

Profile computations for elliptic problems in domains with small holes

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4th International workshop on the multiscale modeling
and methods in biology and medicine

Moscow – october 29th 2014

Introduction

Multiscale representation for small defects

- ▶ Holes in concrete



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- ▶ Model problems : linear elasticity in a perturbed domain

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Introduction

Multiscale representation for small defects

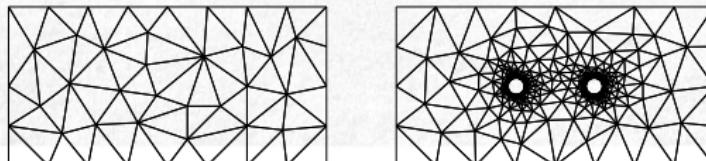
- ▶ Holes in concrete



- ▶ Model problems : linear elasticity in a perturbed domain



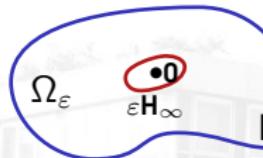
- ▶ Numerical issues



Introduction

Multiscale representation for small defects

- ▶ Example with the Laplace equation



$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \partial_n u_\varepsilon = 0 & \text{on } \Gamma_\varepsilon = \partial \Omega_\varepsilon. \end{cases}$$

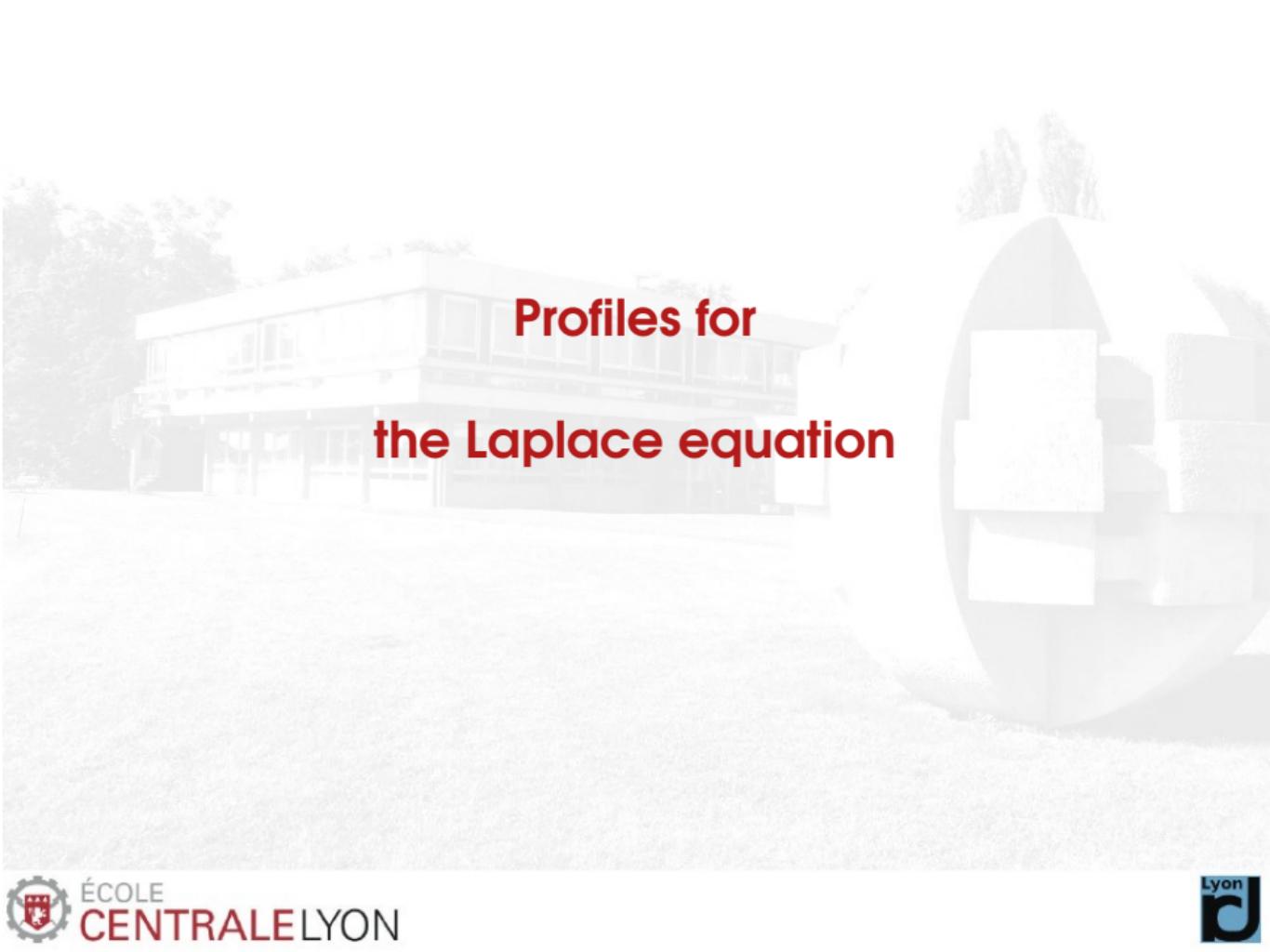
$$\begin{cases} -\Delta \mathbf{V} = 0 & \text{in } \mathbf{H}_\infty, \\ \partial_n \mathbf{V}(\mathbf{y}) = \mathbf{n} & \text{for } \mathbf{y} \in \partial \mathbf{H}_\infty, \\ \mathbf{V} \rightarrow 0 & \text{as } |\mathbf{y}| \rightarrow \infty. \end{cases}$$

