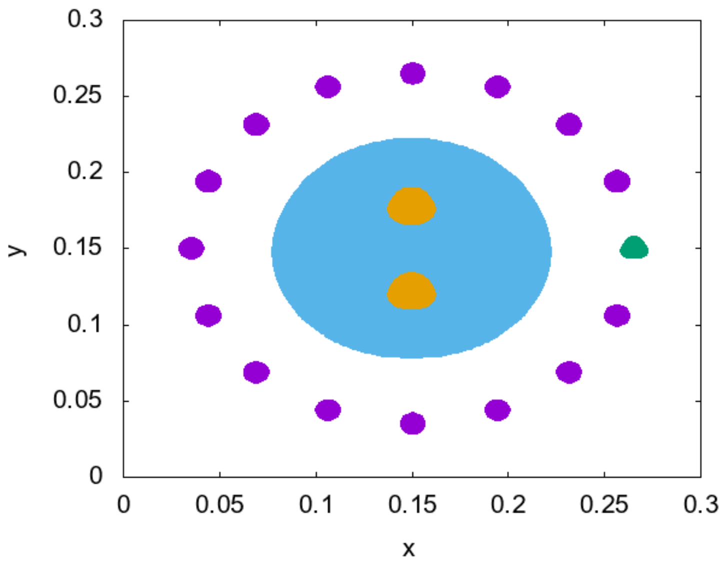
One turn – new iteration

The influence only from **one** source

2) Modified version

$$\rho_{n+1} = \rho_n - \alpha \sum_{j=1}^{K} J_i'(\rho_n)$$

Each iteration – sum of gradients from all sources in all positions in circle
Other positions - receivers



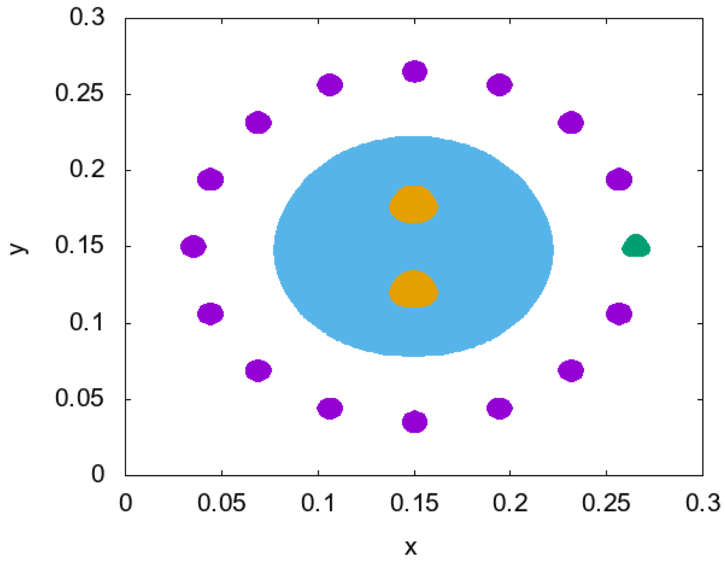
15 turns – new iteration

The influence from **all** sources

Algorithm for solving inverse problem

- Solve direct problem for **exact density** and gather data for each position of source $\forall j = 1 \dots K$

$$f_j(x_i, y_i, t) = p_j(x_i, y_i, t; \rho_{exact}) \quad \forall\, j = 1 \dots K\ \forall i = 1 \dots N$$



- Set initial approximation ρ_0

1) For each source $\forall\, j = 1 \dots K$
    1) Solve direct problem and gather trace $p_j(x_i, y_i, t; \rho_n)$
    2) Compute difference $p_j(x_i, y_i, t; \rho_n) - f_j(x_i, y_i, t)$
    3) Solve adjoint problem
    4) Compute gradient $J_j' = J_j{}'(\rho_n)$
2) Summarize all gradients
3) Make descent step

$$\rho_{n+1} = \rho_n - \alpha \sum_{j=1}^{K} J_j{}'(\rho_n)$$

4) Check residual and relative error. If not enough, go to point 1.

Numerical results

Numerical model. Different amount of receivers.

