Stable numerical schemes for flows in time-dependent domains

Yuri Vassilevski[1,2,3]
Maxim Olshanskii[4] Alexander Danilov[1,2,3]
Alexander Lozovskiy[1] Victoria Salamatova[1,2,3]

[1]Marchuk Institute of Numerical Mathematics RAS
[2]Moscow Institute of Physics and Technology
[3]Sechenov University
[4]University of Houston

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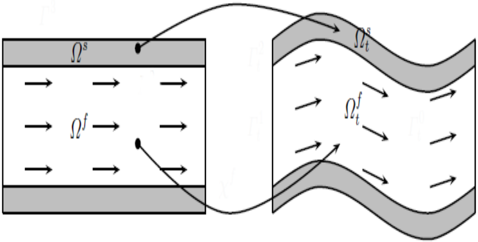
Fluid-Structure Interaction

Fluid-Structure Interaction problem

Prerequisites for FSI



- reference subdomains Ω_f, Ω_s
- transformation ξ maps Ω_f, Ω_s to $\Omega_f(t)$, $\Omega_s(t)$
- $\mathbf{v}$ and $\mathbf{u}$ denote velocities and displacements in $\widehat{\Omega} := \Omega_f \cup \Omega_s$
- $\xi(\mathbf{x}) := \mathbf{x} + \mathbf{u}(\mathbf{x})$, $\mathbf{F} := \nabla\xi = \mathbf{I} + \nabla\mathbf{u}$, $J := \det(\mathbf{F})$
- Cauchy stress tensors $\boldsymbol{\sigma}_f$, $\boldsymbol{\sigma}_s$
- pressures p_f,p_s
- density ρ_f is constant

Fluid-Structure Interaction problem

Universal equations in reference subdomains

Dynamic equations

$$\frac{\partial \mathbf{v}}{\partial t} = \begin{cases} \rho_s^{-1} \operatorname{div}\left(J \boldsymbol{\sigma}_s \mathbf{F}^{-T}\right) & \text{in } \Omega_s, \\ (J\rho_f)^{-1} \operatorname{div}\left(J \boldsymbol{\sigma}_f \mathbf{F}^{-T}\right) - \nabla \mathbf{v} \left(\mathbf{F}^{-1} \left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t} \right) \right) & \text{in } \Omega_f \end{cases}$$

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Kinematic equation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{v} \quad \text{in } \Omega_s$$

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Fluid incompressibility

$$\mathrm{div}\left(J \mathbf{F}^{-1} \mathbf{v}\right) = 0 \quad \text{in } \Omega_f \quad \text{or} \quad J \nabla \mathbf{v} : \mathbf{F}^{-T} = 0 \quad \text{in } \Omega_f$$

Fluid-Structure Interaction problem

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Constitutive relation for the fluid stress tensor

$$\boldsymbol{\sigma}_f = -p_f \mathbf{I} + \mu_f((\nabla \mathbf{v})\mathbf{F}^{-1} + \mathbf{F}^{-T}(\nabla \mathbf{v})^T) \quad \text{in } \Omega_f$$

FSI problem

User-dependent equations in reference subdomains

Constitutive relation for the solid stress tensor

$$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s(J, \mathbf{F}, p_s, \lambda_s, \mu_s, \dots) \quad \text{in } \Omega_s$$

[1] Michler et al (2004), Hubner et al (2004), Hron&Turek (2006),...

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Monolithic approach[1]: Extension of the displacement field to the fluid domain

$$\begin{aligned} G(\mathbf{u}) &= 0 \quad \text{in } \Omega_f, \\ \mathbf{u} &= \mathbf{u}^* \quad \text{on } \partial\Omega_f \end{aligned}$$

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for example, vector Laplace equation or elasticity equation

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for example, vector Laplace equation or elasticity equation

$+$ Initial, boundary, interface conditions $(\boldsymbol{\sigma}_f \mathbf{F}^{-T}\mathbf{n} = \boldsymbol{\sigma}_s \mathbf{F}^{-T}\mathbf{n})$

[1] Michler et al (2004), Hubner et al (2004), Hron&Turek (2006),...

