X conference on Biomath.

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INM

RAS

Towards Poiseuille type flow for axisymmetric flow with thin stiff elastic wall

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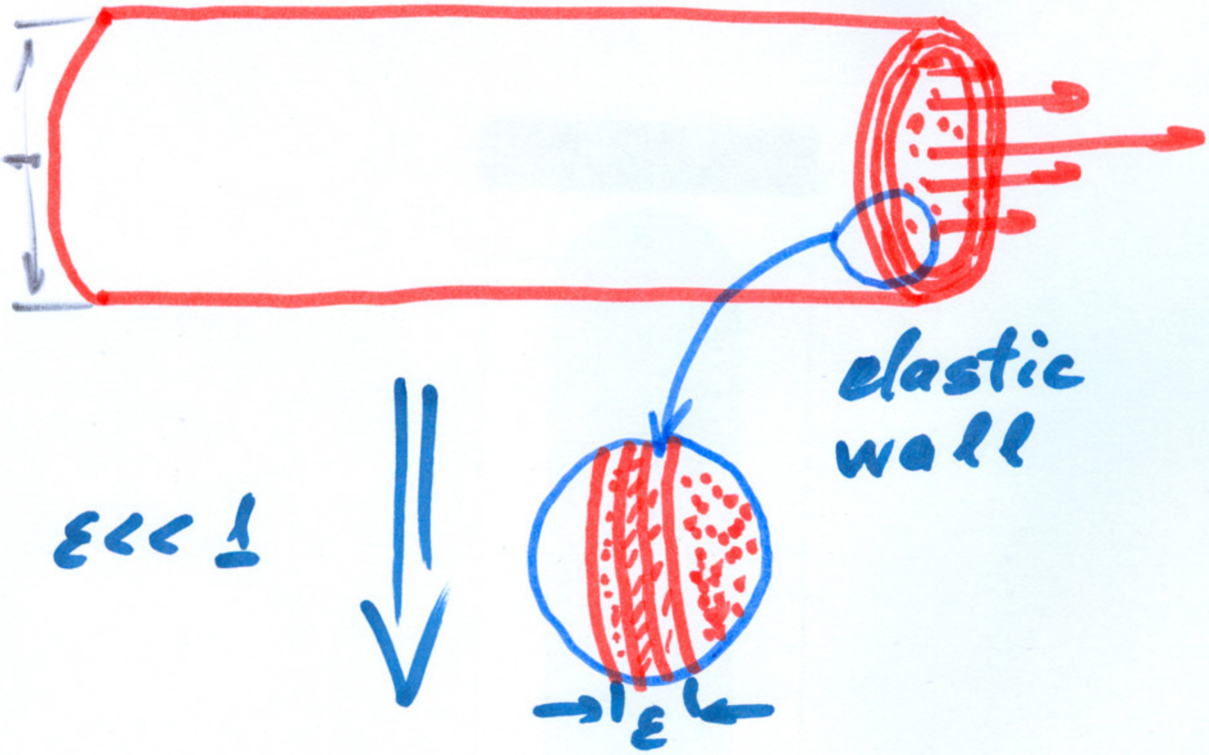
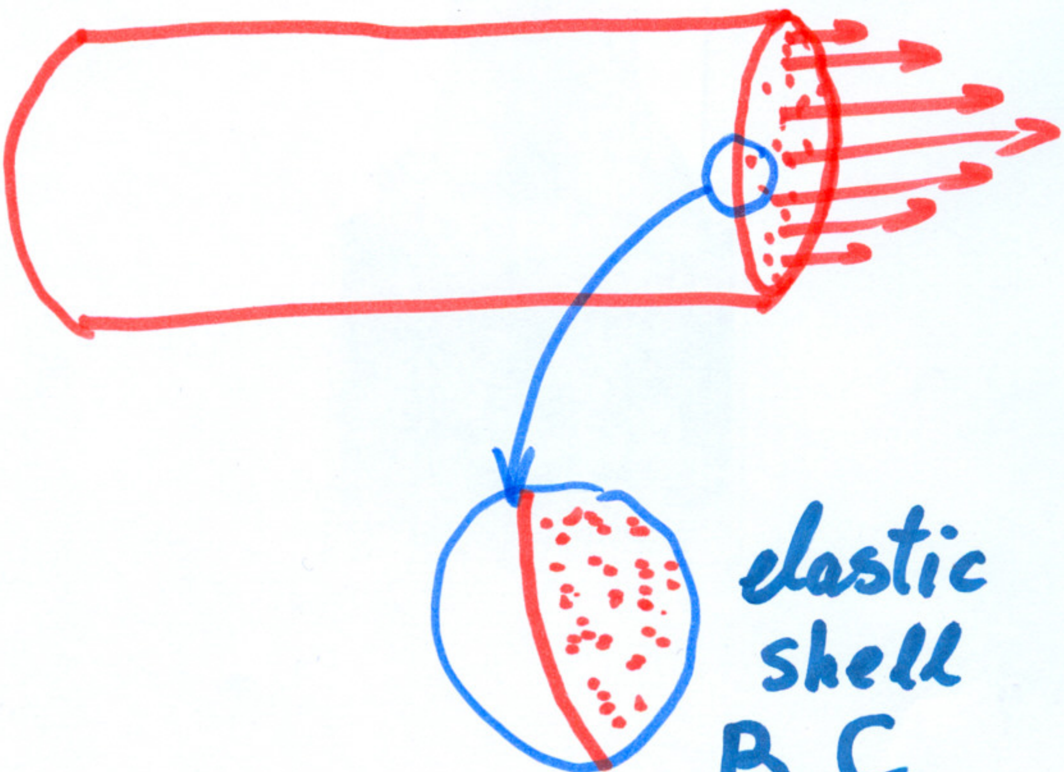
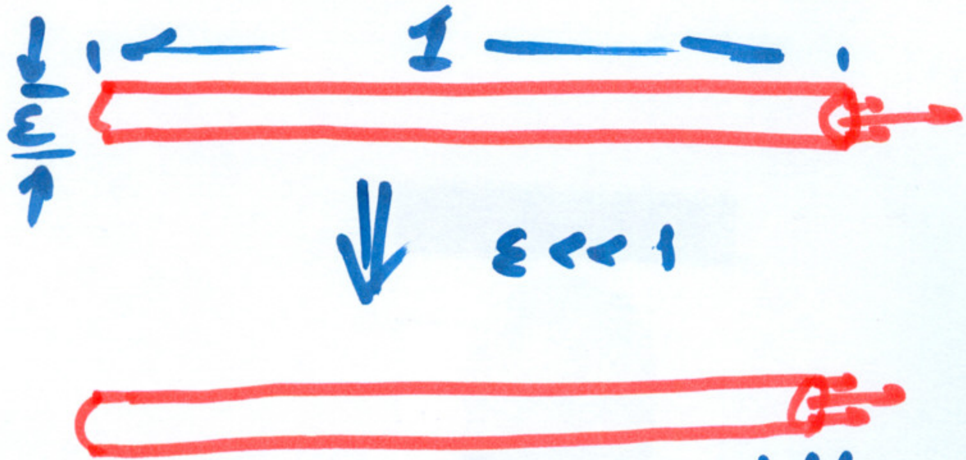
Stage 1





