Numerical validation of an asymptotic model for two viscoelastic layers

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Joint work with:

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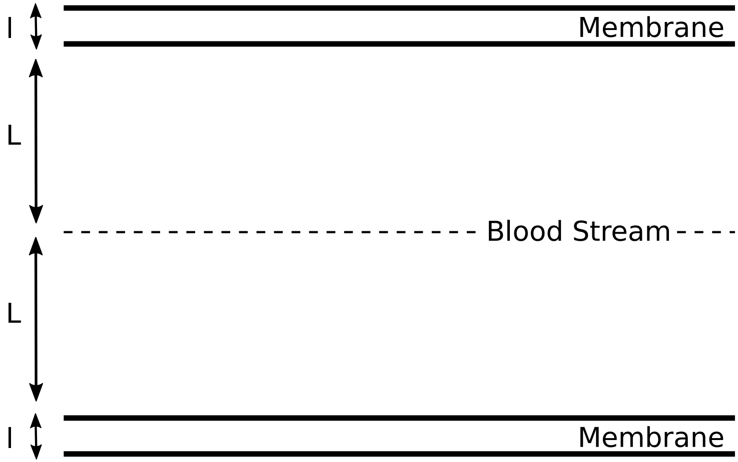
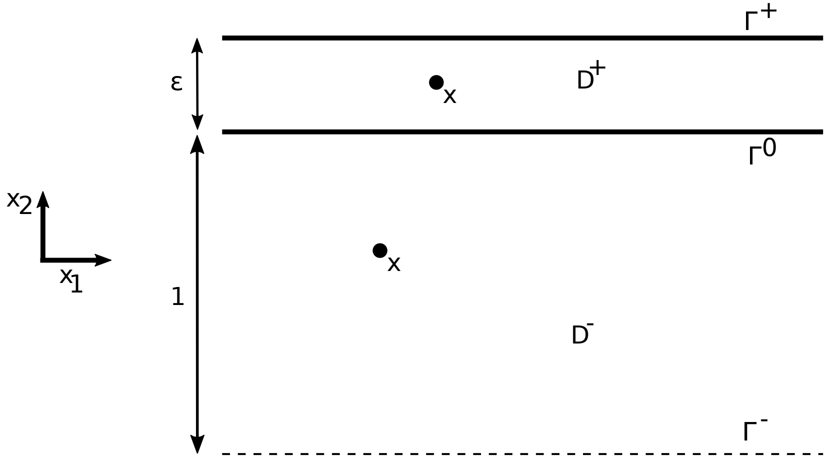
Flow in a capillary



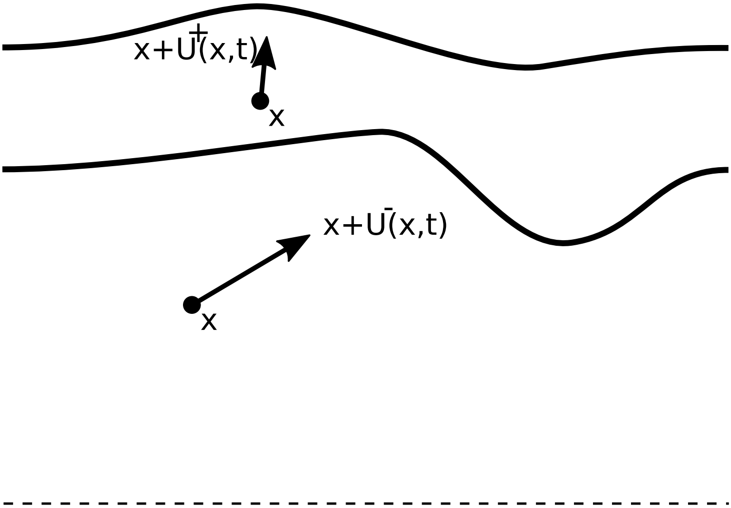
$$\frac{l}{L} = \varepsilon \ll 1$$

Adimensionalized setting



Periodic boundary conditions along x_1

Displacement



Visco-elasticity equations

The two layers are submitted to linear viscoelasticity.