Numerical scheme

- Conformal triangular or tetrahedral mesh Ω_h in $\widehat{\Omega}$
- LBB-stable pair for velocity and pressure P_2/P_1, P_2 for displacements
- Fortran open source software `Ani2D`, `Ani3D` (Advanced numerical instruments 2D/3D, K.Lipnikov, Yu.Vassilevski et al.)
  `http://sf.net/p/ani2d/` `http://sf.net/p/ani3d/`:
  - mesh generation
  - FEM systems
  - algebraic solvers

Numerical scheme

Find $\{\mathbf{u}^{k+1}, \mathbf{v}^{k+1}, p^{k+1}\} \in \mathbb{V}_h^0 \times \mathbb{V}_h \times \mathbb{Q}_h$ s.t.

$$\mathbf{v}^{k+1} = \mathbf{g}_h(\cdot, (k+1)\Delta t) \text{ on } \Gamma_{f0}, \quad \left[\frac{\partial \mathbf{u}}{\partial t}\right]_{k+1} = \mathbf{v}^{k+1} \text{ on } \Gamma_{fs}$$

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where

$\mathbb{V}_h \subset H^1(\widehat{\Omega})^3, \mathbb{Q}_h \subset L^2(\widehat{\Omega}), \mathbb{V}_h^0 = \{\mathbf{v} \in \mathbb{V}_h : \mathbf{v}|_{\Gamma_{s0} \cup \Gamma_{f0}} = \mathbf{0}\}, \mathbb{V}_h^{00} = \{\mathbf{v} \in \mathbb{V}_h^0 : \mathbf{v}|_{\Gamma_{fs}} = \mathbf{0}\}$

$$\left[\frac{\partial \mathbf{f}}{\partial t}\right]_{k+1} := \frac{3\mathbf{f}^{k+1} - 4\mathbf{f}^k + \mathbf{f}^{k-1}}{2\Delta t}$$

Numerical scheme

$$\int_{\Omega_s} \rho_s \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \,\mathrm{d}\Omega + \int_{\Omega_s} J_k \mathbf{F}(\widetilde{\mathbf{u}}^k)\mathbf{S}(\mathbf{u}^{k+1}, \widetilde{\mathbf{u}}^k) : \nabla \psi \,\mathrm{d}\Omega +$$
$$\int_{\Omega_f} \rho_f J_k \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \,\mathrm{d}\Omega + \int_{\Omega_f} \rho_f J_k \nabla \mathbf{v}^{k+1} \mathbf{F}^{-1}(\widetilde{\mathbf{u}}^k) \left(\widetilde{\mathbf{v}}^k - \widetilde{\left[\frac{\partial \mathbf{u}}{\partial t}\right]}_k\right) \psi \,\mathrm{d}\Omega +$$
$$\int_{\Omega_f} 2\mu_f J_k \mathbf{D}_{\widetilde{\mathbf{u}}^k} \mathbf{v}^{k+1} : \mathbf{D}_{\widetilde{\mathbf{u}}^k} \psi \,\mathrm{d}\Omega - \int_{\Omega} p^{k+1} J_k \mathbf{F}^{-T}(\widetilde{\mathbf{u}}^k) : \nabla \psi \,\mathrm{d}\Omega = 0 \quad \forall \psi \in \mathbb{V}_h^0$$

$$J_k := J(\widetilde{\mathbf{u}}^k), \quad \widetilde{\mathbf{f}}^k := 2\mathbf{f}^k - \mathbf{f}^{k-1}, \quad \mathbf{D}_{\mathbf{u}}\mathbf{v} := \{\nabla \mathbf{v} \mathbf{F}^{-1}(\mathbf{u})\}_s, \quad \{\mathbf{A}\}_s := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