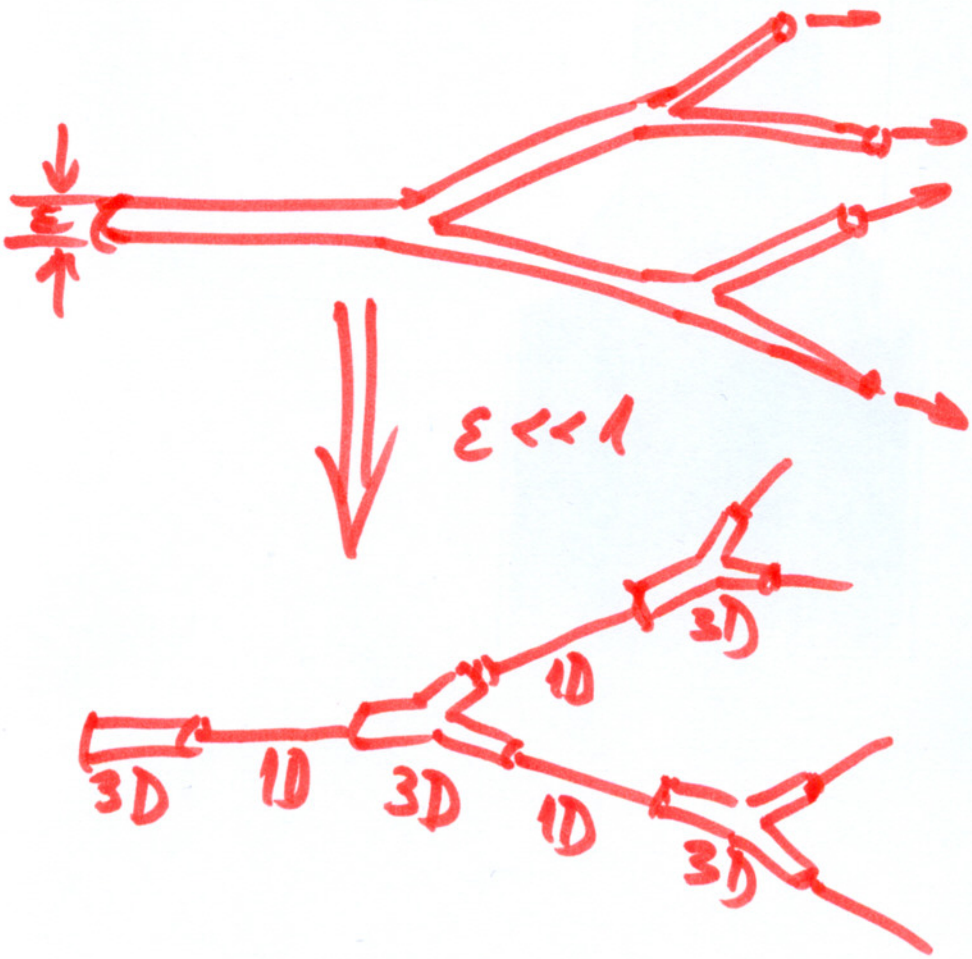
G. Panasenko, R. Stavre, Applicable Analysis
2018

