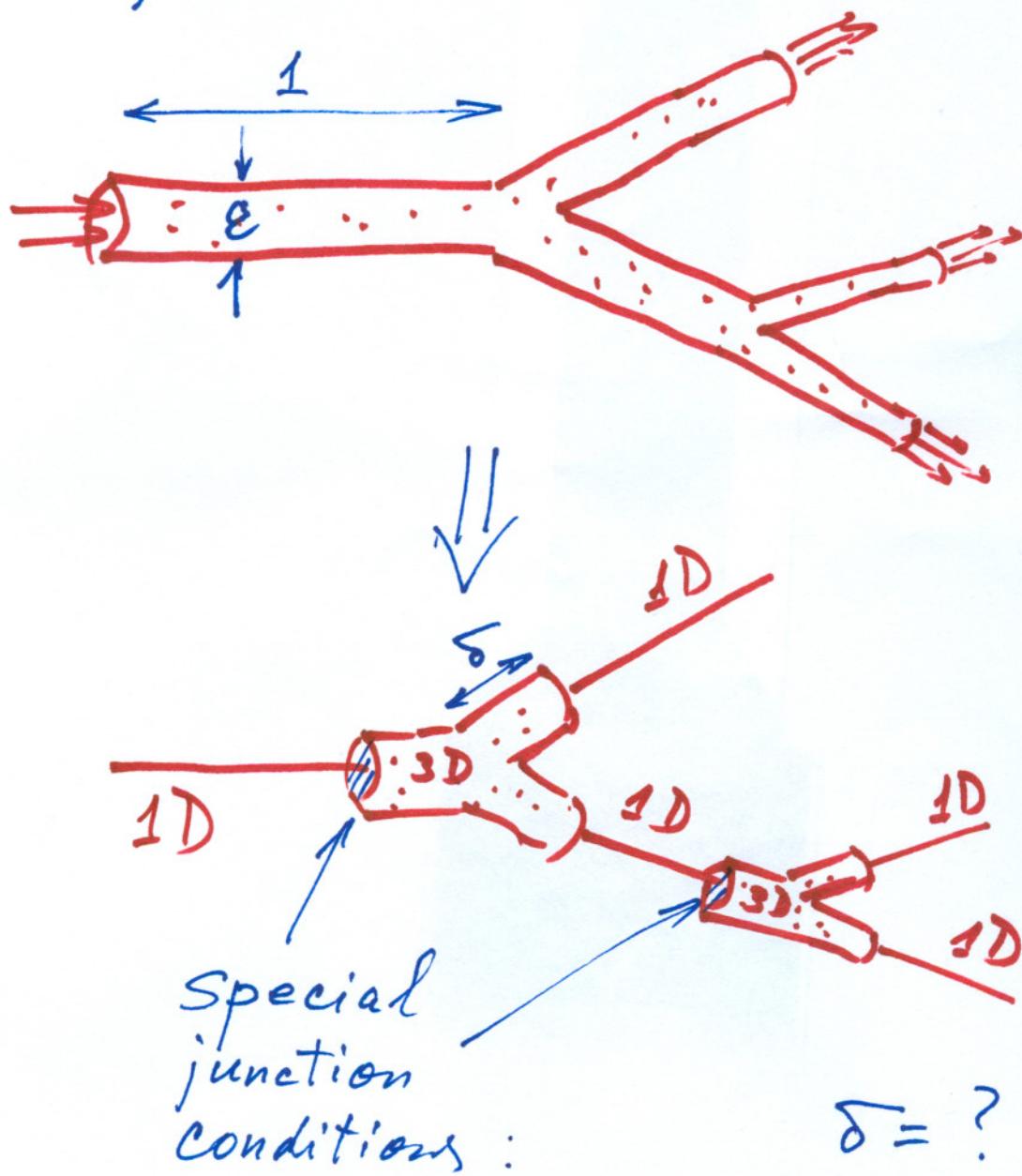


## Outline .

- Motivation: dimension reduction in big systems of vessels
- Definition of thin structures
- Non-steady time-periodic Navier Stokes equations in thin structures
  - setting
  - main results
- Partial dimension reduction
- Case of the tree-like graph: complete decomposition and parallelisation.