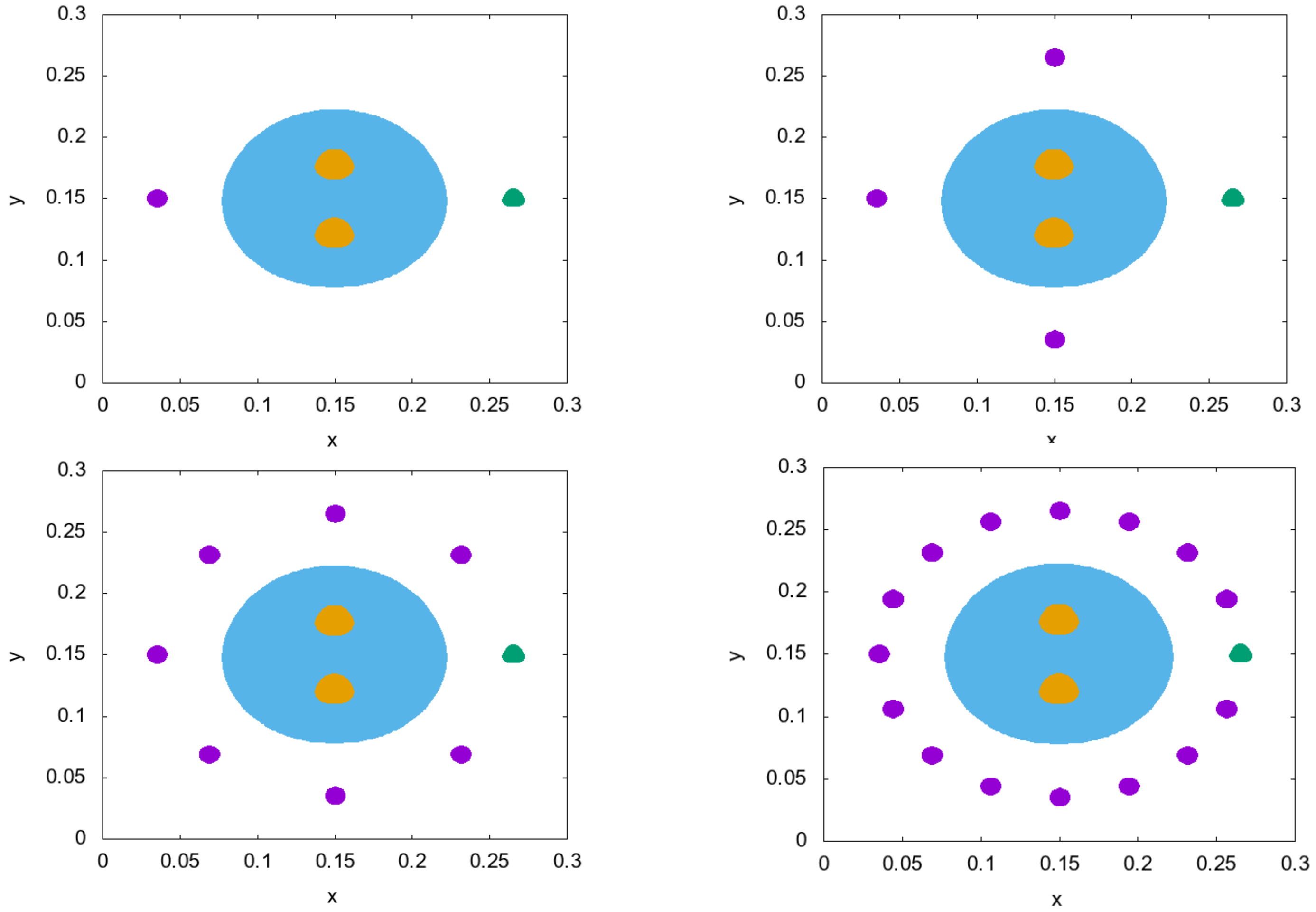
Numerical grid: 500 x 500. $\boldsymbol{L_x} = \boldsymbol{L_y} = 0.3$. $\boldsymbol{R_{rec}} = 0.01$. Velocity is <u>constant</u> in the whole domain.

Synthetic data

$$f_j(x_i, y_i, t) = p_j(x_i, y_i, t; \rho_{exact})$$

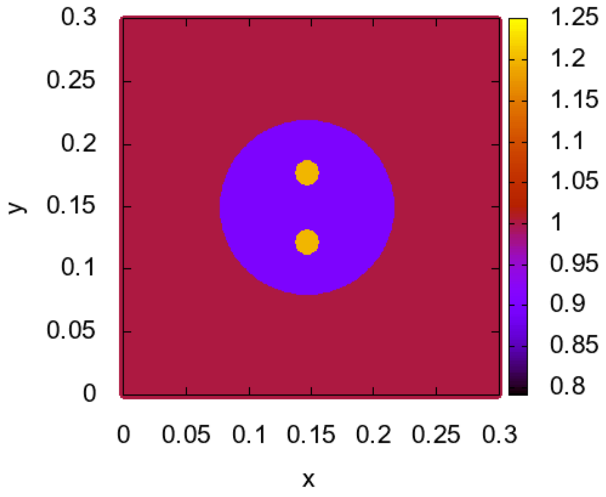
Data with noise

$$f_j(x_i, y_i, t) = p_j(x_i, y_i, t; \rho_{exact}) + (\max - \min)\, \alpha\, \frac{NS}{100}$$