Numerical scheme

$$\int_{\Omega_s} \rho_s \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \, \mathrm{d}\Omega + \int_{\Omega_s} J_k \mathbf{F}(\widetilde{\mathbf{u}}^k)\mathbf{S}(\mathbf{u}^{k+1}, \widetilde{\mathbf{u}}^k) : \nabla \psi \, \mathrm{d}\Omega +$$
$$\int_{\Omega_f} \rho_f J_k \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \, \mathrm{d}\Omega + \int_{\Omega_f} \rho_f J_k \nabla \mathbf{v}^{k+1} \mathbf{F}^{-1}(\widetilde{\mathbf{u}}^k) \left(\widetilde{\mathbf{v}}^k - \widetilde{\left[\frac{\partial \mathbf{u}}{\partial t}\right]_k} \right) \psi \, \mathrm{d}\Omega +$$
$$\int_{\Omega_f} 2\mu_f J_k \mathbf{D}_{\widetilde{\mathbf{u}}^k} \mathbf{v}^{k+1} : \mathbf{D}_{\widetilde{\mathbf{u}}^k} \psi \, \mathrm{d}\Omega - \int_{\Omega} p^{k+1} J_k \mathbf{F}^{-T}(\widetilde{\mathbf{u}}^k) : \nabla \psi \, \mathrm{d}\Omega = 0 \quad \forall \psi \in \mathbb{V}_h^0$$

$$\int_{\Omega_s} \left[\frac{\partial \mathbf{u}}{\partial t}\right]_{k+1} \phi \, \mathrm{d}\Omega - \int_{\Omega_s} \mathbf{v}^{k+1} \phi \, \mathrm{d}\Omega + \int_{\Omega_f} G(\mathbf{u}^{k+1}) \phi \, \mathrm{d}\Omega = 0 \quad \forall \phi \in \mathbb{V}_h^{00}$$

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Numerical scheme

$$\int_{\Omega_s} \rho_s \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \,\mathrm{d}\Omega + \int_{\Omega_s} J_k \mathbf{F}(\widetilde{\mathbf{u}}^k)\mathbf{S}(\mathbf{u}^{k+1}, \widetilde{\mathbf{u}}^k) : \nabla\psi \,\mathrm{d}\Omega +$$
$$\int_{\Omega_f} \rho_f J_k \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \,\mathrm{d}\Omega + \int_{\Omega_f} \rho_f J_k \nabla\mathbf{v}^{k+1}\mathbf{F}^{-1}(\widetilde{\mathbf{u}}^k)\left(\widetilde{\mathbf{v}}^k - \widetilde{\left[\frac{\partial \mathbf{u}}{\partial t}\right]}_k\right)\psi \,\mathrm{d}\Omega +$$
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$$\int_{\Omega_f} J_k \nabla\mathbf{v}^{k+1} : \mathbf{F}^{-T}(\widetilde{\mathbf{u}}^k) q \,\mathrm{d}\Omega = 0 \quad \forall\ q \in \mathbb{Q}_h$$

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Numerical scheme

$$\ldots + \int_{\Omega_s} J_k \mathbf{F}(\widetilde{\mathbf{u}}^k)\mathbf{S}(\mathbf{u}^{k+1}, \widetilde{\mathbf{u}}^k) : \nabla \psi \, \mathrm{d}\Omega + \ldots$$

- St. Venant–Kirchhoff model (geometrically nonlinear):

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \lambda_s \mathtt{tr}(\mathbf{E}(\mathbf{u}_1, \mathbf{u}_2))\mathbf{I} + 2\mu_s \mathbf{E}(\mathbf{u}_1, \mathbf{u}_2);$$
$$\mathbf{E}(\mathbf{u}_1, \mathbf{u}_2) = \{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2) - \mathbf{I}\}_s$$

- inc. Blatz–Ko model:

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \mu_s(\mathtt{tr}(\{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2)\}_s)\mathbf{I} - \{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2)\}_s)$$

- inc. Neo-Hookean model:

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \mu_s \mathbf{I}; \;\; \mathbf{F}(\widetilde{\mathbf{u}}^k) \rightarrow \mathbf{F}(\mathbf{u}^{k+1})$$

$$\{\mathbf{A}\}_s := \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

Numerical scheme

The scheme

- provides strong coupling on interface
- semi-implicit
- produces one linear system per time step
- second order in time

Numerical scheme

The scheme

- provides strong coupling on interface
- semi-implicit
- produces one linear system per time step
- second order in time
- unconditionally stable (no CFL restriction), proved with assumptions:
  - 1st order in time
  - St. Venant–Kirchhoff inc./comp. (experiment: Neo-Hookean inc./comp.)
  - extension of $\mathbf{u}$ to Ω_f guarantees $J_k > 0$
  - Δt is not large