$$\begin{cases} \rho^+ \frac{\partial U^+}{\partial t^2} = F^+ + \sum_{1\leq i,j\leq 2} \frac{\partial}{\partial x_i}\left(A^+_{ij}\frac{\partial U^+}{\partial x_j} + \frac{\partial}{\partial t} B^+_{ij}\frac{\partial U^+}{\partial x_j}\right) & \text{in } D^+ \\ \rho^- \frac{\partial U^-}{\partial t^2} = F^- + \sum_{1\leq i,j\leq 2} \frac{\partial}{\partial x_i}\left(A^-_{ij}\frac{\partial U^-}{\partial x_j} + \frac{\partial}{\partial t} B^-_{ij}\frac{\partial U^-}{\partial x_j}\right) & \text{in } D^- \\ U^+ = U^- & \text{on } D^0 \\ \sum_{1\leq i\leq 2}\left(A^+_{ij}\frac{\partial U}{\partial x_j} + \frac{\partial}{\partial t} B^+_{ij}\frac{\partial U}{\partial x_j}\right) = 0 & \text{on } \Gamma^+ \\ \left.\begin{array}{l} U_1 = 0 \\ \left(\sum_{1\leq i\leq 2}\left(A^-_{ij}\frac{\partial U}{\partial x_j} + \frac{\partial}{\partial t} B^-_{ij}\frac{\partial U}{\partial x_j}\right)\right)_2 = 0 \end{array}\right\} & \text{on } \Gamma^- \end{cases}$$

with

- $D^+ = \{0 < x_2 < \varepsilon\}$
- $D^- = \{-1 < x_2 < 0\}$
- $\Gamma^+ = \{x_2 = \varepsilon\}$
- $\Gamma^0 = \{x_2 = 0\}$
- $\Gamma^- = \{x_2 = -1\}$

We assume that the coefficients in the upper layer are under the following form:

- $\rho^+(x_1, x_2, t) = \overline{\rho^+}(\frac{x_1}{\varepsilon})$
- $\lambda^+(x_1, x_2, t) = \frac{1}{\varepsilon^3}\overline{\lambda^+}(\frac{x_1}{\varepsilon})$
- $\mu^+(x_1, x_2, t) = \frac{1}{\varepsilon^3}\overline{\mu^+}(\frac{x_1}{\varepsilon})$
- $F^+(x_1, x_2, t) = \frac{1}{\varepsilon}\overline{F^+}(\frac{x_1}{\varepsilon}, x_2, t)$

In the lower layer we assume that

- $\rho^-(x_1, x_2, t) = \overline{\rho^-}(x_1)$
- $\lambda^-(x_1, x_2, t) = \overline{\lambda^-}(x_1)$
- $\mu^-(x_1, x_2, t) = \overline{\mu^-}(x_1)$
- $F^-(x_1, x_2, t) = \overline{F^-}(x_1, x_2, t)$

Asymptotic model

Panasenko & Elbert derived an asymptotic model for this system.
U^- can be approximated by $\mathbf{V}$ which satisfies the following system:

$$\begin{cases} \rho^- \frac{\partial \mathbf{V}}{\partial t^2} = F^- + \sum_{1\leq i,j\leq 2} \frac{\partial}{\partial x_i}\left(A_{ij}^- \frac{\partial \mathbf{V}}{\partial x_j} + \frac{\partial}{\partial t} B_{ij}^- \frac{\partial \mathbf{V}}{\partial x_j}\right) & \text{in } D^- \\ \frac{\partial \mathbf{V}_1}{\partial x_2} = 0 & \text{in } \Gamma^0 \\ \int_0^L \left(A_{22}^- \frac{\partial \mathbf{V}}{\partial x_2} + \frac{\partial}{\partial t} B_{22}^- \frac{\partial \mathbf{V}}{\partial x_2}\right)_1 \mathrm{d}x_1 = \int_0^L \mathbf{f}_1^+ \mathrm{d}x_1 & \text{in } \Gamma^0 \\ \left(\sum_{j=1}^{2} A_{2j}^- \frac{\partial \mathbf{V}}{\partial x_j} + \frac{\partial}{\partial t} B_{2j}^- \frac{\partial \mathbf{V}}{\partial x_j}\right)_2 = \mathbf{f}_2^+ - \frac{\Delta}{h_{0,2}^{(1)}} \frac{\partial^4 \mathbf{V}_2}{\partial x_2^4} & \text{in } \Gamma^0 \\ \mathbf{V} = 0 & \text{in } \Gamma^- \end{cases}$$

