Iterative solution of Oseen linear systems and applications to coronary blood flows

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Main ideas

- It is possible to solve FEM discretizations of unsteady Navier–Stokes equations (NSe) for coronary blood flows by iterative methods
- The preconditioners based on two-parameter ILU(τ_1, τ_2) are better then generic ones

- For coronary blood flows the SUPG-stabilization is usually not required
- If the blood viscosity is 1.5 times less then SUPG-stabilization is required (but the linear systems becomes more difficult to solve)

- We solve 1M problems of 3D blood flow on laptop

Prerequisites

- Unsteady Navier–Stokes equations (NSe) are solved numerically either by projection methods or by (semi)-implicit methods
- In projection methods the velocity is predicted (by semi-Lagrangian method or convection-diffusion solver) and then projected
- In (semi)-implicit methods an Oseen-type linear system is solved
- The solution is difficult for high Reynolds numbers $\mathrm{Re} = \frac{UL}{\nu}$ and FEM discretizations on unstructured 3D meshes
  - For instance, block preconditioners (e.g. Kay-Loghin) failed in simulation of coronary blood flows ($\mathrm{Re} \sim 10^2$)
- We found in surprise that our public software (aniILU library from ani2D/ani3D packages) solves the Oseen systems very well
- In this talk we present this "discovery"

I.Konshin, M.Olshanskii, and Yu.Vassilevski. *ILU preconditioners for non-symmetric saddle point matrices with application to the incompressible Navier–Stokes equations*. SIAM J.Sci.Comp., **37** (2015)

Prerequisites

Navier–Stokes equations

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{f} \ \ \textbf{in } \Omega \times (0, T] \\
\textbf{div } \mathbf{u} &= 0 \ \ \textbf{in } \Omega \times [0, T] \\
\mathbf{u} &= \mathbf{g} \ \ \textbf{on } \Gamma_0 \times [0, T] \\
-\nu(\nabla \mathbf{u}) \cdot \mathbf{n} + p\mathbf{n} &= \mathbf{0} \ \ \textbf{on } \Gamma_{\mathrm{N}} \times [0, T] \\
\mathbf{u}(\mathbf{x}, 0) &= \mathbf{u}_0(\mathbf{x}) \ \ \textbf{in } \Omega
\end{aligned}
$$

u is the velocity, p is the pressure, ν is the kinematic viscosity

$\partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_{\mathrm{N}}$ **and** $\Gamma_0 \neq \varnothing$

Prerequisites

Implicit time discretization and linearization

$$\begin{aligned} \alpha\mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{w}\cdot\nabla)\mathbf{u} + \nabla p &= \hat{\mathbf{f}} && \text{in } \Omega \\ \operatorname{div}\mathbf{u} &= \hat{g} && \text{in } \Omega \\ \mathbf{u} = \mathbf{0} \ \text{ on } \Gamma_0, \quad -\nu(\nabla\mathbf{u})\cdot\mathbf{n} + p\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{N}} \end{aligned}$$

- **w is a known velocity field**
- $\alpha \sim (\Delta t)^{-1}$ **(**$\alpha = 0$ **for a steady problem)**
- $\hat{\mathbf{f}}$, $\hat{g}$ **account for non-homogeneous boundary conditions in the nonlinear problem**

Let $\mathbf{V} := \{\mathbf{v} \in H^1(\Omega)^3 \; : \; \mathbf{v}|_{\Gamma_0} = \mathbf{0}\}$**.**

Given $\mathbf{f} \in \mathbf{V}'$**, the problem is to find** $\mathbf{u} \in \mathbf{V}$ **and** $p \in L^2(\Omega)$ **such that**

$$\mathcal{L}(\mathbf{u}, p; \mathbf{v}, q) \; = \; (\mathbf{f}, \mathbf{v})_* + (g, q) \qquad \forall\, \mathbf{v} \in \mathbf{V},\; q \in L^2(\Omega)\,,$$

$$\mathcal{L}(\mathbf{u}, p; \mathbf{v}, q) \; := \; \alpha(\mathbf{u}, \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{w} \cdot \nabla)\, \mathbf{u}, \mathbf{v}) - (p, \mathrm{div}\, \mathbf{v}) + (q, \mathrm{div}\, \mathbf{u})\,,$$

where $(\cdot, \cdot)$ **denotes the** $L^2(\Omega)$ **inner product and** $(\cdot, \cdot)_*$ **is the duality paring for** $\mathbf{V}' \times \mathbf{V}$**.**

Prerequisites

FEM discretization

Let T_h be a regular tetrahedral mesh: $\max_{\tau \in T_h} \mathbf{diam}(\tau)/\rho(\tau) \leq C_T$**.**