A.Lozovskiy, M.Olshanskii, V.Salamatova, Yu.Vassilevski. An unconditionally stable semi-implicit FSI finite element method. *Comput.Methods Appl.Mech.Engrg., 297, 2015*

3D: silicone filament in glycerol

Benchmark challenge for CMBE 2015, Paris

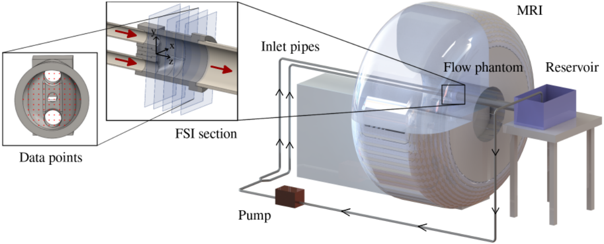

Image from A. Hessenthaler et al. Experiment for validation of fluid-structure interaction models and algorithms. *Int. J. for Numer. Meth. Biomed. Engng.*, 2017

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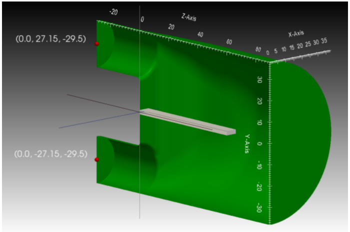
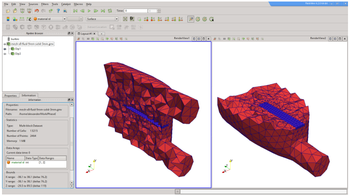
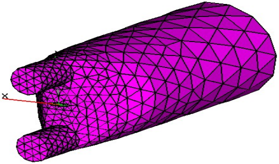


Meshed volume: original and extended domains.



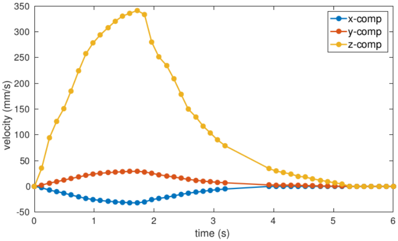


3D: silicone filament in glycerol

- Steady and pulsatile flow regimes

	Phase I	Phase II
velocity	stationary	pulsatile
ρ_f	$1.1633 \cdot 10^{-3}$ g mm^3	$1.164 \cdot 10^{-3}$ g mm^{-3}
μ_f	$12.5 \cdot 10^{-3}$ g mm^{-1}s^{-1}	$13.37 \cdot 10^{-3}$ g mm^{-1}s^{-1}

- Inflow velocities for one cycle of frequency $1/6$ Hz for phase II:



- Simulation was run with $\Delta t = 10^{-2}$ s, $t \in [0, 12]$
- $\#Tets(\Omega_s) = 733$, $\#Tets(\Omega_f) = 28712$, $\#unknowns = 254439$
- The filament (SVK) is lighter than the fluid and deflects upward
- Linear elasticity model is used for the **update** of the displacement extension in Ω_f, the Lame parameters are element-volume dependent

3D: silicone filament in glycerol

displacement extension in Ω_f

M.Landajuela et al. Coupling schemes for the FSI forward prediction challenge: comparative study and validation. *Int. J. for Numer. Meth. in Biomed. Engng.*, 33, 2017.

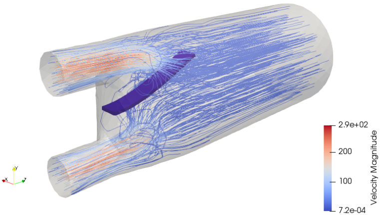
- Linear elasticity model is used for the **update** of the displacement extension in Ω_f

$$
\begin{aligned}
-\operatorname{div}\Bigg[J\Bigg(\lambda_m \mathrm{tr}\left(\nabla\left[\frac{\partial \mathbf{u}}{\partial t}\right]^k \mathbf{F}^{-1}\right)\mathbf{I} \\
+\mu_m\left(\nabla\left[\frac{\partial \mathbf{u}}{\partial t}\right]^k \mathbf{F}^{-1} + \left(\nabla\left[\frac{\partial \mathbf{u}}{\partial t}\right]^k \mathbf{F}^{-1}\right)^T\right)\Bigg)\mathbf{F}^{-T}\Bigg] = 0 \quad \text{in } \Omega_f,
\end{aligned}
$$

