Correctness of mathematical models of the alive systems described by delay differential equations

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1. Introduction.

The term «a correctness of mathematical models» is well–known: existence, uniqueness and continuous dependence of model solutions on input data.

For the mathematical models describing dynamics of quantity of elements of alive systems, the term «a correctness» it is possible to understand so:

- existence and uniqueness of model solutions on any finite time interval $[0, \tau)$ and, as a corollary — on a semiaxis $[0, \infty)$ (a global solvability);
- nonnegativity of model solutions on semiaxis $[0, \infty)$ at nonnegative initial data;
- the continuous dependence of model solutions on initial data on finite intervals $[0, \tau)$.

Let us consider a correctness problem in relation to rather wide family of mathematical models of alive systems.

Denote:

$x(t) = (x_1(t), \dots, x_m(t))^T$ — quantity of elements of some alive system at time t;

$x_i(t)$ — quantity of elements of type i;

$dx_i(t)/dt$ — right–hand derivative, $1 \leqslant i \leqslant m$;

$\omega \geqslant 0$ — some constant, $I_\omega = [-\omega, 0]$;

$x_t : I_\omega \to R^m$ — variable with delay, determined by the rule:

$$x_t(\theta) = x(t+\theta),\ \theta \in I_\omega,\ \ t \geqslant 0.$$

We accept that dynamics of $x_i(t)$ is described by formulas

$$\frac{dx_i(t)}{dt} = f_i(t, x_t) - (\mu_i + g_i(t, x_t))x_i(t), \;\; t \geqslant 0,$$

$$x_i(t) = \psi_i(t), \;\; t \in I_\omega,$$

$f_i(t, x_t)$ — rate of reproduction or immigrating in system from external sources of elements of type i,

$\mu_i + g_i(t, x_t)$ — intensity of death of elements of type i or their transition to another (not considered) systems,

$\psi_i(t)$ — quantity of earlier existing elements of type i (elements which were in system to $t = 0$).

The elementary examples ($m = 1,\ x(t) = x_1(t)$).

I) The simplist model of blood cells production:

$$\frac{dx(t)}{dt} = f(t, x_t) - (\mu + g(t, x_t))x(t) =$$
$$= \frac{ax(t-\omega)}{b + x^n(t-\omega)} - \mu x(t), \quad t \geqslant 0, \quad x(t) = \psi(t),\ t \in I_\omega,$$

$$a, b = const > 0,\ n = const > 1, \quad g(t, x_t) \equiv 0.$$

If $x_t = x(t-\omega) \geqslant 0$, then $0 \leqslant f(t, x_t) \leqslant const$.

If n is odd, then $f(t, x_t)$ is not defined at $b + x^n(t-\omega) = 0$ and $f(t, x_t) < 0$ at $x(t-\omega) < 0,\ b + x^n(t-\omega) > 0$.

II) Combination of Bazykin's and Volterra models of population dynamics:

$$\frac{dx(t)}{dt} = f(t, x_t) - (\mu + g(t, x_t))x(t) =$$
$$= \frac{a\,x^2(t)}{1 + b\,x(t)} - \Big(\mu + \int_{-\omega}^{0} \varphi(\theta)\,x(t+\theta)\,d\theta\Big)x(t),$$
$$x(t) = \psi(t),\ \ t \in I_\omega,$$

where $a, b = const > 0,\ \ \varphi(\theta) \geqslant 0$ — continuous function.

$\forall\ x_t = x(t) \geqslant 0$ inequality $0 \leqslant f(t, x_t) \leqslant (a/b)\,x(t)$ is right,

$x(t) = -1/b < 0$ is a point of discontinuity of $f(t, x_t)$,

$g(t, x_t)x(t)$ does not satisfy a global Lipschitz condition.