Asymptotic expansion at order 1 : $u_\varepsilon(\mathbf{x}) \simeq u_0(\mathbf{x}) - \varepsilon \nabla u_0(\mathbf{0}) \cdot \mathbf{V}\left(\frac{\mathbf{x}}{\varepsilon}\right)$.

- ▶ Similar representations for other operators, boundary conditions, geometries, multiple inclusions... [e.g. Maz'ya-Nazarov-Plamenevskij]
- ▶ **Question :** accurate computation of profiles \mathbf{V} ?

Overview

- ▶ Toy problem : the Laplace equation
 - ▶ Profile properties
 - ▶ Artificial boundary conditions
- ▶ Profile computation for linear elasticity
 - ▶ Artificial boundary conditions
 - ▶ Non-coercive Ventcel problems
- ▶ Some other non-coercive Ventcel-type problems



Profiles for the Laplace equation

Laplace equation

Profiles properties

- ▶ Circular perturbation

H_∞



$$\left\{ \begin{array}{ll} -\Delta V = 0 & \text{in } H_\infty, \\ V(\mathbf{y}) = \mathbf{n} & \text{for } \mathbf{y} \in \partial H_\infty, \\ V \rightarrow 0 & \text{at infinity.} \end{array} \right.$$

Laplace equation

Profiles properties

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\mathbf{H}_∞



$$\begin{cases} -\Delta V_1 = 0 & \text{in } \mathbf{H}_\infty, \\ \partial_n V_1 = \cos \theta & \text{for } \mathbf{y} \in \partial \mathbf{H}_\infty, \\ V_1 \rightarrow 0 & \text{at infinity.} \end{cases}$$

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Explicit solution :

$$V_1(r, \theta) = \frac{\cos \theta}{r}.$$

Laplace equation

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Explicit solution :

$$V_1(r, \theta) = \frac{\cos \theta}{r}.$$

- ▶ General case

No explicit formula, only asymptotic expansion at infinity :

$$V(r, \theta) \underset{\infty}{=} \sum_{k \geq 1} \frac{a_k \cos k\theta + b_k \sin k\theta}{r^k}$$

Laplace equation

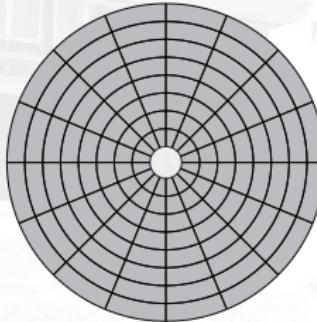
Artificial conditions

H_∞



$$\begin{cases} -\Delta \mathbf{V} = 0 & \text{in } H_\infty, \\ \mathbf{V}(\mathbf{y}) = \mathbf{n} & \text{for } \mathbf{y} \in \partial H_\infty, \\ \mathbf{V} \rightarrow 0 & \text{at infinity.} \end{cases}$$

- Truncated domain H_∞^R at radius R .



Laplace equation

Artificial conditions

\mathbf{H}_∞



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- ▶ Truncated domain \mathbf{H}_∞^R at radius R .
- ▶ Boundary conditions ("absorbing") for $|\mathbf{y}| = R$.

Laplace equation

Artificial conditions

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$$V(r, \theta) = \sum_{k=1}^{\infty} \frac{a_k \cos k\theta + b_k \sin k\theta}{r^k}$$

Laplace equation

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- ▶ Elementary construction :

Laplace equation

Artificial conditions

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Laplace equation

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Indeed $V_j \sim r^{-1}(a_1 \cos \theta + b_1 \sin \theta)$,

and $\partial_n V_j \sim -r^{-2}(a_1 \cos \theta + b_1 \sin \theta)$.