Given conforming FE spaces $\mathbb{V}_h \subset \mathbf{V}$ and $\mathbb{Q}_h \subset L^2(\Omega)$, the Galerkin FE discretization is:

Find $\{\mathbf{u}_h, p_h\} \in \mathbb{V}_h \times \mathbb{Q}_h$ such that

$$\mathcal{L}(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = (\mathbf{f}, \mathbf{v}_h)_* + (g, q_h) \qquad \forall\, \mathbf{v}_h \in \mathbb{V}_h,\ q_h \in \mathbb{Q}_h$$

We use P2-P1 Taylor–Hood FE pair:

- **it satisfies the LBB compatibility condition [Girault,Raviart,1979]**
- **ensures well-posedness and full approximation order for the FE linear problem**

Prerequisites

FEM discretization

Enumerating velocity unknowns first and pressure unknowns next, we get the system

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

- u **and** p **represent the discrete velocity and pressure,**
- $A \in \mathbb{R}^{n \times n}$ **is the discretization of the diffusion, convection, and time-dependent terms,**
- $B^T \in \mathbb{R}^{n \times m}$ **is the discrete gradient,**
- B **is the (negative) discrete divergence,**
- $C \in \mathbb{R}^{m \times m} = 0$ **(for stabilized FEM pairs** $C = C^T \geq 0$**)**

LU factorization

Existence of LU factorization

- **In general, LU factorization of such matrices requires pivoting (rows and columns permutations) for stability reasons.**
- **Due to block structure and the properties of blocks A and C, factorization exists without pivoting**

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & -U_{22} \end{pmatrix}$$

once

- $A = L_{11}U_{11}$ **(holds true for pos.def. A)**
- $S := BA^{-1}B^T + C = L_{22}U_{22}$ **(holds true for pos.def. S)**
- $U_{12} = L_{11}^{-1}B^T$, $\quad L_{21} = BU_{11}^{-1}$

LU factorization

Stability of LU factorization

Numerical stability depends on how large is the skew-symmetric part $A_{\mathrm{N}} = A - A_{\mathrm{S}}$ of A comparing to the symmetric part $A_{\mathrm{S}} = \frac{1}{2}(A + A^T)$.

Golub,Van Loan,1996: Denote $|C| = \{|c_{ij}|\}$ for a matrix $C = \{c_{ij}\}$

$$\||L_{11}||U_{11}|\|_F \leq n \left(\|A_{\mathrm{S}}\| + \|A_{\mathrm{N}} A_{\mathrm{S}}^{-1} A_{\mathrm{N}}\| \right)$$

$$\|A_{\mathrm{N}} A_{\mathrm{S}}^{-1} A_{\mathrm{N}}\| \leq \|A_{\mathrm{S}}^{-\frac{1}{2}} A_{\mathrm{N}} A_{\mathrm{S}}^{-\frac{1}{2}}\|^2 \|A\| \Rightarrow$$

$$\frac{\||L_{11}||U_{11}|\|_F}{\|A\|} \leq n \left(1 + \|A_{\mathrm{S}}^{-\frac{1}{2}} A_{\mathrm{N}} A_{\mathrm{S}}^{-\frac{1}{2}}\|^2 \right)$$

Similarly,

$$\frac{\||L_{22}||U_{22}|\|_F}{\|S\|} \leq m \left(1 + \|S_{\mathrm{S}}^{-\frac{1}{2}} S_{\mathrm{N}} S_{\mathrm{S}}^{-\frac{1}{2}}\|^2 \right)$$

Lemma. $\|S_{\mathrm{S}}^{-\frac{1}{2}} S_N S_{\mathrm{S}}^{-\frac{1}{2}}\| \leq \|A_{\mathrm{S}}^{-\frac{1}{2}} A_{\mathrm{N}} A_{\mathrm{S}}^{-\frac{1}{2}}\|$

LU factorization

Stability of LU factorization

Let $C_A := \|A_S^{-\frac{1}{2}} A_N A_S^{-\frac{1}{2}}\|$ **and** $c_A := \lambda_{\min}(A_S)$.

Then

$$\frac{\|U_{12}\|_F + \|L_{21}\|_F}{(\|U_{11}\| + \|L_{11}\|)\|B\|_F} \leq \frac{m(1 + C_A)}{c_A}$$

Thus the relative sizes of entries of L **and** U **are bounded if** $C_A \leq Q < \infty$, $c_A \geq q > 0$.

