Pulses and waves for reaction-diffusion systems

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VIII-th conference on mathematical models and numerical methods in biomathematics, Moscow, October 31, 2016

1 - Introduction - Scalar equations

2 - System of two equations : model problem

3 - More general systems

Application to a model of competition of species

M. Marion and V. Volpert, 'Existence of pulses for a monotone reaction-diffusion system" , J. Pure and Applied Functional Analysis, 2016

M. Marion and V. Volpert, 'Existence of pulses for the system of competition of species"

Reaction-diffusion systems

- Unknown : $\boldsymbol{v} = (v_1, v_2, ..., v_m)$

- Equations

$$\begin{cases} \frac{\partial v_1}{\partial t} = d_1 \frac{\partial^2 v_1}{\partial x^2} + F_1(v_1, v_2, ..., v_m), \\ \qquad\qquad ... \\ \frac{\partial v_m}{\partial t} = d_m \frac{\partial^2 v_m}{\partial x^2} + F_m(v_1, v_2, ..., v_m) \end{cases}$$

or

$$\frac{\partial \boldsymbol{v}}{\partial t} = D \frac{\partial^2 \boldsymbol{v}}{\partial x^2} + \boldsymbol{F}(\boldsymbol{v})$$

Applications

- Combustion

- Population dynamics

- Epidemiology

- Physiology

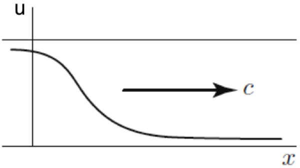
Waves and pulses

$$\frac{\partial \mathbf{v}}{\partial t} = D\frac{\partial^2 \mathbf{v}}{\partial x^2} + \mathbf{F}(\mathbf{v}), \quad x \in \mathbb{R}, \quad t > 0$$

- Waves

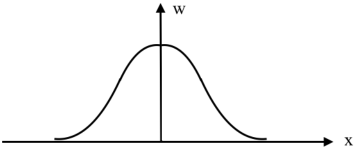
$$\mathbf{v}(x,t) = \mathbf{u}(x - ct) = \mathbf{u}(\xi)$$

$$\begin{cases} D\mathbf{u}'' + c\mathbf{u}' + \mathbf{F}(\mathbf{u}) = \mathbf{0} \\ \mathbf{u}(\infty) = \mathbf{0}, \quad \mathbf{u}(-\infty) = \mathbf{1} \\ \mathbf{u} \geq \mathbf{0} \end{cases}$$



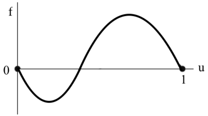
- Pulses

$$\begin{cases} D\mathbf{w}'' + \mathbf{F}(\mathbf{w}) = \mathbf{0} \\ \mathbf{w}(\infty) = \mathbf{w}(-\infty) = \mathbf{0} \\ \mathbf{w}(x) = \mathbf{w}(-x), \\ \mathbf{w}'(x) < \mathbf{0} \text{ for } x > 0 \end{cases}$$



Scalar equation : $\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} + f(u)$

Bistable case



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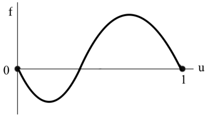

- Existence of a unique wave (u, c)

$$c > 0 \Leftrightarrow \int_0^1 f(s)ds > 0$$

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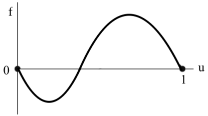

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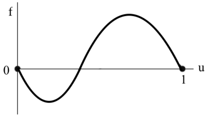

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$$\Leftrightarrow c > 0$$

Systems of equations

Model problem :

$$\begin{cases} \frac{\partial v_1}{\partial t} = \frac{\partial^2 v_1}{\partial x^2} + F_1(v_1, v_2) \\ \frac{\partial v_2}{\partial t} = \frac{\partial^2 v_2}{\partial x^2} + F_2(v_1, v_2) \end{cases}$$

$$F_1(v_1, v_2) = -v_1 + f_1(v_2), \quad F_2(v_1, v_2) = -v_2 + f_2(v_1)$$

Assumptions on the reaction term :