Motivation: partial dimension reduction:

- blood circulation
- hydraulic installations



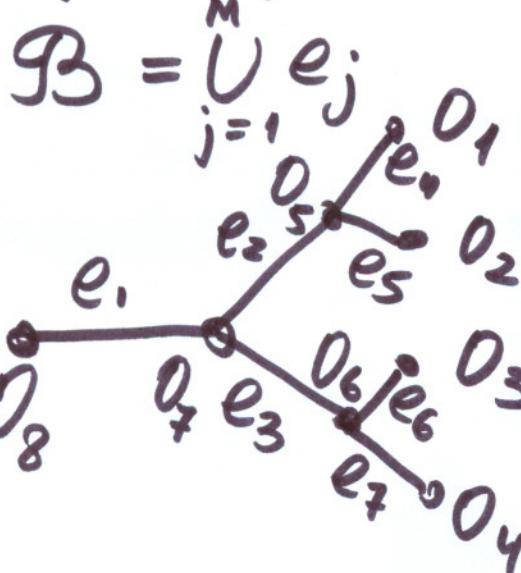
$\forall N$  error  $O(\epsilon^N)$

# Definition of a thin structure

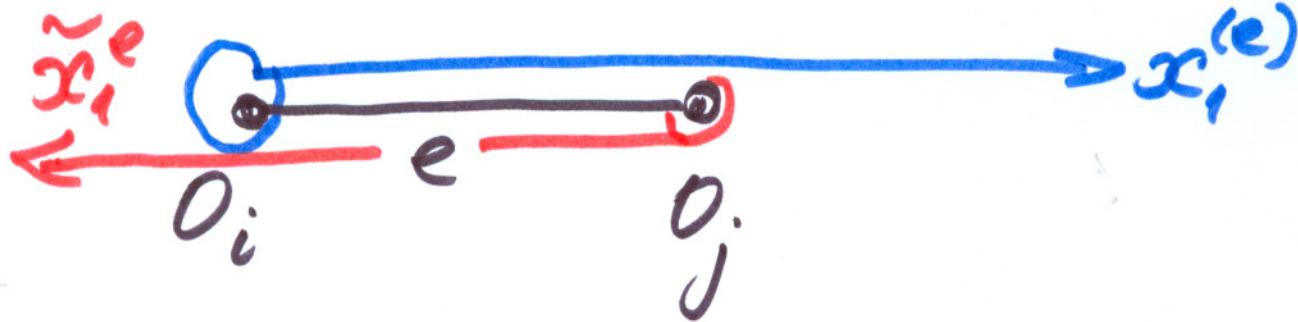
1. Graph.  $O_1, \dots, O_N \in \mathbb{R}^n$ ,  $n \in \{2, 3\}$

$e_1, \dots, e_M$  closed segments

$e_j = \overline{O_{ij} O_{kj}}$ ,  $e_j \cap e_{j_2} \subset \{O_1, \dots, O_N\}$ .



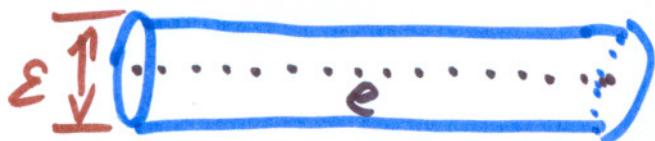
2. Local coordinate systems



### 3. Thin structure

$e_j = e$

$\sigmā^j = \sigmā^{(e)}$



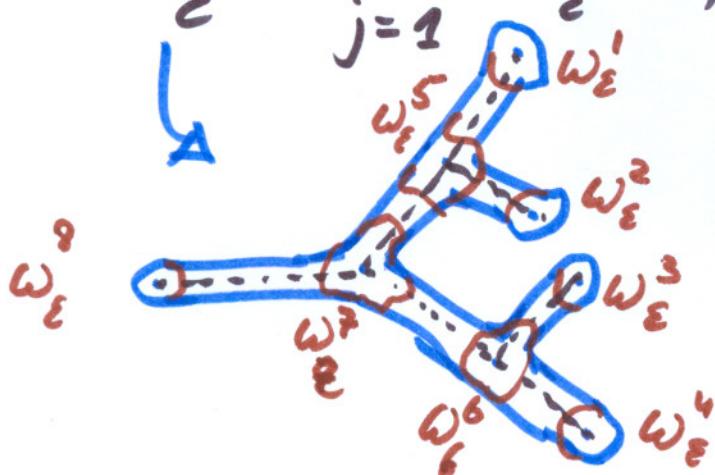
$$B_\epsilon^{(e)} = \{x^{(e)} \mid x_1^{(e)} \in (0, |e|), \frac{x^{(e)}/\epsilon}{\epsilon} \in \sigma^{(e)}\}$$