Stability of LU factorization for the Oseen matrix

What are estimates for stability and ellipticity constants?

$$C_A = \|A_{\mathrm{S}}^{-\frac{1}{2}} A_{\mathrm{N}} A_{\mathrm{S}}^{-\frac{1}{2}}\| \leq Q \qquad c_A = \lambda_{\min}(A_S) \geq q$$

Stability of LU factorization for the Oseen matrix

Let $\{\varphi_i\}_{1\le i\le n}$ and $\{\psi_j\}_{1\le j\le m}$ be bases of $\mathbb{V}_h$ and $\mathbb{Q}_h$. For any $v \in \mathbb{R}^n$ and $\mathbf{v}_h = \sum_{i=1}^n v_i\varphi_i$ it holds

$$\langle Av, v\rangle = \alpha\|\mathbf{v}_h\|^2 + \nu\|\nabla\mathbf{v}_h\|^2 + \frac{1}{2}\int_{\Gamma_{\mathrm{N}}} (\mathbf{w}\cdot\mathbf{n})|\mathbf{v}_h|^2\,ds + \frac{1}{2}((\operatorname{div}\mathbf{w})\mathbf{v}_h, \mathbf{v}_h)$$

Difficulties of the estimate:

- **often $(\mathbf{w}\cdot\mathbf{n}) < 0$ on a *part* of Γ_{N}**
- **often $(\operatorname{div}\mathbf{w}, q_h) = 0 \quad \forall\, q \in \mathbb{Q}_h$ does *not* imply $\operatorname{div}\mathbf{w} = 0$ pointwise**

Stability of LU factorization for the Oseen matrix

A priori estimates

- **Trace inequality** $\int_{\Gamma_\mathrm{N}} |\mathbf{v}_h|^2 \, ds \le C_0 \|\nabla \mathbf{v}_h\|^2 \quad \forall\, \mathbf{v}_h \in \mathbb{V}_h$
- **FE trace inequality** $\int_{\partial\tau} \mathbf{v}_h^2 \, ds \le C_\mathrm{tr} h_\tau^{-1} \|\mathbf{v}_h\|_\tau^2$
- **Friedrichs inequality** $\|\mathbf{v}_h\| \le C_\mathrm{f} \|\nabla \mathbf{v}_h\| \quad \forall\, \mathbf{v}_h \in \mathbb{V}_h$
- **Inflow for** Γ_N $C_\mathbf{w} := \|(\mathbf{w}\cdot\mathbf{n})_-\|_{L^\infty(\Gamma_\mathrm{N})}$
- M, K **are velocity mass and stiffness matrices,** M_p **is pressure mass matrix,** $S = BA^{-1}B^T$
- $\alpha \sim (\Delta t)^{-1}$
- **viscosity** ν
- **minimal mesh size** h_min
- **Assume that**

$$C_\mathbf{w} C_\mathrm{tr} h_\mathrm{min}^{-1} \le \frac{\alpha}{4} \ \text{ or } \ C_\mathbf{w} C_0 \le \frac{\nu}{4},$$

$$\|\mathrm{div}\, \mathbf{w}\|_{L^\infty(\Omega)} \le \frac{1}{4} \max\{\alpha, \nu C_f^{-1}\}$$

Stability of LU factorization for the Oseen matrix

A priori estimates

Theorem

Under the above assumptions

$$c_A \geq \frac{1}{4}\lambda_{\min}(\alpha M + \nu K), \qquad C_A \lesssim 1 + \frac{\|\mathbf{w}\|_{L^\infty(\Omega)}}{\sqrt{\nu\alpha} + \nu + h_{\min}\alpha}$$

$$\langle Av, v\rangle \geq \frac{1}{4}\langle(\alpha M + \nu K)v, v\rangle \quad \forall\, v \in \mathbb{R}^n$$

$$\langle Sq, q\rangle \gtrsim \frac{\langle M_p q, q\rangle}{(\nu + \alpha + \|\mathbf{w}\|_{L^\infty(\Gamma_{\mathrm{N}})} + \|\mathrm{div}\,\mathbf{w}\|_{L^\infty(\Omega)})(1 + C_A^2)} \quad \forall\, q \in \mathbb{R}^m$$

Stability of LU factorization for the Oseen matrix

A priori estimates

Are the assumptions

$$C_{\mathbf{w}} C_{\mathrm{tr}} h_{\min}^{-1} \leq \frac{\alpha}{4} \quad \textbf{or} \quad C_{\mathbf{w}} C_0 \leq \frac{\nu}{4},$$

$$\|\mathrm{div}\,\mathbf{w}\|_{L^\infty(\Omega)} \leq \frac{1}{4} \max\{\alpha, \nu C_f^{-1}\}$$

feasible?