$$\mathbf{f}^+(t, x_2) = \int_0^\varepsilon F^+(t, x_1, x_2)\mathrm{d}x_2$$

$$h_{0,2}^{(1)} = \langle \frac{-\overline{E^+}}{1 - (\overline{\nu^+})^2} \rangle$$

$$\Delta = h_{0,2}^{(1)} \langle -\mathcal{A}(\frac{E}{1-\hat{\nu}^2}(\frac{1}{2} - \xi_2)) \rangle - \langle \mathcal{A}(\frac{-E}{1-\hat{\nu}^2}) \rangle \langle -\frac{\overline{E^+}}{1-(\overline{\nu^+})^2}(\frac{1}{2} - \xi_2) \rangle$$

$$\langle f \rangle = \int_0^1 f, \quad \mathcal{A}(f)(\xi_2) = \xi_2 \langle f \rangle - \int_0^{\xi_2} f$$

with $\overline{E^+}$ and ν^+ such that:

$$\overline{\lambda^+} = \frac{\overline{E^+}}{(1 + \overline{\nu^+})(1 - 2\overline{\nu^+})}, \quad \overline{\nu^+} = \frac{\overline{E^+}}{2(1 + \overline{\nu^+})}$$

Weak form : bilinear forms

Let us introduce the following bilinear forms:

$$\mathcal{I}_{\rho^+}(U,V) = \int_{D^+} \rho^+ \langle U, V\rangle$$
$$\mathcal{I}_{A^+}(U,V) = \int_{D^+} \sum_{1\leq i,j\leq 2} \langle \frac{\partial U}{\partial x_i}, A^+_{ij} \frac{\partial V}{\partial x_i} \rangle$$
$$\mathcal{I}_{B^+}(U,V) = \int_{D^+} \sum_{1\leq i,j\leq 2} \langle \frac{\partial U}{\partial x_i}, B^+_{ij} \frac{\partial V}{\partial x_i} \rangle$$

$$\mathcal{I}_{\rho^-}(U,V) = \int_{D^-} \rho^- \langle U, V\rangle$$
$$\mathcal{I}_{A^-}(U,V) = \int_{D^-} \sum_{1\leq i,j\leq 2} \langle \frac{\partial U}{\partial x_i}, A^-_{ij} \frac{\partial V}{\partial x_i} \rangle$$
$$\mathcal{I}_{B^-}(U,V) = \int_{D^-} \sum_{1\leq i,j\leq 2} \langle \frac{\partial U}{\partial x_i}, B^-_{ij} \frac{\partial V}{\partial x_i} \rangle$$

$$\mathcal{I}_0(U,V) = \frac{\Delta}{h^{(1)}_{0,2}} \int_{\Gamma^0} \langle \frac{\partial U}{\partial x_1^2}, \frac{\partial V}{\partial x_1^2} \rangle$$

Weak form of the full problem

$$K^0 = L^2(D^+)^2 \times L^2(D^-)^2 \quad K^1 = H^1(D^+)^2 \times H^1(D^-)^2, \Omega^\pm = D^\pm \times (0, T)$$

$$K_T^0 = L^2(\Omega^+)^2 \times L^2(\Omega^-)^2, K_T^1 = H^1_{per,1}(\Omega^+)^2 \times H^1_{per,1}(\Omega^-)^2.$$