$\omega^i$  bounded domains in  $\mathbb{R}^n$   
 $i = 1, \dots, N$

$\omega_\epsilon^i = \{x \in \mathbb{R}^n \mid \frac{x - O_i}{\epsilon} \in \omega^i\}$

$$B_\epsilon = \left( \bigcup_{j=1}^M B_\epsilon^{(e_j)} \right) \cup \left( \bigcup_{i=1}^N \omega_\epsilon^i \right)$$



$$\partial B_\epsilon \in C^2$$

$$\underline{\underline{\omega_\varepsilon^i}} \cap \gamma_\varepsilon^i = \partial\omega_\varepsilon^i \cap \partial B_\varepsilon,$$

$i = N_1 + 1, \dots, N$ :  $O_i$  vertices

#### 4. Non-steady Navier-Stokes eqs

$$\left| \begin{array}{l} \frac{\partial \vec{u}_\varepsilon}{\partial t} - \gamma \Delta \vec{u}_\varepsilon + (\vec{u}_\varepsilon, \nabla) \vec{u}_\varepsilon + \nabla p_\varepsilon = 0 \\ \operatorname{div} \vec{u}_\varepsilon = 0, \quad x \in B_\varepsilon \\ \vec{u}_\varepsilon = \vec{g}_i \left( \frac{x - O_i}{\varepsilon}, t \right), \quad x \in \gamma_\varepsilon^i \\ \vec{u}_\varepsilon = 0, \quad x \in \partial B_\varepsilon \setminus \gamma_\varepsilon, \quad \gamma_\varepsilon = \bigcup_{i=N_1+1}^N \gamma_\varepsilon^i \\ \vec{u}_\varepsilon \text{ is T-periodic in } t \\ \vec{g}_i \in C^{[\frac{3+1}{2}]+1}([0, T]; W_2^2), \\ \operatorname{div}_x \vec{g}_i = 0, \quad \sum_{i=N_1+1}^N \int_{\gamma_\varepsilon^i} \vec{g}_i \cdot \vec{n} = 0, \\ \gamma > 0. \end{array} \right.$$

Definition of a solution:

Let  $g$  be an extension of  $g_i$  to  $B_\varepsilon$

Such that  $\forall t \in [0, T]$

$$\begin{cases} \|g\|_{L^2} + \|g_t\|_{L^2} + \|g_{tt}\|_{L^2} \leq C\varepsilon^{\frac{n-1}{2}} \\ \|\nabla g\|_{L^2} + \|\nabla g_t\|_{L^2} \leq C\varepsilon^{\frac{n-3}{2}} \\ \|\Delta g\|_{L^2} \leq C\varepsilon^{\frac{n-5}{2}}, \quad n=2,3, \text{ div } g=0 \end{cases}$$

(we prove that such an extension  $\exists$ ).

$(L^2 = L^2(B_\varepsilon))$ . Then  $u_\varepsilon = w + g$ ,  
 $\text{div } w = 0$ ,  $w \in L^2(0, T; H_0^1(B_\varepsilon)) \cap$   
 $\bigcap_{T-\text{per.}} L^\infty(0, T; H_0^1(B_\varepsilon))$ ,  $w_t \in L^2(0, T; L^2(B_\varepsilon))$

$$\int_{B_\varepsilon} (w_t \cdot \eta + \nu \nabla w \cdot \nabla \eta - ((w+g) \cdot \nabla) \eta \cdot w -$$

$$-(w \cdot \nabla) \eta \cdot g \, dx = -\nu \int_{B_\varepsilon} \nabla g \cdot \nabla \eta \, dx \quad (N-S)$$