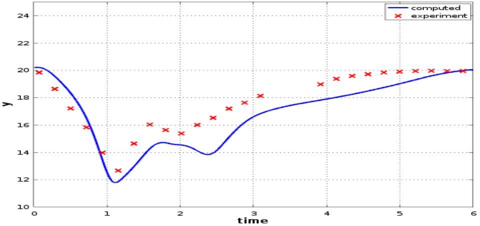
- the Lame parameters are element-volume dependent:

$$\lambda_m = 16\mu_m = 16\frac{\mu_s}{v_e^{1.2}}$$

3D: silicone filament in glycerol



Streamlines colored by the velocity magnitude $t = 0.721s$ (left), $t = 2.017s$ (right)



Track of the computed y-displacement of the point in the structure with coordinate $z \approx 53$, $x = 0$ for $t \in [0, 6]$ and recorded experimental data

A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. Analysis and assessment of a monolithic FSI finite element method. To appear in Computers & Fluids

Incompressible fluid flow in a time-dependent domain

Navier-Stokes equations in a time-dependent domain

Prerequisites

- reference domain Ω_0
- transformation $\boldsymbol{\xi}$ mapping Ω_0 to $\Omega(t)$ is given
- $\mathbf{v}$ and $\mathbf{u}$ denote velocities and displacements in Ω_0
- $\boldsymbol{\xi}(\mathbf{x}) := \mathbf{x} + \mathbf{u}(\mathbf{x})$, $\mathbf{F} := \nabla\boldsymbol{\xi} = \mathbf{I} + \nabla\mathbf{u}$, $J := \det(\mathbf{F})$
- Cauchy stress tensor $\boldsymbol{\sigma}$
- pressure p
- density ρ is constant

Incompressible fluid flow in a moving domain

Navier-Stokes equations in reference domain Ω_0

Let $\boldsymbol{\xi}$ mapping Ω_0 to $\Omega(t)$, $\mathbf{F} = \nabla\boldsymbol{\xi} = \mathbf{I} + \nabla\mathbf{u}$, $J = \det(\mathbf{F})$ be given

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$$\frac{\partial \mathbf{v}}{\partial t} = (J\rho_f)^{-1}\mathrm{div}\left(J\boldsymbol{\sigma}_f\mathbf{F}^{-T}\right) - \nabla\mathbf{v}\left(\mathbf{F}^{-1}\left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t}\right)\right) \quad \text{in } \Omega_0$$

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Fluid incompressibility

$$\mathrm{div}\left(J\mathbf{F}^{-1}\mathbf{v}\right) = 0 \quad \text{in } \Omega_0 \quad \text{or} \quad J\nabla\mathbf{v} : \mathbf{F}^{-T} = 0 \quad \text{in } \Omega_0$$

Incompressible fluid flow in a moving domain

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Mapping $\boldsymbol{\xi}$ does not define material trajectories $\rightarrow$ quasi-Lagrangian formulation

Finite element scheme

Let $\mathbb{V}_h, \mathbb{Q}_h$ be Taylor-Hood P_2/P_1 finite element spaces.
Find $\{\mathbf{v}_h^k, p_h^k\} \in \mathbb{V}_h \times \mathbb{Q}_h$ satisfying b.c.
("do nothing" $\boldsymbol{\sigma}\mathbf{F}^{-T}\mathbf{n} = 0$ or no-penetration no-slip $\mathbf{v}^k = (\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1})/\Delta t$)