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Laplace equation

Artificial conditions



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[Engquist-Majda, Halpern-Rauch, ...]

Laplace equation

Artificial conditions

- ▶ Second order approximate problem :

$$\begin{cases} -\Delta V = 0 & \text{in } \mathbf{H}_\infty^R, \\ \partial_n V(\mathbf{y}) = n_j & \text{for } \mathbf{y} \in \partial\mathbf{H}_\infty, \\ V + \frac{3R}{2}\partial_n V - \frac{R^2}{2}\Delta_\tau V = 0 & \text{on } \Gamma_R = \{\mathbf{y} : |\mathbf{y}| = R\}. \end{cases}$$

Laplace equation

Artificial conditions

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- ▶ Variational formulation

$$\int_{\mathbf{H}_\infty^R} \nabla V \cdot \nabla W + \frac{2}{3R} \int_{\Gamma_R} V W + \frac{R}{3} \int_{\Gamma_R} \nabla_{\Gamma_R} V \cdot \nabla_{\Gamma_R} W = \int_{\mathbf{H}_\infty^R} f W + \int_{\partial \mathbf{H}_\infty} n_j W$$

coercive in

$$\mathfrak{W} = \left\{ W \in \mathbf{H}^1(\mathbf{H}_\infty^R) ; W|_{\Gamma_R} \in \mathbf{H}^1(\Gamma_R) \right\}.$$

Laplace equation

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- ▶ Well-posed problem !

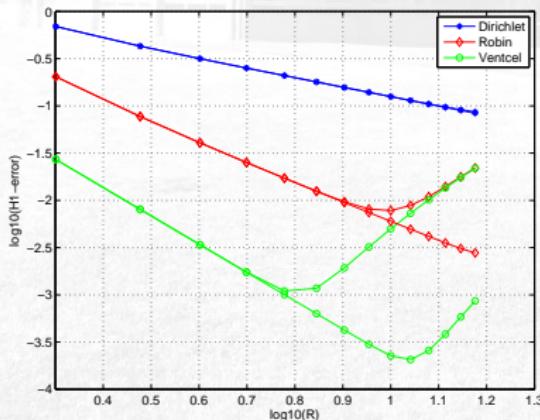
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- ▶ Order 0 : $\mathbf{V} = 0$ for $r = R$.
- ▶ Order 1 : $\mathbf{V} + R \partial_n \mathbf{V} = 0$ for $r = R$.
- ▶ Order 2 : $\mathbf{V} + \frac{3R}{\partial_n} \mathbf{V} - \frac{R^2}{2} \Delta_\tau \mathbf{V} = 0$ for $r = R$.



H^1 -errors vs R
(loglog)

Q_6 and Q_{10} .

