Mathematical model of a fluid flow in collapsible tubes

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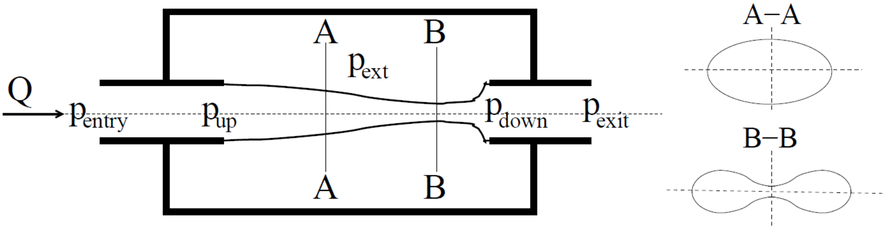
Starling resistor



M. Heil, O. E. Jensen (2003). *Flows in deformable tubes and channels: Theoretical models and biological applications*. In: P. W. Carpenter, T. J. Pedley (eds.). Flow Past Highly Compliant Boundaries and in Collapsible Tubes.

One-dimensional model

Averaging over cross-section:

$$\frac{\partial A}{\partial t} + \frac{\partial (Au)}{\partial x} = 0,$$

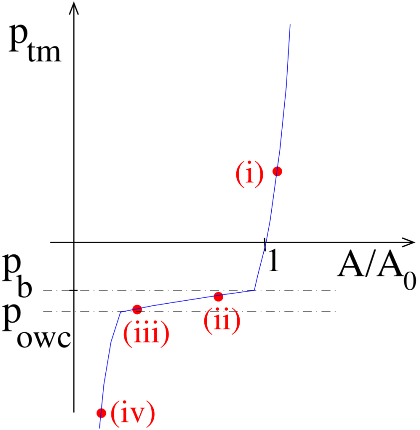
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - Ru,$$

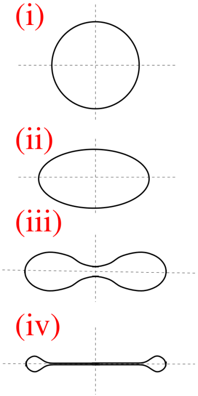
where A is the cross-sectional area, u is the mean velocity, p is the transmural pressure ($p - p_{\text{ext}}$).

The "equation-of-state" ("tube law"):

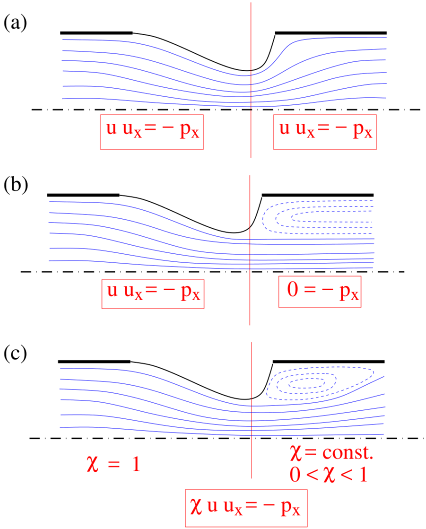
$$p = p(t, x, A(t, x)).$$

Tube law





Self-excited oscillations

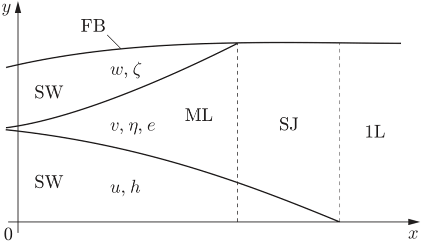


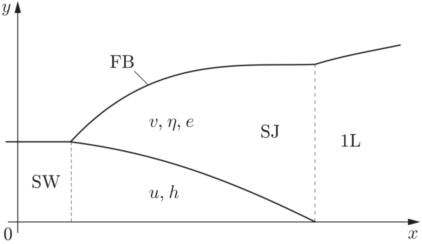
(a) Attached flow; no pressure loss. Choking.

(b) Ideal separation; no pressure recovery. Steady flow.

(c) An intermediate case: partial pressure recovery.
Self-excite oscillations.
[Cancelli, Pedley, 1985]

Multi-layer models





Water waves: mixing layer, near-surface turbulent layer, turbulent bore.

Gas dynamics: pseudo-shocks.

[Liapidevskii, Chesnokov, *et al.*]

Axisymmetric flow

Axisymmetric flow:

$$\begin{aligned} u_t + uu_x + vu_r + \rho^{-1} p_x &= 0, \\ v_t + uv_x + vv_r + \rho^{-1} p_r &= 0, \\ (ru)_x + (rv)_r &= 0. \end{aligned} \tag{1}$$

Long-wave approximation

Long-wave approximation ($\varepsilon^2 \equiv R_0^2/L_0^2 \ll 1$):

$$\begin{aligned} u_t + uu_x + vu_r + \rho^{-1} p_x &= 0, \\ p_r &= 0, \\ (ru)_x + (rv)_r &= 0. \end{aligned} \tag{2}$$

Pressure:

$$p = p(t, x).$$

Vorticity (up to $\mathcal{O}(\varepsilon^2)$):

$$\omega = -u_r.$$

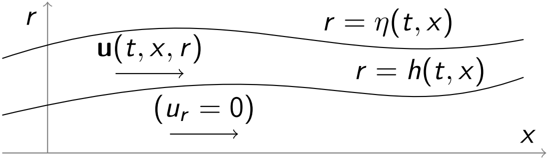
Two-layer flow

$r \in (0, h)$: potential core ($\omega = -u_r = 0$).
$r \in (h, \eta)$: turbulent boundary layer.
Areas:

$$s(t,x) = \pi h^2(t,x), \quad \bar{s}(t,x) = \pi\big(\eta^2(t,x) - h^2(t,x)\big).$$