- F has three zeros : $(0,0)$, $(1,1)$ and $\bar{w} = (\bar{w}_1, \bar{w}_2)$, $0 < \bar{w}_i < 1$
- $(0,0)$ and $(1,1)$ stable, $\bar{w}$ unstable
- $f_i' > 0$

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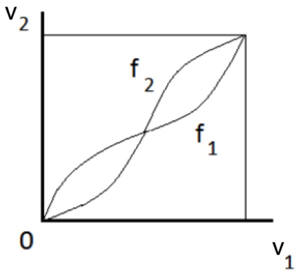

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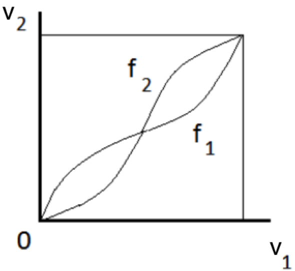
Monotone system

$\frac{\partial F_i}{\partial v_j}(v_1, v_2) > 0, \quad i \neq j$



Model problem

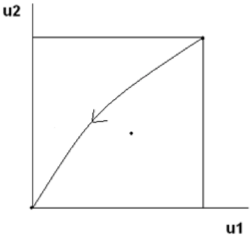
Travelling waves : $\boldsymbol{u}'' + c\boldsymbol{u}' + \boldsymbol{F}(\boldsymbol{u}) = \mathbf{0}$

$\boldsymbol{u}(\infty) = (0,0), \quad \boldsymbol{u}(-\infty) = (1,1), \quad \boldsymbol{u} \geq \mathbf{0}$

Existence and uniqueness

of a wave $(\boldsymbol{u}, c)$

Volpert et Al, Trans of AMS 1994



Pulses : $\boldsymbol{w}'' + \boldsymbol{F}(\boldsymbol{w}) = \mathbf{0}$

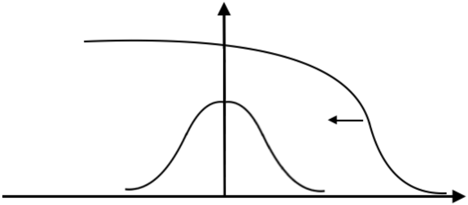
$\boldsymbol{w}(\pm\infty) = \mathbf{0}$

$\boldsymbol{w}(x) = \boldsymbol{w}(-x), \quad \boldsymbol{w}'(x) < \mathbf{0} \text{ for } x > 0$

Existence of a pulse $\Leftrightarrow c > 0$

Sketch of the proof

- Existence of pulse $\Rightarrow c > 0$



Sketch of the proof : $c > 0 \Rightarrow$ existence of pulse

$$(\mathcal{P}) \quad \begin{cases} \boldsymbol{w}'' + \boldsymbol{F}(\boldsymbol{w}) = \boldsymbol{0} \\ \boldsymbol{w}'(0) = \boldsymbol{0}, \quad \boldsymbol{w} \geq \boldsymbol{0}, \quad \boldsymbol{w}(\infty) = \boldsymbol{0} \end{cases}$$

$E^1 = \{\boldsymbol{w} \in C^{2+\alpha}(\mathbb{R}_+)^2, \ \boldsymbol{w}'(0) = \boldsymbol{0}\}$

$E^1_\mu =$ weighted space, $\mu(x) = \sqrt{1 + x^2}$ with norm $\|\boldsymbol{w}\|_{E^1_\mu} = \|\boldsymbol{w}\mu\|_{E^1}$

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I - A priori estimates

$c > 0 \ \Rightarrow \|\boldsymbol{w}^M\|_{E^1_\mu} \leq R$ for any monotone solutions $\boldsymbol{w}^M$ of $(\mathcal{P})$

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II - Separation between monotone and non monotone solutions of $(\mathcal{P})$

$$\|w^M - w^N\|_{E^1_\mu} \geq r > 0, \ \|w^M\|_{E^1_\mu} > r > 0$$

Sketch of the proof : $c > 0 \Rightarrow$ existence of pulse

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III - Topological degree

D bounded subset of E^1_μ containing all monotone solutions but no non-monotone ones and not the trivial one. We want to show that

$$\gamma(A, D) \neq 0$$

$\rightarrow$ Homotopy argument

Sketch of the homotopy argument

$$\boldsymbol{w}'' + \boldsymbol{F}^\tau(\boldsymbol{w}) = \boldsymbol{0}, \quad F_1^\tau(w) = -w_1 + f_1^\tau(w_2), \quad F_2^\tau(w) = -w_2 + f_2^\tau(w_1)$$