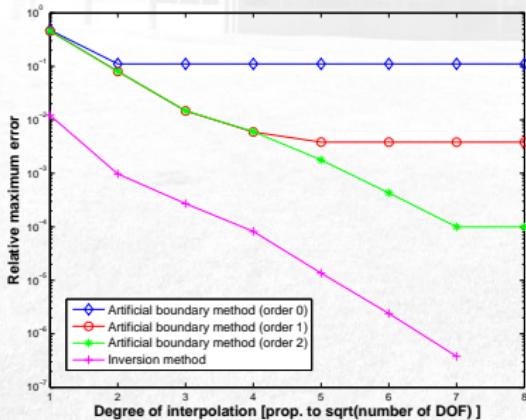
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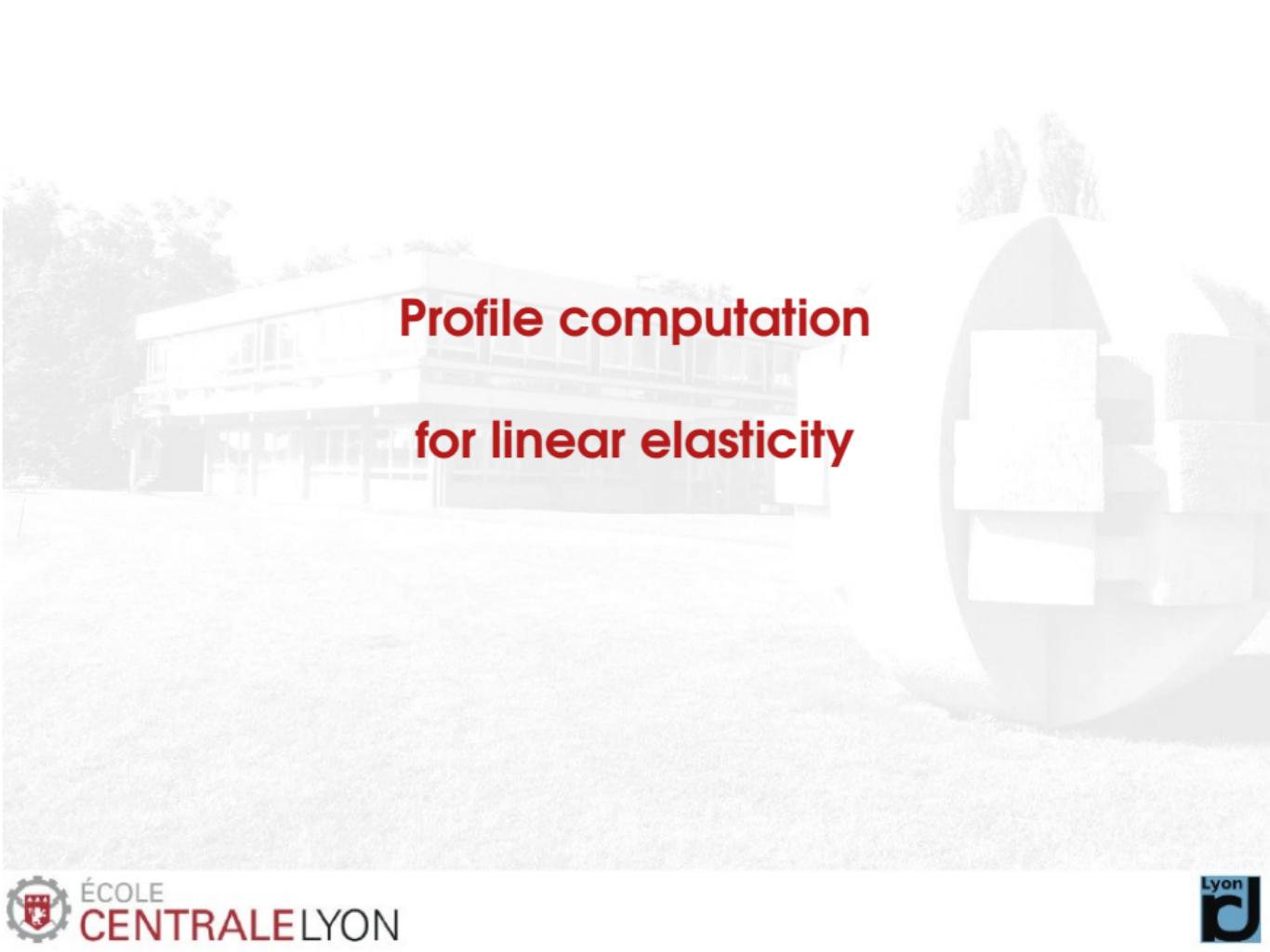


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L^∞ -error vs DoF
(semilog)



Profile computation for linear elasticity

Linear elasticity

Profiles for interior inclusion

$$\begin{cases} -\mu \Delta \mathbf{V} - (\lambda + \mu) \mathbf{grad} \operatorname{div} \mathbf{V} = \mathbf{0} \text{ in } \mathbf{H}_\infty, \\ \boldsymbol{\sigma}(\mathbf{V}) \cdot \mathbf{n} = \mathbf{G} \text{ on } \partial \mathbf{H}_\infty, \\ \mathbf{V} \rightarrow \mathbf{0} \text{ at infinity.} \end{cases}$$

- ▶ Singularities at infinity for the profile problem [Kondrat'ev, Grisvard]

- ▶ $\mathfrak{S}^q(r, \theta) = r^q \begin{pmatrix} \varphi_r(\theta) \\ \varphi_\theta(\theta) \end{pmatrix}$ for exponents $q \in \mathbb{Z}$.

- ▶ For $q = -1$,

$$\begin{cases} \varphi_r(\theta) = A \cos 2\theta + B \sin 2\theta \\ \varphi_\theta(\theta) = \frac{\mu}{\lambda + 2\mu} (B \cos 2\theta - A \sin 2\theta) + B \frac{\lambda + \mu}{\lambda + 2\mu} \end{cases}$$

- ▶ Expansion at infinity :

$$\mathbf{V} = \tilde{\mathfrak{S}}^{-1} + \tilde{\mathfrak{S}}^{-2} + \dots$$

Linear elasticity

Artificial conditions

- ▶ Expansion at infinity :

$$\mathbf{V} = \sum_{\infty} \tilde{\mathfrak{S}}^{-1} + \tilde{\mathfrak{S}}^{-2} + \dots$$