- **If $\Gamma_{\mathrm{N}} = \varnothing$ or Γ_{N} is outflow boundary then $C_{\mathbf{w}} = 0$**
- **Otherwise, $C_{\mathbf{w}} C_0 \leq \frac{\nu}{4}$ holds for large ν (creeping flows) or for Courant-type condition $c \leq (\Delta t)^{-1} h_{\min}$**
- $\|\mathrm{div}\,\mathbf{w}\|_{L^\infty(\Omega)} \leq \widetilde{C}_\nu (h^2 + (\Delta t)^2)$
- **but one can choose such small Δt that $\alpha/4 \geq \|\mathrm{div}\,\mathbf{w}\|_{L^\infty(\Omega)}$**

Threshold incomplete LU factorization

- any threshold ILU factorization $A = LU - E$ with an error matrix E
- $\|E\|$ depends on threshold $\tau > 0$
- Kaporin,2007: estimates on GMRES convergence in terms of $\|E\|$
- Kuznetsov,1968,1969: formulation of GMRES
- Golub,Van Loan,1996: for pos.def. A with $c_A = \lambda_{\min}(A_S)$ LU is also pos.def. and factorization is numerically stable if $\tau < c_A n^{-1}$
- actual τ is not that small: non-positive or close to zero pivots may encounter, resulting in breakdown
- Benzi,2002: review of remedies (pivot modification, diagonal shifting, matrix scaling, unknowns reordering, Ajiz-Jennings modification)
- our choice: two-sided scaling of matrix

Threshold ILU factorization for positive definite matrices

Motivation

- **Consider (1,1)-block A of the Oseen matrix for small ν and large Δt, $\|\mathbf{w}\|$**
- **In this case $c_A \ll 1$ and $C_A \gg 1$**
- **For the sake of stability τ must be smaller and fill-in larger**
- **Two-parameter Tismenetsky-Kaporin ILU(τ_1, τ_2) factorization [Kaporin,1998] allows to increase τ and reduce fill-in**
- **ILU(τ_1, τ_2) was suggested and analyzed for SPD matrices, but has been applied successfully to general matrices**

Two-parameter threshold ILU factorization for positive definite matrices

ILU(τ_1, τ_2):

$$A = LU + LR_u + R_\ell U - E$$

- R_u **and** R_ℓ **are strictly upper and lower triangular matrices:**

$$\tau_2 < |R_{u\,ij}| \le \tau_1 \textbf{ or } R_{u\,ij} = 0, \quad \tau_2 < |R_{\ell\,ij}| \le \tau_1 \textbf{ or } R_{\ell\,ij} = 0$$

- U **and** L **are upper and lower triangular matrices:**

$$|U_{ij}| > \tau_1 \textbf{ or } U_{ij} = 0, \quad |L_{ij}| > \tau_1 \textbf{ or } L_{ij} = 0$$

- E **is an error matrix with entries of order** $O(\tau_2)$

ILU(τ) can be viewed as ILU(τ_1, τ_2) with $R_u = R_\ell = 0$ **and** $\tau_1 = \tau_2 = \tau$

Two-parameter threshold ILU factorization for positive definite matrices

ILU(τ_1, τ_2):

$$A = LU + LR_u + R_\ell U - E$$

Benefits over a generic ILU(τ):

- **fill-in of L and U is ruled by τ_1, while preconditioning quality is ruled by τ_2, once $\tau_1^2 \lesssim \tau_2$**
- **for $\tau_2 = \tau_1^2 := \tau^2$ the fill-in of ILU(τ_1, τ_2) is similar to ILU(τ), convergence is similar to ILU(τ^2)** (Kaporin,1998:proved for SPD A, $L^T = U$, $R_\ell^T = R_u$)
- **computing L and U factors is more costly than for ILU(τ_1) and less expensive than for ILU(τ_2)**
- **stability of system solution with matrices L,U is ruled by τ_1^2 and τ_2**

  $$|L_{ii}U_{ii}| \geq c_A - \|R_\ell R_u\| - \|E\|$$

  where $\|R_\ell R_u\| \sim \tau_1^2$, $\|E\| \sim \tau_2$

Algorithmic ideas of ILU(τ_1, τ_2)