$$\int_{\Omega_0} J_k \frac{\mathbf{v}_h^k - \mathbf{v}_h^{k-1}}{\Delta t} \cdot \psi \, \mathrm{d}\mathbf{x} + \int_{\Omega_0} J_k \nabla \mathbf{v}_h^k \mathbf{F}_k^{-1} \left(\mathbf{v}_h^{k-1} - \frac{\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1}}{\Delta t} \right) \cdot \psi \, \mathrm{d}\mathbf{x} -$$
$$\int_{\Omega_0} J_k p_h^k \mathbf{F}_k^{-T} : \nabla \psi \, \mathrm{d}\mathbf{x} + \int_{\Omega_0} J_k q \mathbf{F}_k^{-T} : \nabla \mathbf{v}_h^k \, \mathrm{d}\mathbf{x} +$$
$$\int_{\Omega_0} \nu J_k (\nabla \mathbf{v}_h^k \mathbf{F}_k^{-1} \mathbf{F}_k^{-T} + \mathbf{F}_k^{-T} (\nabla \mathbf{v}_h^k)^T \mathbf{F}_k^{-T}) : \nabla \psi \, \mathrm{d}\mathbf{x} = 0$$

$$\int_{\Omega_0} J_k \nabla \mathbf{v}^k : \mathbf{F}_k^{-T} q \, \mathrm{d}\Omega = 0$$

for all ψ and q from the appropriate FE spaces

Finite element scheme

The scheme

- semi-implicit
- produces one linear system per time step
- first order in time (may be generalized to the second order)

Finite element scheme

The scheme

- semi-implicit
- produces one linear system per time step
- first order in time (may be generalized to the second order)
- unconditionally stable (no CFL restriction) and 2nd order accurate, proved with assumptions:
  - $\inf_Q J \geq c_J > 0, \quad \sup_Q(\|\mathbf{F}\|_F + \|\mathbf{F}^{-1}\|_F) \leq C_F$
  - LBB-stable pairs (e.g. P_2/P_1)
  - Δt is not large

A.Danilov, A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A finite element method for the Navier-Stokes equations in moving domain with application to hemodynamics of the left ventricle. *Russian J. Numer. Anal. Math. Modelling, 32, 2017*

A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A quasi-Lagrangian finite element method for the Navier-Stokes equations in a time-dependent domain. Comput. Methods Appl. Mech. Engrg. 333, 2018

3D: left ventricle of a human heart

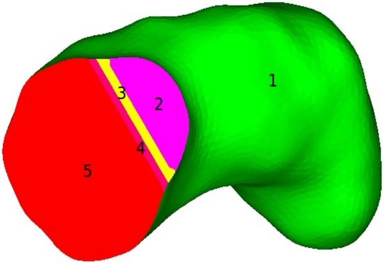


Figure: Left ventricle

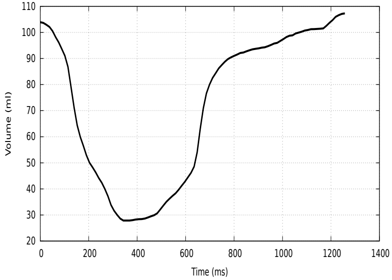
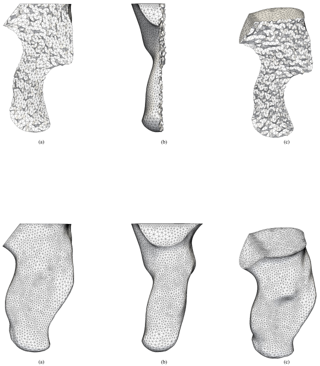


Figure: Ventricle volume

The law of motion for the ventricle walls is known thanks to ceCT scans $\rightarrow$ 100 mesh files with time gap 0.0127 s $\rightarrow$ $\mathbf{u}$ given as input $\rightarrow$ FSI reduced to NSE in a moving domain

- 2 - aortic valve (outflow)
- 5 - mitral valve (inflow)

3D: left ventricle of a human heart



- Quasi-uniform mesh: 14033 vertices, 69257 elements, 88150 edges.
- Boundary conditions: Dirichlet $\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$ except:
  - Do-nothing on aortal valve during systole
  - Do-nothing on mitral valve during diastole
- Time step 0.0127 s is too large! $\Longrightarrow$ refined to $\Delta t = 0.0127/20$ s $\Longrightarrow$ Cubic-splined $\mathbf{u}$.
- Blood parameters: $\rho_f = 10^3$ kg/m^3, $\mu_f = 4 \cdot 10^{-3}$ Pa ·s.