$$-\int_{B_\varepsilon} ((g \cdot \nabla) g + g_t) \cdot \eta \, ds$$

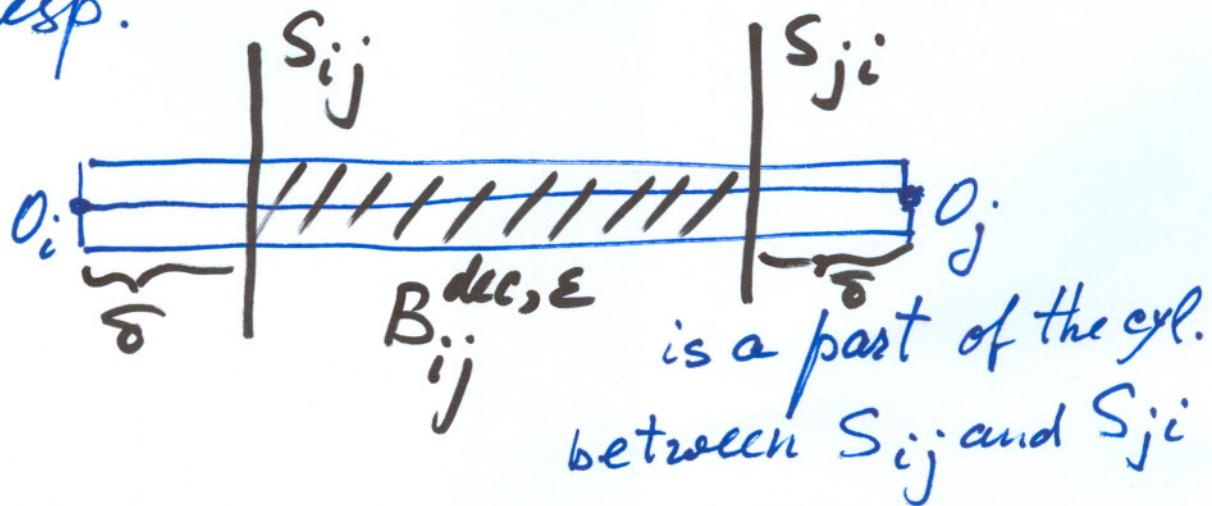
$\forall \eta \in H_0^1(B_\varepsilon)$ ,  $\text{div } \eta = 0$

Main results  
(G.P., K.Pileckas)

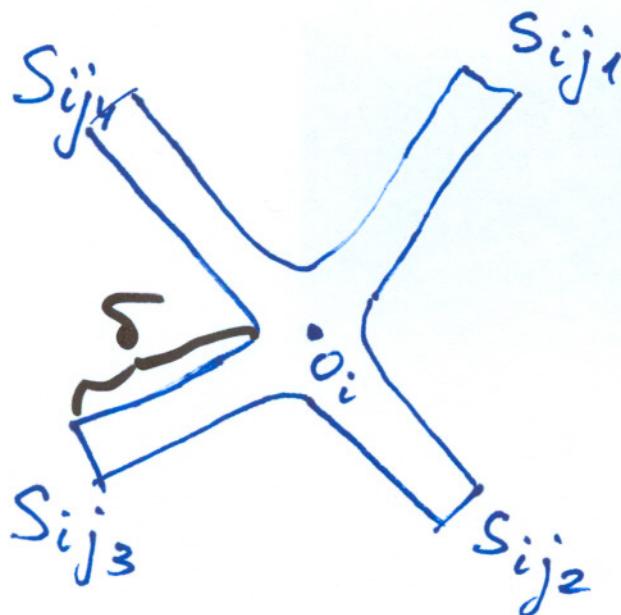
- 1)  $\exists!$  solution (for sufficiently small  $\epsilon$ ).
- 2) Complete asymptotic expansion of the solution is constructed and justified.
- 3) Hybrid dimension model, partially reducing dimension, is constructed and justified.

✓ edge  $e = \overline{O_i O_j}$  introduce 2 hyperplanes  $S_{ij}$  and  $S_{ji} \perp e$  at the distance  $\delta$  from  $O_i$  and  $O_j$ .

resp.