Preprocessing by two-sided scaling

Kaporin,1998: Derivation of ILU(τ_1, τ_2) in SPD case requires $A_{ii} = 1$

Kaporin,2007: In general case one balances (nearly) Euclidean norms of rows and columns by

$$A' = \mathrm{diag}(\ell) A \, \mathrm{diag}(r)$$

Scaling vectors $\ell, r \in \mathbb{R}^n$ **are found by Sinkhorn algorithm applied to** $F = [A_{kj}^2]_{kj=1}^n$**: starting with vector** $\ell^{(0)}$ **of all ones, perform**

$$\begin{aligned} \mathrm{diag}(r^{(k+1)}) &= \mathrm{diag}(F^T \ell^{(k)})^{-1}, \\ \mathrm{diag}(\ell^{(k+1)}) &= \mathrm{diag}(F r^{(k+1)})^{-1}, \end{aligned}$$

$$L'U' \approx A' \Rightarrow LU \approx A, \qquad L = (\mathrm{diag}(\ell))^{-1} L', \qquad U = U'(\mathrm{diag}(r))^{-1}$$

At least one Sinkhorn iteration is needed for the Oseen matrices

(we take 5)

Algorithmic ideas of ILU(τ_1, τ_2)

Rowwise ILU(τ_1, τ_2) factorization by Sergei Goreinov *(ani2D/ani3D packages)*

($i+1$)th step (skipping the error matrix):

$$\begin{bmatrix} A^i & a^i & \widetilde{A}^i \\ \widehat{a}^i & \alpha^i & \widetilde{a}^i \\ * & * & * \end{bmatrix} = \begin{bmatrix} L^i & & \\ l^i & \lambda^i & \\ * & * & * \end{bmatrix} \begin{bmatrix} U^i & u^i & \widetilde{U}^i \\ & \mu^i & \widetilde{u}^i \\ & & * \end{bmatrix}$$

$$+ \begin{bmatrix} L^i & & \\ l^i & \lambda^i & \\ * & * & * \end{bmatrix} \begin{bmatrix} R^i_u & r^i & \widetilde{R}^i \\ & 0 & \widetilde{r}^i \\ & & * \end{bmatrix} + \begin{bmatrix} R^i_\ell & & \\ \widehat{r}^i & 0 & \\ * & * & * \end{bmatrix} \begin{bmatrix} U^i & u^i & \widetilde{U}^i \\ & \mu^i & \widetilde{u}^i \\ & & * \end{bmatrix}$$

scalar (Greek), vectors (Latin lowercase), matrices (Latin uppercase)

$$\begin{aligned} \widehat{a}^i &= (l^i + \widehat{r}^i)U^i + l^i R^i_u, \\ \alpha^i &= (l^i + \widehat{r}^i)u^i + l^i r^i + \lambda^i \mu^i, \\ \widetilde{a}^i &= (l^i + \widehat{r}^i)\widetilde{U}^i + l^i \widetilde{R}^i_u + \lambda^i(\widetilde{u}^i + \widetilde{r}^i) \end{aligned}$$

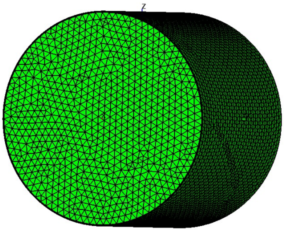
$$l^i, \widehat{r}^i \to \mu^i, \widetilde{u}^i \to \lambda^i$$

If $\tau_2 = \tau_1$, ILU(τ_1, τ_2) is similar to ILUT(n, τ_1) (Saad,2003)

Numerical experiment: Pipe flow

Cylinder of diameter 1, length 2. Poiseuille flow on inlet, do-nothing b.c. on outlet, $max|\mathbf{w}| = 1$

$\text{fill}_{LU} = (\text{nz}(L) + \text{nz}(U))/\,\text{nz}(A)$



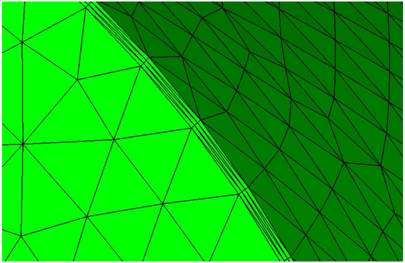
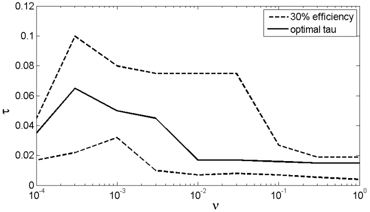


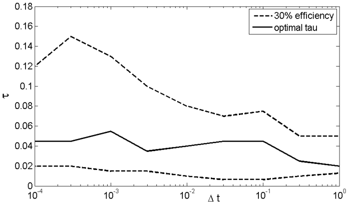
Table : The dependence of ILU(τ,τ) performance on τ; $\nu = 10^{-3}, \alpha = 1$