- ▶ Derivation of artificial conditions on $|\mathbf{y}| = R$

- ▶ Order 0 :

$$\mathbf{V} = 0.$$

- ▶ Order 1 (polar coordinates) :

$$\sigma(\mathbf{V}) \cdot \mathbf{n} + \frac{A_1}{R} \mathbf{V} + \frac{1}{2R} \begin{bmatrix} A_2 & 0 \\ 0 & A_3 \end{bmatrix} \partial_\theta^2 \mathbf{V} = 0.$$

with

$$A_1 = \frac{E}{1+\nu}, \quad A_2 = -\frac{\nu E}{2(1-\nu^2)}, \quad A_3 = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}.$$

Since $E > 0$ and $\nu \in (-1, .5)$, we get $A_i > 0$.

Linear elasticity

Artificial conditions

$$\boldsymbol{\sigma}(\mathbf{V}) \cdot \mathbf{n} + \frac{A_1}{R} \mathbf{V} + \frac{1}{2R} \begin{bmatrix} A_2 & 0 \\ 0 & A_3 \end{bmatrix} \partial_\theta^2 \mathbf{V} = 0.$$

- ▶ Bad sign : $A_i > 0$.
- ▶ Model scalar problem :

$$\begin{cases} -\Delta V = 0 & \text{in } \Omega, \\ R\partial_n V + \alpha V + \beta \partial_\theta^2 V = G & \text{on } \partial\Omega. \end{cases}$$

with $\alpha, \beta > 0$.

- ▶ No variational approach.
- ▶ Existence, uniqueness ?

A model non-coercive Ventcel problem

Case of a ball

For $\Omega = \mathcal{B}_1 \subset \mathbb{R}^2$:

$$\begin{cases} -\Delta V = 0 & \text{in } \mathcal{B}_1, \\ R\partial_n V + \alpha V + \beta\partial_\theta^2 V = G & \text{on } \partial\mathcal{B}_1. \end{cases}$$

- We seek V under the form

$$V(r, \theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n.$$

- We get the relations

$$a_n(-\beta n^2 + n + \alpha) = a_n(G) \quad \text{and} \quad b_n(-\beta n^2 + n + \alpha) = b_n(G),$$

where $a_n(G)$ and $b_n(G)$ are the Fourier coefficients of G .

- Unique solution iff $\alpha \neq \beta n^2 - n$ for every $n \in \mathbb{N}$.

A model non-coercive Ventcel problem

General case

$$\begin{cases} -\Delta V = 0 & \text{in } \Omega, \\ R\partial_n V + \alpha V + \beta R^2 \Delta_\tau V = G & \text{on } \partial\Omega. \end{cases}$$

- ▶ Introduction of DtN operator :

$$\begin{aligned} \Lambda : \quad H^{1/2}(\partial\Omega) &\rightarrow H^{-1/2}(\partial\Omega) \\ \psi &\mapsto \partial_n U, \end{aligned}$$

where U solves

$$\begin{cases} -\Delta U = 0 & \text{in } \Omega \\ U = \psi & \text{on } \partial\Omega \end{cases}$$

- ▶ Rewriting as equation on $\partial\Omega$:

$$\beta R^2 \Delta_\tau w + R\Lambda w + \alpha w = G$$

(the Dirichlet datum w is then lifted to obtain U).

A model non-coercive Ventcel problem

General case

$$\beta R^2 \Delta_\tau w + R \Lambda w + \alpha w = G$$

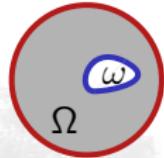
Résult. The problems admits a unique solution iff $\alpha \neq \alpha_n$ ($\forall n \in \mathbb{N}$), where (α_n) is a sequence increasing to infinity ($\beta > 0$ fixed).

IDEA OF PROOF.

- ▶ The operator Δ_τ is pseudo. diff. elliptic bounded from below, selfadjoint of order 2.
- ▶ The operator Λ is pseudo. diff. elliptic bounded from below, selfadjoint of order 1.
- ▶ The operator $-\beta R^2 \Delta_\tau - R \Lambda$ is pseudo. diff. elliptic bounded from below, selfadjoint of order 2.
- ▶ The (α_n) are the eigenvalues of $-\beta R^2 \Delta_\tau - R \Lambda$.

A model non-coercive Ventcel problem

Case of an interior inclusion



$$\left\{ \begin{array}{rcl} -\Delta V & = & 0 & \text{in } \Omega \setminus \overline{\omega}, \\ R\partial_n V + \alpha V + \beta R^2 \Delta_\tau V & = & 0 & \text{on } \partial\Omega, \\ V & = & G & \text{on } \partial\omega. \end{array} \right.$$

- ▶ **Question :** if $\Omega = \mathcal{B}_R$ do we have existence for R large enough ?