Remaining part of  $B_\epsilon$  consists of connected sets  $B_i^{\epsilon, \delta} \ni O_i$



Consider the subspace  $H_{\text{dec}}^J(B_\varepsilon)$  of the space  $\{H_0^1(B_\varepsilon) \mid \text{div} = 0\}$  such that  $\nabla B_{ij}^{\text{dec } \varepsilon}$ , vector function  $V$  has a form  $\left( \sum_{\lambda=0}^{[J/2]} a_{2\lambda} U_{2\lambda}\left(\frac{\tilde{x}'}{\varepsilon}\right) \right)$  in local

coordinates,  $x'$  are transversal local coordinates and  $U_{2\lambda}$  are solutions of problems:

$$\begin{cases} -\nu \Delta' U_0(y') = 1, y' \in \sigma \\ U_0|_{\partial\sigma} = 0 \end{cases}$$

$$\begin{cases} -\nu \Delta' U_{22}(y') = -U_{22-2}(y'), y' \in \sigma \\ U_{22}|_{\partial\sigma} = 0, 2 \geq 1. \end{cases}$$

Define partially decomposed problem:

Find  $w_d$ :  $\operatorname{div} w_d = 0$ ,  $w_d \in L^2(0,T; H_{\text{dec}}^J)$   
 $\wedge L^\infty(0,T; H_{\text{dec}}^J)$ ,  $w_d \in L^2(0,T; L^2(B_\varepsilon))$   
 satisfying (NS)  $\# \gamma \in H_{\text{dec}}^J$ .  
 $u_d := w_d + \gamma$ .

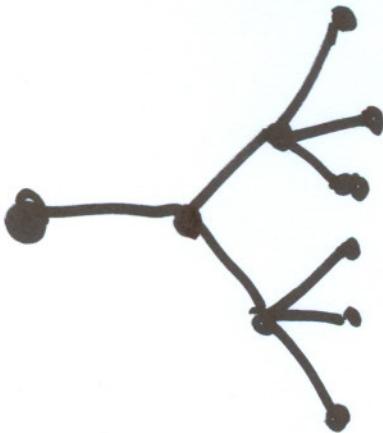
Theorem 1.  $\exists!$  solution (for sufficiently small  $\varepsilon$ ).

Theorem 2.  $\|u_\varepsilon - u_d\| = \|w - w_d\| =$   
 $= O(\varepsilon^{J-2})$ ,

where  $\|\varphi\| = \sup_{0 \leq t \leq T} \frac{\|\varphi\|_{L^2(B_\varepsilon)} + \|\nabla \varphi\|_{L^2(0,T; L^2)}}$

$$\boxed{\delta = \text{const.} \cdot J \cdot \varepsilon |\ln \varepsilon|}$$

# Tree - like graph.



In this case  
in every node we  
can write the  
flux balance :

$$\sum F_j = 0$$

while for the vertices we get

$$F_j = \int_{\partial E} S_j g_j \vec{n} ds$$

This system defines all fluxes  $F_j$ .

For any  $e$ , on  $B_{ij}^{dec \varepsilon}$  we can calculate  $u_d$  explicitly

(see G.P., K. Pileckas, Applic. Analysis, 2011) : in local variables the longitudinal component of the velocity

$$v_{long} = \sum_{k=0}^{[J/2]} \varepsilon^{2k} \tilde{V}_{2k} \left( \frac{x'}{\varepsilon}, t \right),$$

$$V_{2k}(y, t) = \beta_{2k}(t) U_0(y) + \frac{d S_{2k-2}}{dt} V_2(y) + \dots + \frac{d^k \beta_0(t)}{dt^k} U_{2k}(y), \quad (*)$$

where  $\beta_0(t) = \frac{1}{\alpha_0} F_j(t)$ ,

$$\beta_{2k} = -\alpha_0^{-1} \alpha_2 \frac{d \beta_{2k-2}}{dt} - \dots - \alpha_0^{-1} \alpha_{2k} \frac{d^k \beta_0}{dt^k},$$

$$\alpha_{2s} = \int_0^1 U_{2s}(y) dy \neq 0 \quad (s \geq 0)$$

(Moreover, the pressure  $p = -s(t)x_n + a(t)$ ,

$$-s(t) = \sum_{k=0}^{[J/2]} \varepsilon^{2k-2} \beta_{2k}(t).$$

So, in this case the problem is completely decomposed and on every  $B_i^{\varepsilon, \delta}$  we get a standard N-S problem with Dirichlet's conditions given by  $(*)$ . Computations may be parallelized!