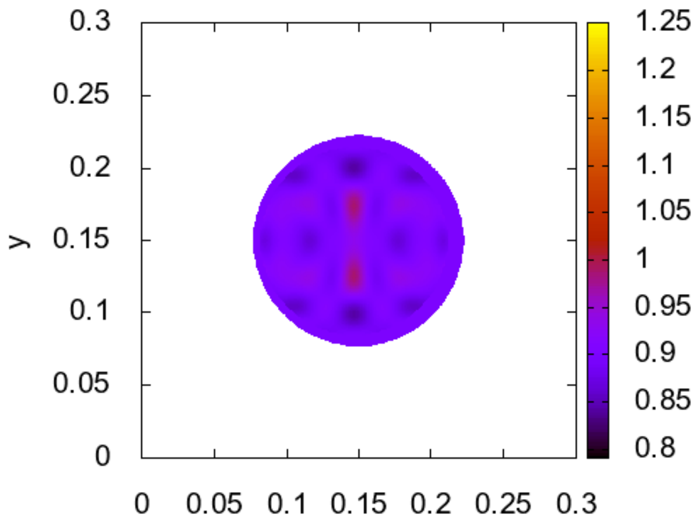
Numerical results. K = 2 and K = 4.

Exact solution $R^2(x,t)$

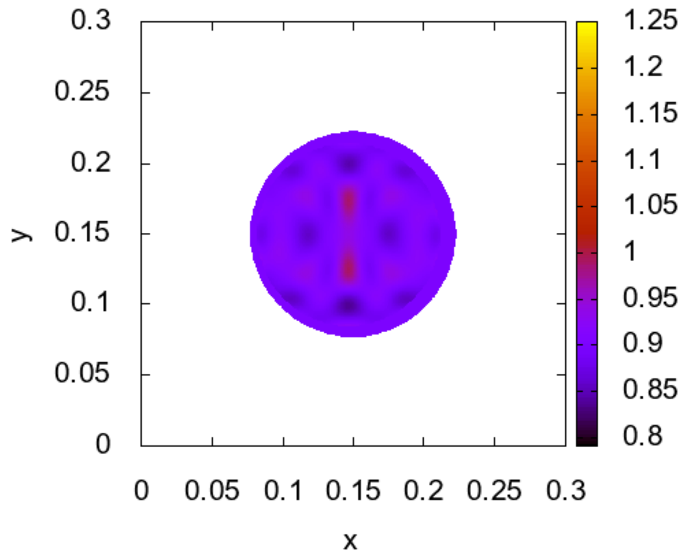


$N_{iterations} = 1000$

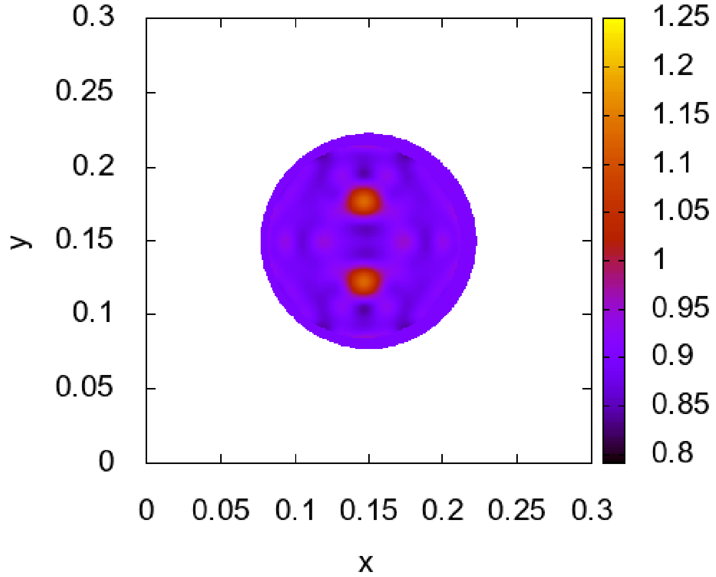
$K = 2$



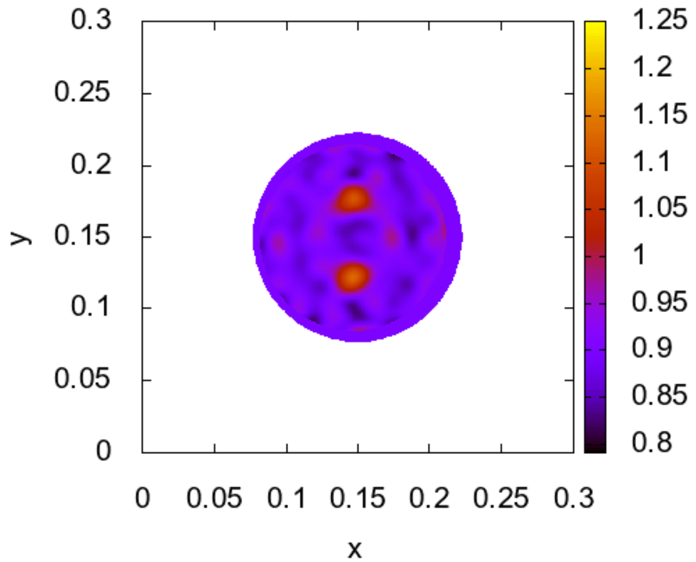
$K = 2$. **Noise 10 %**



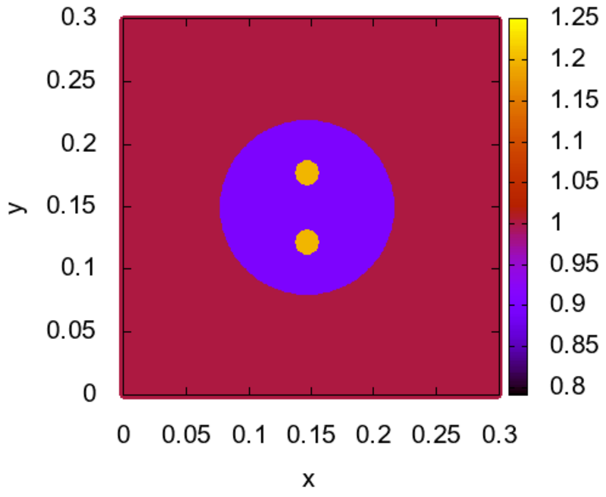
$K = 4$



$K = 4$. **Noise 10 %**

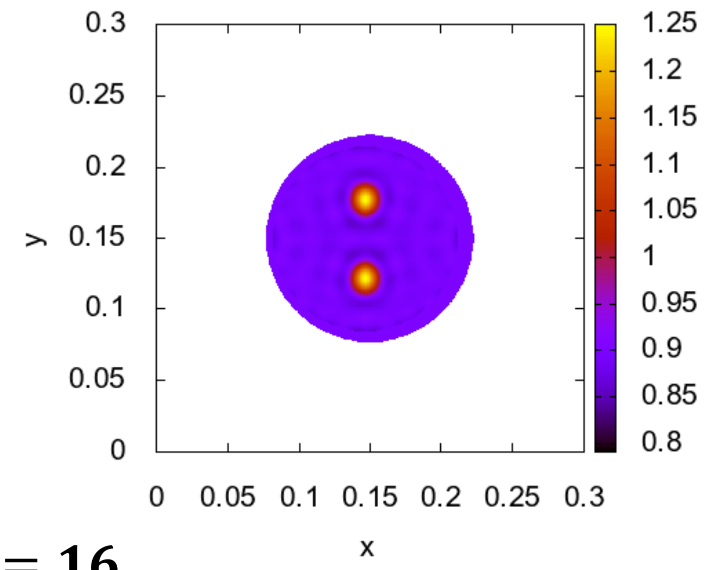


Exact solution $R^2(x,t)$

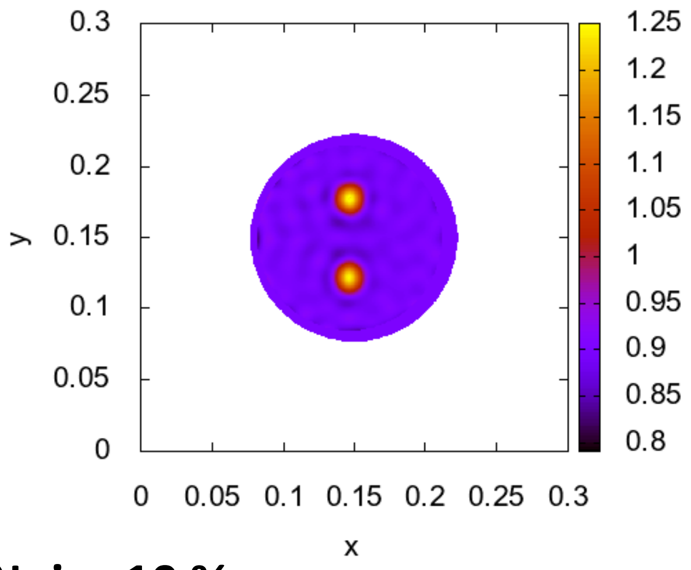


$N_{iterations} = 1000$

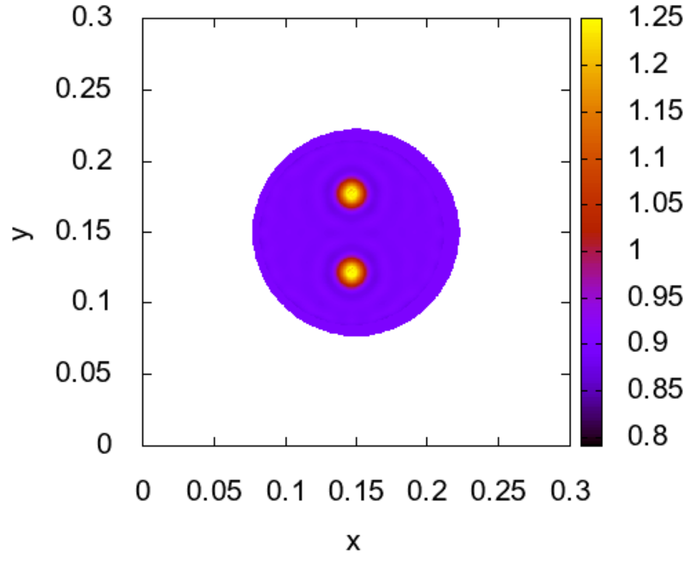