A model non-coercive Ventcel problem

Case of an interior inclusion



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- ▶ **Question :** if $\Omega = \mathcal{B}_R$ do we have existence for R large enough ?
- ▶ Particular case of a ring : $\omega = \mathcal{B}_1$.
 - ▶ Representation as a Laurent series :

$$V(r, \theta) = d + c \ln r + \sum_{n \in \mathbb{Z}^*} (a_n r^n \cos n\theta + b_n r^n \sin n\theta)$$

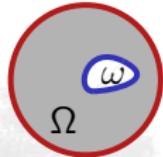
- ▶ Uniqueness condition

$$R \notin \left\{ \gamma_n = \left(\frac{\beta n^2 + n - \alpha}{\beta n^2 - n - \alpha} \right)^{1/2n} \text{ for } n \in \mathbb{N} \right\}$$

⇒ uniqueness for $R > R_c$.

A model non-coercive Ventcel problem

Case of an interior inclusion



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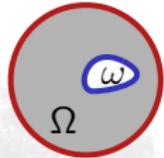
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- ▶ General case : uniqueness for R large enough.

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Case of an interior inclusion



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► **Question :** if $\Omega = \mathcal{B}_R$ do we have existence for R large enough ?

► General case.

► With the homotopy $x \mapsto x/R$, we are led to

$$\omega = \varepsilon \tilde{\omega} \quad \text{and} \quad \Omega = \mathcal{B}_1,$$

with $\varepsilon = 1/R$.

► The operator of interest is then

$$L_D = \alpha \text{Id} + \Lambda_D + \beta \Delta_\tau \quad \text{on } \partial\mathcal{B}_1,$$

with $\Lambda_D(\psi) = U$ where U satisfies

$$\begin{cases} -\Delta U = 0 & \text{in } \mathcal{B}_1 \setminus \bar{D} \\ U = \psi & \text{on } \partial\mathcal{B}_1 \\ U = 0 & \text{on } \partial D \end{cases}$$

A model non-coercive Ventcel problem

Case of an interior inclusion

$$L_D = \alpha \text{Id} + \Lambda_D + \beta \Delta_\tau \quad \text{on } \partial\mathcal{B}_1,$$

with $\Lambda_D(\psi) = U$ where U satisfies

$$\begin{cases} -\Delta U &= 0 & \text{in } \mathcal{B}_1 \setminus \overline{D} \\ U &= \psi & \text{on } \partial\mathcal{B}_1 \\ U &= 0 & \text{on } \partial D \end{cases}$$

- ▶ $D = \emptyset$ is known : L_\emptyset is invertible iff $\alpha \notin \{\alpha_n\}$.
- ▶ Asymptotic result :

$$\|\Lambda_{\varepsilon\tilde{\omega}} - \Lambda_\emptyset\|_{\mathcal{L}(\mathsf{H}^{1/2}(\partial\Omega), \mathsf{H}^{-1/2}(\partial\Omega))} \leq \frac{C}{|\ln \varepsilon|}.$$

- ▶ Conclusion : if ε is small enough, $L_{\varepsilon\tilde{\omega}}$ is generically invertible.

A model non-coercive Ventcel problem

Case of an interior inclusion

$$\begin{cases} -\Delta u = 0 & \text{in } \mathcal{B}_R \setminus \overline{\omega}, \\ R\partial_n u + \alpha u + \beta R^2 \Delta_\tau u = g & \text{sur } \partial \mathcal{B}_R, \\ u = 0 & \text{on } \partial \omega, \end{cases}$$

and the operator L corresponds to the problem

$$Lu = -\beta R^2 \Delta_\tau u - \alpha u - R\Lambda u \quad \text{on } \partial \mathcal{B}_R$$

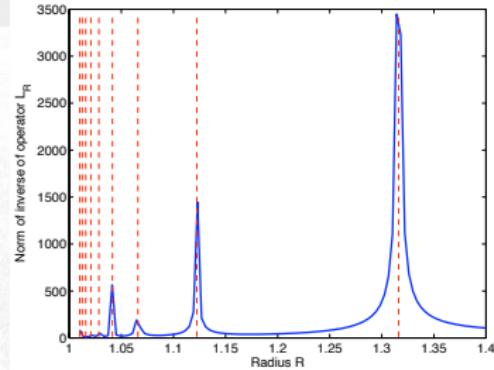
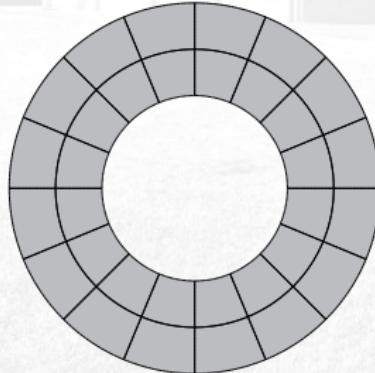
A model non-coercive Ventcel problem