Let $\mathbb{V} = \{(V^+, V^-) \in K^1 | V^+ = V^- \text{ on } \Gamma^0, \quad V_2^- = 0 \text{ on } \Gamma^-\}$.
We look for a solution $(U^+, U^-) \in K_T^1$ such that $(U^+, U^-)' \in K_T^1$, $(U^+, U^-)'' \in K_T^0$, for $t \in [0, T]$ we have $u(., t) \in \mathbb{V}$ and for $(V^-, V^+) \in \mathbb{V}$:

$$\begin{aligned} &\mathcal{I}_{\rho^+}(U^+, (V^+)'') + \mathcal{I}_{B^+}(U^+, (V^+)') + \mathcal{I}_{A^+}(U^+, V^+) \\ &\quad + \mathcal{I}_{\rho^-}(U^-, (V^-)'') + \mathcal{I}_{B^-}(U^-, (V^-)') + \mathcal{I}_{A^-}(U^-, V^-) \\ &\qquad = (\int_{D^+} F^+ V^+) + (\int_{D^-} F^- V^-) \end{aligned}$$

Weak form of the asymptotic problem

$$K^0 = L^2(D^+)^2 \times L^2(D^-)^2 \quad K^1 = H^1(D^+)^2 \times H^1_{per,1}(D^-)^2, \Omega^\pm = D^\pm \times (0,T)$$

$$\overline{K^0_T} = L^2(\Omega^-)^2, \overline{K^1_T} = H^1_{per,1}(\Omega^-)^2.$$

Let $\mathbb{V} = \{V^- \in K^1 | U^+ = U^- \text{ on } \Gamma^0, \quad U^-_2 = 0 \text{ on } \Gamma^-\}$.
We look for a solution $(U^+, U^-) \in K^1_T$ such that $(U^+, U^-)' \in K^1_T$, $(U^+, U^-)'' \in K^0_T$, for $t \in [0,T]$ we have $u(.,t) \in \mathbb{V}$ and for $(V^-, V^+) \in \mathbb{V}$:

$$\mathcal{I}_{\rho^-}(U^-, (V^-)'') + \mathcal{I}_{B^-}(U^-, (V^-)') + \mathcal{I}_{A^-}(U^-, V^-) + \mathcal{I}_0(U^-, V^-) = (\int_{\Gamma^0} \mathbf{f}^+ V^+) + (\int_{D^-} f^- V^-)$$

Finite elements

To compare numerically these two numerical scheme, we used:

- Crank-Nicholson for time discretization (Implicit, constant).
- $\mathbb{P}^3$-type elements for the space discretization.

Why $\mathbb{P}^3$ and not $\mathbb{P}^1, \mathbb{P}^2$?

- More accurate.
- $\mathbb{P}^3(\mathcal{T}_h^-) \cap C^1(\Gamma^0) \subset H^1(D^-) \cap H^2(\Gamma^0)$ where $\mathcal{T}_h$ is a triangulation of D^-.
- Implemented in usual Finite Elements Solvers.

Drawbacks:

- Less sparse matrices (but still sparser than with Argyris elements).

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Complete system

Let k be the time step and $\mathcal{T}_h$ be the mesh. $\mathbb{V}_h = \{(U_h^+, U_h^-) \in \mathbb{P}^3(\mathcal{T}_h^+) \times \mathbb{P}^3(\mathcal{T}_h^-) | U_h^\pm \; x_1 - \text{periodic}, U_h^+ = U_h^- \text{ on } \Gamma^+, (U_h^-)_2 = 0 \text{ on } \Gamma^-\}$.
We look for $(U_h^{n+}, U_h^{n-})_n \in \mathbb{V}_h^{\mathbb{N}}$ such that:

$$\mathcal{I}_{\rho^+}(\frac{U_h^{(n+1)+} - 2U_h^{n+} + U_h^{(n-1)+}}{k^2}, V_h^+) + \mathcal{I}_{B^+}(\frac{U_h^{(n+1)+} - U_h^{(n-1)+}}{2k}, V_h^+)$$