stage 2



1D quasi-Poiseuille

stage 3 (in progress)



Method of Asymptotic Partial Decomposition of Domain

DENOTE :

$$
\left\{
\begin{aligned}
L\vec{u}\cdot e_3 &= \frac{\partial}{\partial x_3}\left((\lambda+2\mu)\frac{\partial u_3}{\partial x_3}+\lambda\left(\frac{\partial u_r}{\partial r}+\frac{1}{r}u_r\right)\right)\\
&\quad+\frac{\partial}{\partial r}\left(\mu\left(\frac{\partial u_3}{\partial r}+\frac{\partial u_r}{\partial x_3}\right)\right)+\frac{\mu}{r}\left(\frac{\partial u_3}{\partial r}+\frac{\partial u_r}{\partial x_3}\right)\\
L\vec{u}\cdot e_r &= \frac{\partial}{\partial x_3}\left(\mu\left(\frac{\partial u_3}{\partial r}+\frac{\partial u_r}{\partial x_3}\right)\right)+\frac{\partial}{\partial r}\left(\lambda\left(\frac{\partial u_3}{\partial x_3}+\frac{u_r}{r}\right)\right.\\
&\quad\left.+(\lambda+2\mu)\frac{\partial u_r}{\partial r}\right)+\frac{2\mu}{r}\left(\frac{\partial u_r}{\partial r}-\frac{u_r}{r}\right)\\
\operatorname{div}_c\vec{u} &= \frac{\partial u_3}{\partial x_3}+\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\\
\operatorname{div}_c S &= \left(\frac{\partial S_{33}}{\partial x_3}+\frac{1}{r}\frac{\partial}{\partial r}\left(r S_{r3}\right)\right)e_3+\\
&\quad+\left(\frac{\partial S_{3r}}{\partial x_3}+\frac{1}{r}\frac{\partial}{\partial r}\left(r S_{rr}\right)-\frac{S_{\theta\theta}}{r}\right)e_r\\
\nabla_c\vec{u} &= \begin{pmatrix} \frac{\partial u_3}{\partial x_3} & 0 & \frac{\partial u_r}{\partial r} \\ 0 & \frac{u_r}{r} & 0 \\ \frac{\partial u_r}{\partial x_3} & 0 & \frac{\partial u_r}{\partial r} \end{pmatrix},\\
D_c(\vec{u}) &= \frac{1}{2}\left(\nabla_c\vec{u}+(\nabla_c\vec{u})^T\right).
\end{aligned}
\right.
$$