Case of an interior inclusion

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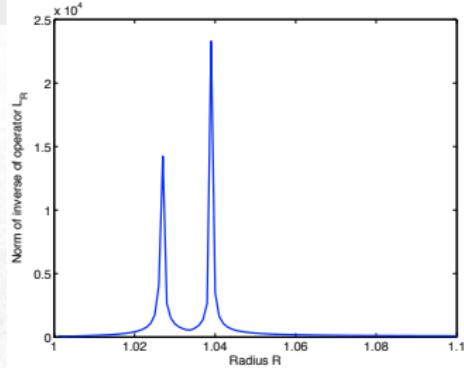
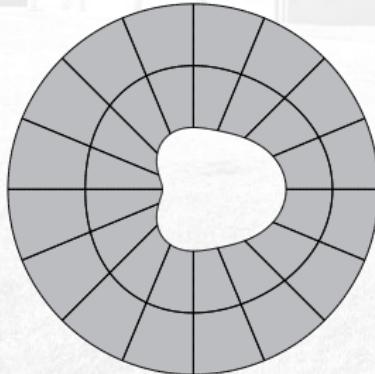
A model non-coercive Ventcel problem

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Ventcel problem for linear elasticity

Case of a circular inclusion

► Profile problem

$$\left\{ \begin{array}{l} -\mu \Delta \mathbf{u} - (\lambda + \mu) \mathbf{grad} \operatorname{div} \mathbf{u} = \mathbf{0} \quad \text{in } \mathcal{B}_R \setminus \omega \\ \sigma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{G} \quad \text{on } \partial \omega \\ \frac{R(1+\nu)}{E} \sigma(\mathbf{u}) \cdot \mathbf{n} + \frac{R^2}{2} \begin{bmatrix} \frac{-\nu}{2(1-\nu)} & 0 \\ 0 & \frac{1-\nu}{1-2\nu} \end{bmatrix} \Delta_\tau \mathbf{u} + \mathbf{u} = \mathbf{0} \quad \text{on } \partial \mathcal{B}_R \end{array} \right.$$

Ventcel problem for linear elasticity

Case of a circular inclusion

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$$\left\{ \begin{array}{l} -\mu \Delta \mathbf{u} - (\lambda + \mu) \mathbf{grad} \operatorname{div} \mathbf{u} = \mathbf{0} \quad \text{in } \mathcal{B}_R \setminus \omega \\ \sigma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{G} \quad \text{on } \partial \omega \\ \frac{R(1+\nu)}{E} \sigma(\mathbf{u}) \cdot \mathbf{n} + \frac{R^2}{2} \begin{bmatrix} \frac{-\nu}{2(1-\nu)} & 0 \\ 0 & \frac{1-\nu}{1-2\nu} \end{bmatrix} \Delta_\tau \mathbf{u} + \mathbf{u} = \mathbf{0} \quad \text{on } \partial \mathcal{B}_R \end{array} \right.$$

► Case of a ball : $\omega = \mathcal{B}_1$

Theorem. There exists a countable set \mathcal{S} s.t.

$\forall \nu \notin \mathcal{S}, \quad \exists \mathcal{R}_\nu$ countable s.t.. $\forall R \notin \mathcal{R}_\nu,$

$\forall \mathbf{G} \in H^{1/2}(\partial \omega), \quad \exists \mathbf{u} \in H^2(\mathcal{B}_R \setminus \omega)$ solution.

Ventcel problem for linear elasticity

Case of a circular inclusion

Again we consider the equation on $\partial\mathcal{B}_R$:

$$\frac{1}{2} \begin{bmatrix} \frac{-\nu}{2(1-\nu)} & 0 \\ 0 & \frac{1-\nu}{1-2\nu} \end{bmatrix} \partial_\theta^2 \varphi + \varphi + \Lambda_R(\varphi) = \frac{-R(1+\nu)}{E} \boldsymbol{\sigma}(\mathbf{u}_0) \cdot \mathbf{n} \text{ on } \partial\mathcal{B}_R,$$

with $\Lambda_R(\varphi) = \left[\frac{R(1+\nu)}{E} \boldsymbol{\sigma}(\mathbf{v}) \cdot \mathbf{n} \right] \Big|_{\partial\mathcal{B}_R}$, where