Denote: $\psi(t) = (\psi_1(t), \ldots, \psi_m(t))^T$,

$$f(t, x_t) = (f_1(t, x_t), \ldots, f_m(t, x_t))^T,$$
$$g(t, x_t) = \operatorname{diag}(g_1(t, x_t), \ldots, g_m(t, x_t)),$$
$$\mu = \operatorname{diag}(\mu_1, \ldots, \mu_m),$$

and model equations will study in a vector form

$$\frac{dx(t)}{dt} = f(t, x_t) - (\mu + g(t, x_t))x(t),\ t \geqslant 0, \qquad (1)$$

$$x(t) = \psi(t),\ t \in I_\omega. \qquad (2)$$

Let T — finite $[0, \tau)$ or infinite $[0, \infty)$ interval.

Function x is called to be a solution of model (1), (2) on interval T, if x is continuous on interval $I_\omega \bigcup T$, satisfies the initial condition (2) and equation (1) $\forall\, t \in T$.

2. Assumptions and results.

For $c, u, v, w \in R^m$ inequalities $c < 0$, $u > 0$, $v \leqslant w$ are understood in a component-wise form.

Let $C(I_\omega, R^m)$ — set of all continuous functions $z : I_\omega \to R^m$, $||z|| = \max_{\theta \in I_\omega} ||z(\theta)||_{R^m}$.

If $z_1, z_2 \in C(I_\omega, R^m)$, then inequality $z_1 \leqslant z_2$ is understood as inequality $z_1(\theta) \leqslant z_2(\theta)$ between vectors at each fixed $\theta \in I_\omega$.

Functional $h : C(I_\omega, R^m) \to R^m$ is called to be isotone, if $\forall\, z_1, z_2 \in C(I_\omega, R^m)$, $z_1 \leqslant z_2$, inequality $h(z_1) \leqslant h(z_2)$ is carried out.

Let $x, y : R_+ \to R^m$. $\forall\, t \in R_+$ inequality $x(t) \leqslant y(t)$ is understood as inequality between vectors.

• Group of assumptions (H1)

(H1.1): $\exists\ \boldsymbol{c} \in \boldsymbol{R}^m$, $\boldsymbol{c} < 0$, such, that mappings

$$f_i,\ g_i : \boldsymbol{R}_+ \times C(I_\omega, [c_1, \infty) \times \cdots \times [c_m, \infty)) \to \boldsymbol{R}$$

are continuous and satisfy nonnegativity condition:

$$f_i,\ g_i : \boldsymbol{R}_+ \times C(I_\omega, \boldsymbol{R}_+^m) \to \boldsymbol{R}_+,\ \ 1 \leqslant i \leqslant m;$$

(H1.2): mappings f_i, g_i are locally Lipschitzian: for each fixed $\boldsymbol{d} \in \boldsymbol{R}^m$, $\boldsymbol{d} > 0$, for all $0 \leqslant t < \infty$ and $\forall\, z_1, z_2 \in C(I_\omega, \boldsymbol{R}^m)$, $\boldsymbol{c} \leqslant z_1, z_2 \leqslant \boldsymbol{d}$, inequalities

$$|f_i(t, z_1) - f_i(t, z_2)| \leqslant L_f^{(i)}(\boldsymbol{c}, \boldsymbol{d})\, ||z_1 - z_2||,$$
$$|g_i(t, z_1) - g_i(t, z_2)| \leqslant L_g^{(i)}(\boldsymbol{c}, \boldsymbol{d})\, ||z_1 - z_2||,$$

are valid, where $L_f^{(i)}(\boldsymbol{c}, \boldsymbol{d}) > 0$, $L_g^{(i)}(\boldsymbol{c}, \boldsymbol{d}) > 0$ – Lipschitz constants, $1 \leqslant i \leqslant m$;

(H1.3): constant $\mu_i > 0, \;\; 1 \leqslant i \leqslant m$;

(H1.4): function $\psi_i(t)$ is continuous and nonnegative, $t \in I_\omega$, $1 \leqslant i \leqslant m$.

THEOREM 1. Suppose, that assumptions (H1) are valid. Then, if model (1), (2) has on finite interval $[0, \tau)$ solution $x \geqslant 0$, then this solution is unique.