$K = 8$



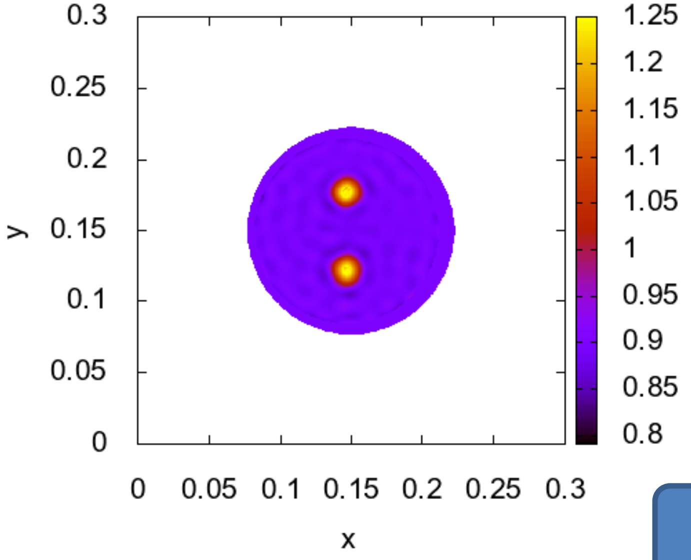
$K = 8$. Noise 10 %



$K = 16$



$K = 16$. Noise 10 %



Relative errors

$N_{iterations} = 1000$

$$Rel.error = \frac{\|\rho_{exact} - \rho_{numerical}\|_{L_2}}{\|\rho_{exact}\|_{L_2}} \times 100$$

K	Relative error (exact data)	Relative error (noisy data)
2	5.7 %	5.9 %
4	4.1 %	4.5 %
8	2.8 %	3.1 %
16	2.6 %	2.8 %

Thank you for attention!

How to compute gradient from one source?

$$J'(\rho) = J'(\rho)(x, y) = \int_0^T -U\psi_{1_t} - V\psi_{2_t} + \frac{1}{\rho}\psi_3(U_x + V_y)dt$$

U, V - solution of **direct** problem

ψ_1, ψ_2, ψ_3 - solution of **adjoint** problem

$$\frac{\partial \psi_1}{\partial t} + \frac{1}{\rho}\frac{\partial \psi_3}{\partial x} = 0$$

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$$\frac{\partial \psi_3}{\partial t} + \rho c^2\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y}\right) = \sum_{i=1}^{N} \delta(x - x_i, y - y_i)[p - f_i]dxdydt$$

$$\psi_1, \psi_2, \psi_3 \Big|_{(x,y)\in\partial\Omega} = 0$$

$$\psi_1, \psi_2, \psi_3 \Big|_{t=T} = 0$$