τ	fill_{LU}	#it	T_{build}	T_{it}	T_{CPU}	fill_{LU}	#it	T_{build}	T_{it}	T_{CPU}
	Mesh regular, d.o.f.$\sim$ 300 000					Mesh with b.layer, d.o.f.$\sim$ 530 000				
0.080	0.497	94	1.5	10.2	11.7	0.385	129	2.3	23.3	25.6
0.060	0.667	60	1.9	7.3	9.2	0.519	69	2.9	13.7	16.6
0.050	0.793	52	2.3	6.8	9.1	0.640	62	3.4	13.2	16.6
0.040	0.969	49	2.9	7.0	9.9	0.798	52	4.2	12.1	16.4
0.030	1.239	44	3.9	7.2	11.1	1.003	43	5.4	11.2	16.6
0.010	2.917	22	12.3	6.1	18.4	2.209	24	15.0	10.0	25.0
0.005	4.700	16	25.1	6.2	31.3	3.384	16	28.8	8.9	37.7
0.003	6.472	13	41.6	6.3	47.9	4.520	12	46.5	8.2	54.7
0.001	11.954	9	115.5	7.0	122.5	8.007	10	125.4	10.6	135.9

Numerical experiment: Pipe flow

Cylinder of diameter 1, length 2. Poiseuille flow on inlet, do-nothing b.c. on outlet, $max|\mathbf{w}| = 1$ $\quad \text{fill}_{LU} = (\text{nz}(L) + \text{nz}(U))/\,\text{nz}(A)$





- τ_{opt} is almost grid independent
- τ_{opt} is not very sensitive to ν and $\alpha = (\Delta t)^{-1}$ $\rightarrow$ $\tau_{opt} = 0.03$
- ILU($\tau_1 = 0.03, \tau_2 = 7\tau_1^2$) is largely comparable to ILU(τ_{opt})
- at least 1 two-sided balancing iteration is obligatory

Numerical experiment: Pipe flow

Cylinder of diameter 1, length 2. Poiseuille flow on inlet, do-nothing b.c. on outlet, $max|\mathbf{w}| = 1$

$$\text{fill}_{LU} = (\text{nz}(L) + \text{nz}(U))/\,\text{nz}(A)$$

Table : The performance of ILU preconditioners for the pipe flow test case on the anisotropic mesh

	ν:	1		10^{-1}		10^{-2}		10^{-3}		10^{-4}	
	α:	10	100	10	100	10	100	10	100	10	100
	ILU($\tau = 0.03$)										
fill_{LU}		0.83	0.81	0.80	0.70	0.73	0.73	1.00	0.99	1.91	1.02
#it		177	67	59	36	32	50	43	59	99	136
T_{CPU}		51.4	19.6	17.7	11.4	10.7	15.3	16.9	20.5	51.8	41.9
	ILU($\tau_1 = 0.03, \tau_2 = 7\tau_1^2$)										
fill_{LU}		0.86	0.83	0.83	0.71	0.74	0.71	0.97	0.88	1.74	0.94
#it		127	55	42	29	22	36	32	53	45	83
T_{CPU}		39.7	21.5	18.3	13.1	12.2	15.5	24.2	24.3	68.3	34.3

Numerical experiment: Ethier-Steinman vortex

analog of Taylor vortex problem

Solution to unsteady NSe in $[-1,1]^3$ is

$$
\begin{aligned}
u_1 &= -a\left(e^{ax}\sin(ay+dz)+e^{az}\cos(ax+dy)\right)e^{-\nu d^2 t}\\
u_2 &= -a\left(e^{ay}\sin(az+dx)+e^{ax}\cos(ay+dz)\right)e^{-\nu d^2 t}\\
u_3 &= -a\left(e^{az}\sin(ax+dy)+e^{ay}\cos(az+dx)\right)e^{-\nu d^2 t}\\
p &= -\frac{a^2}{2}\big(e^{2ax}+e^{2ay}+e^{2az}+2\sin(ax+dy)\cos(az+dx)e^{a(y+z)}\\
&\quad +2\sin(ay+dz)\cos(ax+dy)e^{a(z+x)}\\
&\quad +2\sin(az+dx)\cos(ay+dz)e^{a(x+y)}\big)e^{-2\nu d^2 t}.
\end{aligned}
$$