• • • PROBLEM:

at what additional assumptions it is possible to speak about a correctness of model (1), (2) ? ? ?

- Assumption (H2): $\forall\ (t, z) \in R_+ \times C(I_\omega, R_+^m)$ inequality $f(t, z) \leqslant q$ is right, where $q \in R_+^m$ — some vector.

<u>THEOREM 2.</u> Suppose, that assumptions (H1) and (H2) are valid. Then:

A) model (1), (2) has on interval $[0, \infty)$ unique solution x, and

$$0 \leqslant x(t) \leqslant \max\Big\{\sup_{t \in I_\omega} \psi(t);\ \mu^{-1} q\Big\}, \quad t \in [0, \infty);$$

B) solution of model (1), (2) on each finite interval $[0, \tau)$ depends continuously on function ψ.

- Assumption (H3): $\forall\ (\boldsymbol{t}, \boldsymbol{z}) \in \boldsymbol{R}_+ \times \boldsymbol{C}(\boldsymbol{I}_\omega, \boldsymbol{R}_+^m)$ estimate

$$\boldsymbol{f}(\boldsymbol{t}, \boldsymbol{z}) \leqslant \boldsymbol{p} + \int_{-\omega}^{0} \boldsymbol{d\nu}(\boldsymbol{\theta})\boldsymbol{z}(\boldsymbol{\theta}) \quad \text{is carried out, where}$$

$\boldsymbol{p} \in \boldsymbol{R}_+^m$ — some vector, elements of matrix $\boldsymbol{\nu}(\boldsymbol{\theta})$ do not decrease on $\boldsymbol{I}_\omega$, matrix $\boldsymbol{\Delta\nu} = \boldsymbol{\nu}(0) - \boldsymbol{\nu}(-\boldsymbol{\omega})$ contains at least one positive element.

<u>THEOREM 3.</u> Suppose, that assumptions (H1) and (H3) are valid. Then:

A) model (1), (2) has on interval $[0, \infty)$ unique solution $\boldsymbol{x}$, and

$$0 \leqslant \boldsymbol{x}(\boldsymbol{t}) \leqslant \boldsymbol{\rho}\, \boldsymbol{e}^{\eta t}, \quad \boldsymbol{t} \in [0, \infty),$$

where $\boldsymbol{\rho} \in \boldsymbol{R}^m$, $\boldsymbol{\rho} > 0$, $\boldsymbol{\eta} \in \boldsymbol{R}$, $\boldsymbol{\eta} > 0$ depend on $\boldsymbol{p}$, $\boldsymbol{\Delta\nu}$, $\boldsymbol{\mu}$, $\boldsymbol{\psi}$;

B) solution of model (1), (2) on each finite interval $[0, \boldsymbol{\tau})$ depends continuously on function $\boldsymbol{\psi}$.

- Assumption (H4): $\forall\ (t, z) \in R_+ \times C(I_\omega, R_+^m)$ estimate

$$f(t, z) \leqslant h(z) \quad \text{is carried out, where}$$

$h : C(I_\omega, R_+^m) \to R_+^m$ — some continuous isotone functional.

<u>THEOREM 4.</u> Suppose, that assumptions (H1), (H4) are valid and $\exists\ w \in R^m$, $w > 0$, such, that $h(w) \leqslant \mu\, w$. Then:

A) if $\psi(t) \leqslant w$, $t \in I_\omega$, then model (1), (2) has on interval $[0, \infty)$ unique solution x, and

$$0 \leqslant x(t) \leqslant w, \quad t \in [0, \infty);$$

B) solution of model (1), (2) on each finite interval $[0, \tau)$ depends continuously on function ψ, provided that $\psi(t) \leqslant w$, $t \in I_\omega$.

3. Conclusion.

Assumptions (H1) in combination with (H2) or (H3) or (H4) prove a possibility of application of model (1), (2) for studying the dynamics of various alive systems.