$$+ \mathcal{I}_{A^+}(\frac{U_h^{(n+1)+} + U_h^{(n-1)+}}{2}, V_h^+)$$

$$\mathcal{I}_{\rho^-}(\frac{U_h^{(n+1)-} - 2U_h^{n-} + U_h^{(n-1)-}}{k^2}, V_h^-) + \mathcal{I}_{B^-}(\frac{U_h^{(n+1)-} - U_h^{(n-1)-}}{2k}, V_h^-)$$

$$+ \mathcal{I}_{A^-}(\frac{U_h^{(n+1)-} + U_h^{(n-1)-}}{2}, V_h^-)$$

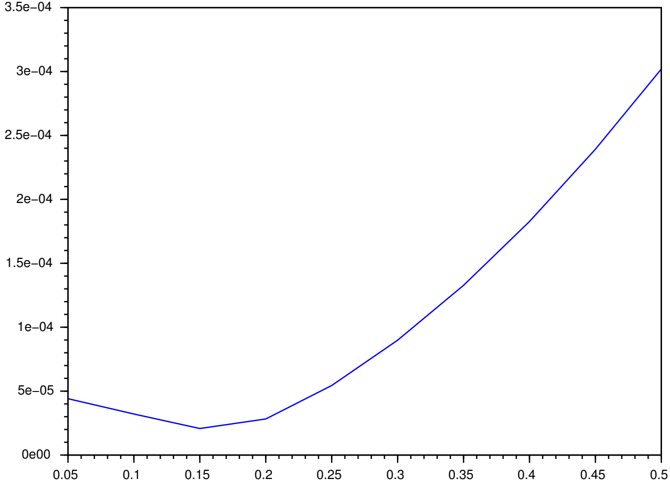
$$= (\int_{D^+} F^+ V_h^+) + (\int_{D^-} F^- V_h^-)$$

Asymptotic system

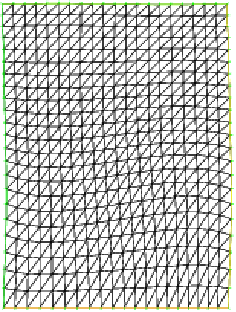
Let k be the time step and $\mathcal{T}_h$ be the mesh. $\mathbb{V}_h = \{U_h^- \in P^3(\mathcal{T}_h^-)^2 | (U_h)_1 = \text{cte}, \quad (U_h)_2 \in H^2_{per}(\Gamma^0), \quad (U_h^-)_2 = 0 \text{ on } \Gamma^-\}$.
We look for solutions $(U_h^{n-})_n \in V_h^{\mathbb{N}}$ such that, for all V_h

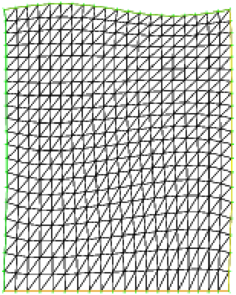
$$\mathcal{I}_{\rho^-}(\frac{U_h^{(n+1)-} - 2U_h^{n-} + U_h^{(n-1)-}}{k^2}, V_h^-) + \mathcal{I}_{B^-}(\frac{U_h^{(n+1)-} - U_h^{(n-1)-}}{2k}, V_h^-)$$
$$+ \mathcal{I}_{A^-}(\frac{U_h^{(n+1)-} + U_h^{(n-1)-}}{2}, V_h^-) + \mathcal{I}_0(\frac{U_h^{(n+1)-} + U_h^{(n-1)-}}{2}, V_h^-)$$
$$= (\int_{\Gamma^0} \mathbf{f}^+ V_h^-) + (\int_{D^-} F^- V_h^-)$$

L^2 distance between the results obtained with the full system and the asymptotic model in the permanent regime



Simulation with the full model on the left and the asymptotic model on the right





Conclusion

Advantages:

- Flat triangles avoided and the complexity of the membrane taken in account
- Unstructured mesh can be used

Drawbacks:

- Need of a high-order method
- Bad condition number due to the fourth-order derivative (the full system is also also badly conditionned when $\varepsilon \to 0$)

Possible solutions:

- Discontinuous Galerkin methods
- Multi-precision arithmetics

Благодарю вас за внимание