$$L^e_\varepsilon = \{(x_3, r) \mid x_3 \in \mathbb{R},\ r \in (1, 1+\varepsilon)\}$$

wall

$$L^f = \{(x_3, r) \mid x_3 \in \mathbb{R},\ r \in (0, 1)\}$$

fluid

$$F^0 = \{(x_3, 0) \mid x_3 \in \mathbb{R}\}$$

$$F^1 = \{(x_3, 1) \mid x_3 \in \mathbb{R}\}$$

$$F^{1+\varepsilon} = \{(x_3, 1+\varepsilon) \mid x_3 \in \mathbb{R}\}$$

Pbl. 1-per in x_3

$$\omega_\rho \rho_e \frac{\partial^2 \vec{u}}{\partial t^2} - \omega_E L\vec{u} = \varepsilon^{-1}\vec{g}, \quad L^e_\varepsilon \times (0,T)$$

$$\begin{cases} \rho_f \dfrac{\partial \vec{v}}{\partial t} - 2\nu\, \mathrm{div}_c D_c(\vec{v}) + \nabla p = \vec{f}, \\ \mathrm{div}_c \vec{v} = 0, \qquad \text{in } L^f \times (0,T) \end{cases}$$

$$v_r = 0 \text{ at } F^0 \times (0,T) \qquad (1)$$

$$\begin{cases} \dfrac{\partial u_3}{\partial r} + \dfrac{\partial u_r}{\partial x_3} = 0, \\ \lambda \dfrac{\partial u_3}{\partial x_3} + (\lambda + 2\mu)\dfrac{\partial u_r}{\partial r} + \dfrac{\lambda}{1+\varepsilon} u_r = 0 \quad \text{at } F^{1+\varepsilon} \times (0,T) \end{cases}$$

$$\begin{cases} \vec{v} = \dfrac{\partial \vec{u}}{\partial t} \\ \nu\left(\dfrac{\partial v_3}{\partial r} + \dfrac{\partial v_r}{\partial x_3}\right) = \omega_E \mu \left(\dfrac{\partial u_3}{\partial r} + \dfrac{\partial u_r}{\partial x_3}\right) \quad \text{at } F^1 \times (0,T) \\ -p + 2\nu \dfrac{\partial v_r}{\partial r} = \omega_E\left(\lambda \dfrac{\partial u_3}{\partial x_3} + (\lambda + 2\mu)\dfrac{\partial u_r}{\partial r} + \lambda u_r\right) \end{cases}$$

$$\vec{u}|_{t=0} = 0, \quad \partial\vec{u}/\partial t\,|_{t=0} = 0, \quad \vec{v}|_{t=0} = 0$$

where ρ_e, λ, μ depend on $\frac{r-1}{\varepsilon} = \xi$,

(λ, μ) coef. Lamé; via mod. Young and coef. Poisson:

$$\lambda = \frac{E\hat{\nu}}{(1+\hat{\nu})(1-2\hat{\nu})}, \quad \mu = \frac{E}{2(1+\hat{\nu})},$$

ρ_e is density of the wall; $E, \hat{\nu}, \rho_e$ are piecewise-smooth, $\lambda, \mu, \rho_e \geq \varkappa_0 > 0$.

$\rho_f, \nu > 0$ (consts). $\vec{f}(x_3, r, t)$, $\vec{g}(x_3, t)$. $\in C^\infty$; $\cdot_3$ component is odd, $\cdot_r$ is even w.r.t. x_3. $\exists\, \tau_0 > 0$: $\vec{f} = 0$, $\vec{g} = 0$ for $t \leq \tau_0$.