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We need $c^\tau > 0$ for $\tau \in [0, 1]$

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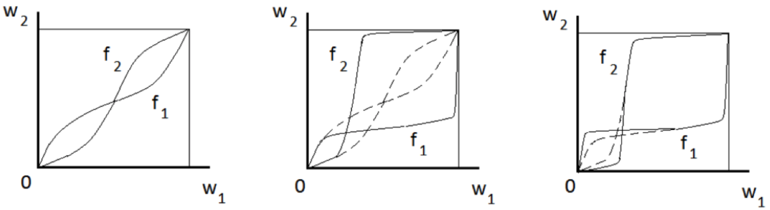


$\tau = 0$

$\tau = 1/2$

$\int_0^1 (f_i(s) - s)ds > 0$

$\tau = 1$

$f_1 = f_2$

$\int_0^1 (f_1(s) - s)ds > 0$

$w_1 = w_2$

$w_1'' - w_1 + f_1(w_1) = 0$

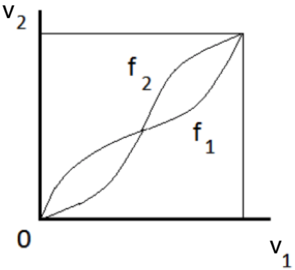
$\gamma(A, D) = -K \leq -1$

More general systems

$$w_1'' + F_1(w_1, w_2) = 0, \quad w_2'' + F_2(w_1, w_2) = 0.$$

Assumptions on the reaction term :

- $\frac{\partial F_i}{\partial w_j}(w) > 0, \quad i, j = 1, 2, \quad i \neq j, \quad w = (w_1, w_2) \in \mathbb{R}^2$
- Three zeros : $(0, 0)$, $(1, 1)$, $\bar{w} = (\bar{w}_1, \bar{w}_2), 0 < \bar{w}_i < 1$
- zeros of F_1 and F_2



Existence of a pulse $\Leftrightarrow c > 0$

Model of competition of species

$$\begin{cases} v_1'' + v_1(1 - v_1 - \alpha v_2) = 0 \\ v_2'' + \rho v_2(1 - \beta v_1 - v_2) = 0 \end{cases} \qquad \alpha > 1 \text{ and } \beta > 1$$

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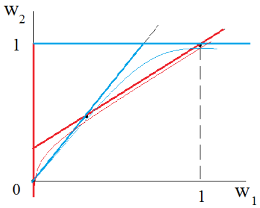
Monotone system : $w_1 = v_1$ and $w_2 = 1 - v_2$

$$\begin{cases} w_1'' + w_1(1 - w_1 - \alpha(1 - w_2)) = 0 \\ w_2'' + \rho(1 - w_2)(\beta w_1 - w_2) = 0 \end{cases}$$



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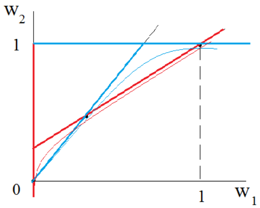

Perturbed system : $\begin{cases} w_1'' + F_1(w_1, w_2) + \epsilon\alpha w_2 = 0 \\ w_2'' + F_2(w_1, w_2) - \epsilon\rho w_2 = 0 \end{cases}$

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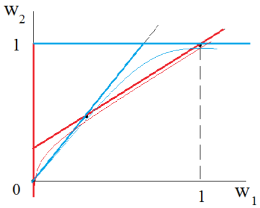

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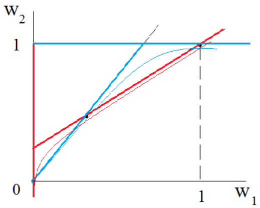

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- $c > 0$ if $\rho = 1$ and $\beta > \alpha$

Thank you for your attention !