In our experiments we set $t=0.1$, $a=\pi/4$, $d=\pi/2$ and vary ν

Numerical experiment: Ethier-Steinman vortex

analog of Taylor vortex problem

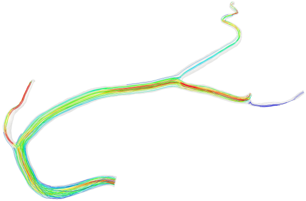
Table : The performance of the ILU($\tau = 0.02$),ILU($\tau_1 = 0.02, \tau_2 = 7\tau_1^2$) preconditioners for the Ethier–Steinman flow, d.o.f.$\sim$ 520 000

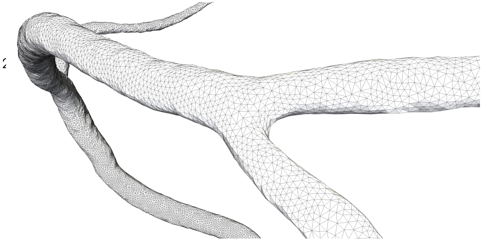
ν:	1			10^{-2}			10^{-3}		
α:	1	10	100	1	10	100	1	10	100
					fill$_{LU}$				
ILU(τ)	1.22	1.22	1.17	1.48	1.27	0.97	n/c	6.53	1.89
ILU(τ_1, τ_2)	1.20	1.20	1.16	1.53	1.30	0.96	9.28	5.33	1.80
					#it(BCGstab)				
ILU(τ)	337	296	50	27	22	31	n/c	83	44
ILU(τ_1, τ_2)	201	158	36	22	24	26	47	38	31
					$T_{\rm CPU}$(BCGstab)				
ILU(τ)	170.1	149.7	31.0	20.8	16.7	16.8	n/c	234.1	38.4
ILU(τ_1, τ_2)	93.8	96.9	35.7	51.2	42.8	35.1	2174	735	89.5

for GMRES(30) the results are similar

The two-parameter ILU factorization leads to more efficient preconditioner in terms of memory usage (fill-in) and iteration counts, compared to the standard ILU(τ).

Numerical experiment: Flow in right coronary artery

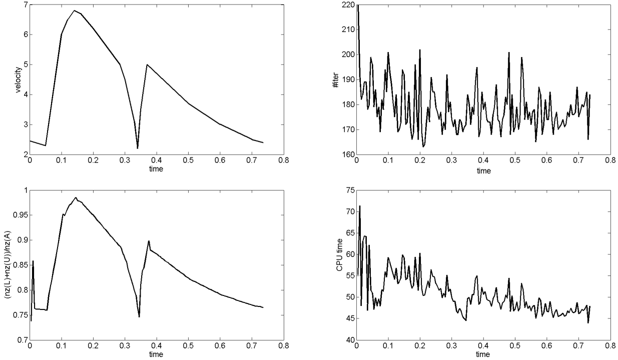




A real patient CT angiography, tetrahedral mesh (120 000 tets), FE systems with 620 000 d.o.f. generated by ani3D package

thanks to Alexander Danilov, Tatiana Dobroserdova

Numerical experiment: Flow in right coronary artery



$\nu = 0.04\,\mathrm{cm}^2/\mathrm{s}$, $\rho = 1\,\mathrm{g/cm}$, **one cardiac cycle period** 0.735 **s, inlet velocity waveform from clinical measurements, Dirichlet b.c. with Poiseuille profile (inlet), Neumann b.c. (outlet)**

$$\Delta t = 0.005, \quad \tau_1 = 0.03, \quad \tau_2 = 7\tau_1^2$$

- **fill-in rate for the LU-factors repeats the waveform of the inflow velocity**
- **due to such adaptivity, the variations of the iteration numbers and computational times per linear solve are rather modest**

SUPG-stabilized FEM

Weak formulation:

$$\mathcal{L}(\mathbf{u}, p; \mathbf{v}, q) \;=\; (\mathbf{f}, \mathbf{v})_* + (g, q) \qquad \forall\, \mathbf{v} \in \mathbf{V},\; q \in L^2(\Omega)\,,$$

$$\mathcal{L}(\mathbf{u}, p; \mathbf{v}, q) \;:=\; \alpha(\mathbf{u}, \mathbf{v}) + \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + ((\mathbf{w}\cdot\nabla)\,\mathbf{u}, \mathbf{v}) - (p, \mathrm{div}\,\mathbf{v}) + (q, \mathrm{div}\,\mathbf{u})$$

Given conforming FE spaces $\mathbb{V}_h \subset \mathbf{V}$ and $\mathbb{Q}_h \subset L^2(\Omega)$, the SUPG-FE discretization is:

Find $\{\mathbf{u}_h, p_h\} \in \mathbb{V}_h \times \mathbb{Q}_h$ such that

$$\mathcal{L}(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) + \sum_{\tau\in T_h} \sigma_\tau(\alpha\mathbf{u}_h - \nu\Delta\mathbf{u}_h + \mathbf{w}\cdot\nabla\mathbf{u}_h + \nabla p_h - \mathbf{f}, \mathbf{w}\cdot\nabla\mathbf{v}_h)_\tau =$$

$$= (\mathbf{f}, \mathbf{v}_h)_* + (g, q_h) \qquad \forall\, \mathbf{v}_h \in \mathbb{V}_h,\; q_h \in \mathbb{Q}_h$$

$$\begin{pmatrix} A & \widetilde{B}^T \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

SUPG parameter σ_τ

- **Mesh Reynolds number** $\mathrm{Re}_\tau := \|\mathbf{w}\|_{L_\infty(\tau)} h_{\mathbf{w}}/\nu$

$$\sigma_\tau = \begin{cases} \bar\sigma \frac{h_{\mathbf{w}}}{2\|\mathbf{w}\|_{L_\infty(\tau)}} \left(1 - \frac{1}{\mathrm{Re}_\tau}\right), & \text{if } \mathrm{Re}_\tau > 1, \\ 0, & \text{if } \mathrm{Re}_\tau \le 1, \end{cases} \quad \text{with } 0 \le \bar\sigma < 1.$$

- $\widetilde{S} := BA^{-1}\widetilde{B}^T + C > 0$ **and the stability of its LU factorization is guaranteed if the perturbation** $E = \widetilde{B} - B$ **is not too large:**

$$\kappa := (1 + C_A)\epsilon_E c_S^{-\frac{1}{2}} < 1,$$

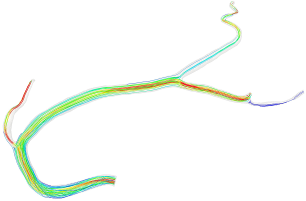
where $\epsilon_E := \|A_S^{-\frac{1}{2}} E^T\|$, $c_S := \frac{1}{2}\lambda_{\min}(S + S^T)$.

- $\kappa < 1$ **is provided by additional restrictions**

$$\sigma_\tau \le \frac{h_\tau^2}{2\nu \bar{C}_{\mathrm{in}}^2}\left(1 + \frac{\alpha h_\tau^2}{\nu C_{\mathrm{in}}^2}\right) \quad \text{and} \quad \sigma_\tau \le \frac{h_\tau}{4\|\mathbf{w}\|_{L^\infty(\tau)} C_{\mathrm{in}}} \quad \forall\, \tau \in T_h$$

I.Konshin, M.Olshanskii, and Yu.Vassilevski. ***LU factorizations and ILU preconditioning for stabilized discretizations of incompressible Navier–Stokes equations*****. Submitted to Num.Lin.Alg.Appl.**

Numerical experiment: Flow in right coronary artery, different viscosities



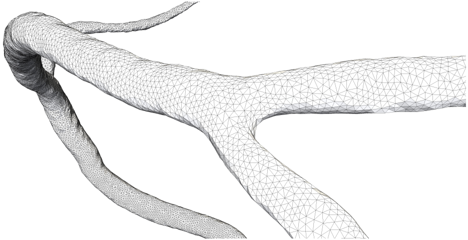


Table : The performance of ILU($\tau_1, \tau_2 = 7\tau_1^2$). The table shows values of τ_1 which allow to run the simulation for the complete cardiac cycle for different parameters $\bar{\sigma}$. '$\star$' means finite element solution blow-up, '–' means intractable systems for any possible τ_1.

ν, cm^2/s \ $\bar{\sigma}$	**0**	**1/96**	**1/48**	**1/24**	**1/12**	**1/6**	**1/3**
0.040	**0.03**	**0.03**	**0.03**	**0.03**	**0.03**	**0.03**	**0.003**
0.025	$\star$	**0.03**	**0.03**	**0.03**	**0.03**	**0.003**	**–**

Conclusions

- **We proved numerical stability bounds for the exact LU factorization of non-symmetric saddle-point matrices**
- **We estimated the dependence of the constants in these bounds on the flow problem and SUPG-stabilization parameters**
- **For considered problems, natural u-p ordering of unknowns and matrix two-side scaling is sufficient for numerically stable factorizations**
- **Two-parameter threshold ILU(τ_1, τ_2) likely covers most laminar flows**
- **Quasi-optimal τ-s can be chosen and successfully used for a wide range of flow, discretization and stabilization parameters**
- **The two-parameter ILU preconditioner was applied successfully in simulation of a blood flow in a right coronary artery reconstructed from a real patient coronary CT angiography**