Main results at the stage 1:

1. $\exists!$ solution to (1). pressure is unique!
2. Complete asymptotic expansion is constructed and justified by error estimates: $\varepsilon \to 0$, $\omega_\rho \leq \omega_E$, $\omega_\rho = \varepsilon^\alpha$, $\omega_E = \varepsilon^\beta$, $\alpha, \beta \in \mathbb{R}$

$\omega_\rho \to 0$, $\omega_\rho \to +\infty$, $\omega_E \to 0$, $\omega_E \to +\infty$

Limit problem

$$\begin{cases} \rho_f \dfrac{\partial \vec{v}_0}{\partial t} - 2\nu \, \mathrm{div}_c \, D_c(\vec{v}_0) + \nabla p_0 = \vec{f} \quad \text{in } L^f \times (0,T) \\ \\ \mathrm{div}_c \, \vec{v}_0 = 0 \\ \vec{v}_0 \big|_{t=0} = 0 \qquad\qquad (2) \\ \\ \text{plus a special boundary condition at } F^1 \end{cases}$$

1.1 Case $\omega_E \gg \varepsilon^{-1}$ (very stiff wall)

$$\vec{v}_0 = 0, \quad \langle p_0 \rangle = -\langle g_2 \rangle, \quad (3_1)$$

$$\langle \cdot \rangle = \int_0^1 \cdot \, dx_3 .$$

1.2. Case $\omega_E = \varepsilon^{-1}$

$$
\begin{cases}
\vec{v}_0 = \dfrac{\partial \vec{w}_0}{\partial t}, \\[2ex]
\dfrac{\omega_S}{\omega_E} \langle \rho_e \rangle \dfrac{\partial^2 \vec{w}_0}{\partial t^2} - E_1 \dfrac{\partial^2 w_{03}}{\partial x_3^2} \vec{e}_3 - E_2 \dfrac{\partial w_{02}}{\partial x_3} \vec{e}_3 \\[2ex]
\quad + E_2 \dfrac{\partial w_{03}}{\partial x_3} \vec{e}_2 + E_3 w_{02} \vec{e}_2 \\[1ex]
\quad + 2\nu D_c(\vec{v}_0) \vec{e}_2 - p_0 \vec{e}_2 = \vec{g} \\[1ex]
\vec{w}_0, \vec{v}_0, p_0 \quad \text{1-per. in } x_3 \\[1ex]
\vec{w}_0 |_{t=0} = 0
\end{cases}
$$

Simplification: $\boxed{w_{03} = 0.}$

1.3. Case $\omega_E << \varepsilon^{-1}$ (very soft wall)

$$2\nu D_c(\vec{v}_0) \vec{e}_2 - p_0 \vec{e}_2 = 0.$$

$$E_1 = \left\langle \frac{E}{1-\hat{\nu}^2} \right\rangle ,$$

$$E_2 = \left\langle \frac{1}{1+\xi\varepsilon} \frac{E\hat{\nu}}{1-\hat{\nu}^2} \right\rangle ,$$

$$E_3 = \left\langle \frac{1}{(1+\xi\varepsilon)^2} \frac{E}{1-\hat{\nu}^2} \right\rangle ,$$

$$\langle \cdot \rangle = \int_0^1 \cdot \, d\xi .$$

Stage 2: radius $= \varepsilon_1$.

Assume that $E_3 \sim \varepsilon_1^{-3}$.

$$v_3 = \frac{\varepsilon_1^2 - r^2}{4\nu}\left(-\frac{\partial q}{\partial x_3} + f_3\right) + O(\varepsilon_1^3), \quad (2.1)$$

$$v_r = r\,\frac{(\varepsilon_1^2 - r^2/2)}{8\nu}\left(\frac{\partial^2 q}{\partial x_3^2} - \frac{\partial f_3}{\partial x_3}\right) + O(\varepsilon_1^4), \quad (2.2)$$

$$p = q + O(\varepsilon_1), \quad (2.3)$$

$$\frac{\partial u_r}{\partial t} = \frac{\varepsilon_1^3}{16\nu}\left(\frac{\partial^2 q}{\partial x_3^2} - \frac{\partial f_3}{\partial x_3}\right) + O(\varepsilon_1^4) \quad (2.4)$$

$$E_3 u_r = q + O(\varepsilon_1) \quad (2.5)$$

$$\Downarrow$$

$$\boxed{\frac{\partial q}{\partial t} = \frac{\varepsilon_1^3 E_3}{16\nu}\left(\frac{\partial^2 q}{\partial x_3^2} - \frac{\partial f_3}{\partial x_3}\right)} \quad (2.6)$$

Remark: (2.1), (2.4) $\Rightarrow \dfrac{\partial S}{\partial t} + \dfrac{\partial (SU)}{\partial x_3} = 0$

Stage 3.

in progress.

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THANK YOU FOR
ATTENTION