Average velocity in the boundary layer and root-mean-square deviation

$$\bar{u}(t,x) = \frac{2\pi}{\bar{s}} \int_h^\eta u(t,x,r)\, r\, dr, \quad q^2(t,x) = \frac{2\pi}{\sigma} \int_h^\eta (u - \bar{u})^2\, r\, dr.$$



Governing equations

$$
\begin{aligned}
s_t + (su)_x &= -\sigma q, \\
\bar{s}_t + (\bar{s}\bar{u})_x &= \sigma q, \\
u_t + uu_x + \rho^{-1} p_x &= 0, \\
(su + \bar{s}\bar{u})_t + \big(su^2 + \bar{s}(\bar{u}^2 + q^2)\big)_x + \rho^{-1}(s + \bar{s}) p_x &= 0, \\
\Big(\frac{su^2 + \bar{s}(\bar{u}^2 + q^2)}{2}\Big)_t + \Big(\frac{su^3 + \bar{s}\bar{u}(\bar{u}^2 + 3q^2)}{2}\Big)_x &+ \\
+ \rho^{-1}(su + \bar{s}\bar{u}) p_x &= -\sigma\kappa q^3.
\end{aligned}
\tag{3}
$$

Parameters σ, κ characterize mixing and energy dissipation.

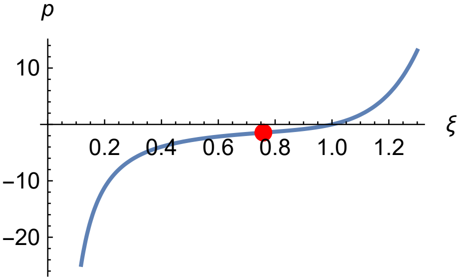
Tube law

Suppose that the tube law has the following form:

$$p = K(x)\bar{p}(S/S_0).$$

$K(x)$ characterizes the stiffness of the tube, $S = s + \bar{s}$, S_0 corresponds to zero transmural pressure.

In the following: $\bar{p}(\xi) = \xi^{10} - \xi^{-3/2}$.



Characteristics

Non-conservative form:

$$\begin{aligned}
&s_t + us_x + su_x = -\sigma q,\\
&\bar{s}_t + \bar{u}\bar{s}_x + \bar{s}\bar{u}_x = \sigma q,\\
&u_t + uu_x + \rho^{-1} p_x = 0,\\
&\bar{u}_t + \bar{u}\bar{u}_x + 2qq_x + \bar{s}^{-1} q^2 \bar{s}_x + \rho^{-1} p_x = 0,\\
&q_t + q\bar{u}_x + \bar{u} q_x = Q,
\end{aligned}$$

where $p_x = K\bar{p}'(S)(s_x + \bar{s}_x)$.

It can be shown that there exist at least three real characteristic velocities.

Numerical experiments

$$L = 30, \quad \sigma = 0.15, \quad \kappa = 6, \quad N = 500.$$

$$K = 1 + 3/(1 + \alpha(x - 0.9L)^8)^2.$$

Case 1:

$$S = 1.2, \quad u_0 = 25.$$

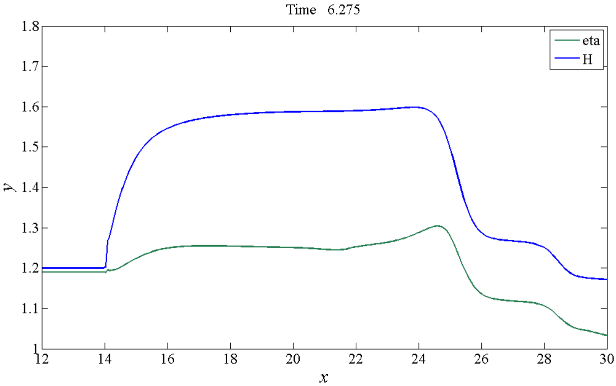
Case 2:

$$S = 0.8, \quad u_0 = 4.$$

The system was solved with Godunov's scheme.

Pseudo-shock

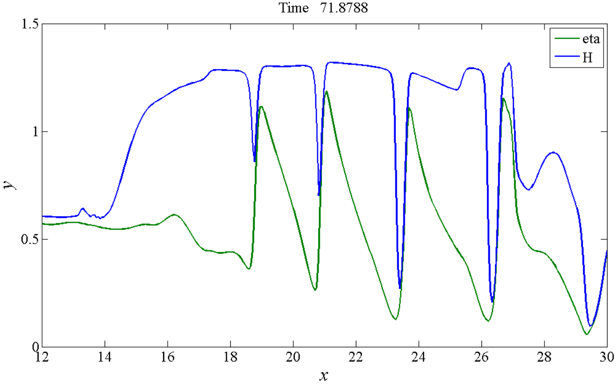
$$S = 1.2, \quad u_0 = 25$$



Smooth pseudo-shock.

Oscillatory regime

$$S = 0.8, \quad u_0 = 4$$



Quasi-periodic waves of large amplitude.

Summary

The mathematical model proposed can describe stationary and non-stationary (self-excited oscillations) regimes.

The model is based on a two-layer flow formulation which is used for description of the pseudo-shocks or smooth transition from supercritical to subcritical flows.