$$\begin{cases} -\mu \Delta \mathbf{v} - (\lambda + \mu) \mathbf{grad} \operatorname{div} \mathbf{v} = 0 & \text{in } \mathcal{B}_R \setminus \omega, \\ \boldsymbol{\sigma}(\mathbf{v}) \cdot \mathbf{n} = 0 & \text{on } \partial\omega, \\ \mathbf{v} = \varphi & \text{on } \partial\mathcal{B}_R. \end{cases}$$

and

$$\begin{cases} -\mu \Delta \mathbf{u}_0 - (\lambda + \mu) \mathbf{grad} \operatorname{div} \mathbf{u}_0 = 0 & \text{in } \mathcal{B}_R \setminus \omega, \\ \boldsymbol{\sigma}(\mathbf{u}_0) \cdot \mathbf{n} = \mathbf{G} & \text{on } \partial\omega, \\ \mathbf{u}_0 = 0 & \text{on } \partial\mathcal{B}_R. \end{cases}$$

Ventcel problem for linear elasticity

Case of a circular inclusion

We look for a solution under the form

$$\varphi = \begin{bmatrix} \varphi_0^r \\ \varphi_\theta^r \\ \varphi_0^\theta \\ \varphi_\theta^\theta \end{bmatrix} + \sum_{n \geq 1} \begin{bmatrix} \varphi_n^r \\ \varphi_\theta^r \\ \varphi_n^\theta \\ \varphi_\theta^\theta \end{bmatrix} \cos n\theta + \sum_{n \geq 1} \begin{bmatrix} \psi_n^r \\ \psi_\theta^r \\ \psi_n^\theta \\ \psi_\theta^\theta \end{bmatrix} \sin n\theta$$

Let $\Phi_n = [\varphi_n^r, \psi_n^r, \varphi_n^\theta, \psi_n^\theta]^\top$ and $f_{n,R}$ the decomposition of the RHS.

The problems reads

$$P_n \Phi_n + \mathcal{R}_{n,R} \Phi_n = f_{n,R}$$

with $\|\mathcal{R}_{n,R}\|_\infty \leq Cn^2 R^{-2n+2}$, $\|\mathcal{R}_{1,R}\|_\infty \leq CR^{-4}$, $\|\mathcal{R}_{0,R}\|_\infty \leq CR^{-2}$

Ventcel problem for linear elasticity

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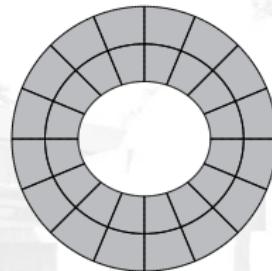
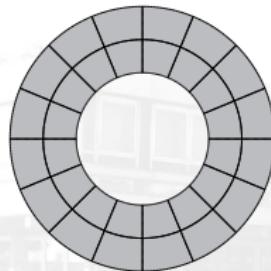
Result. Let $\gamma = \frac{1-2\nu}{2(1-\nu)}$.

1. There exist a countable set \mathcal{S} s.t. for every $\gamma \notin \mathcal{S}$, P_n is invertible with inverse bounded w.r.t. n .
2. For every $\gamma \notin \mathcal{S}$, there exists R_γ such that the Ventcel problems admits a unique solution for $R \geq R_\gamma$.

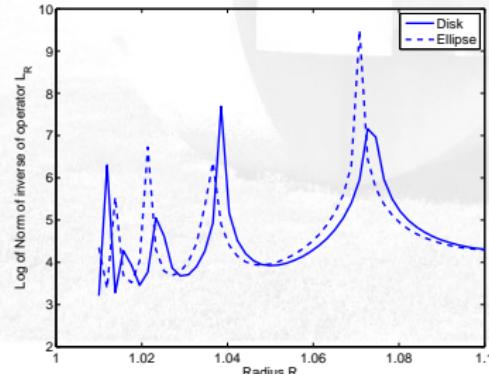
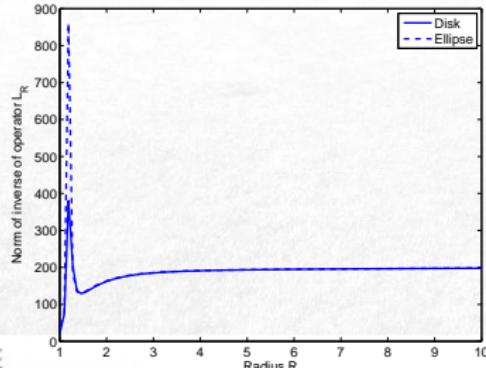
Ventcel problem for linear elasticity

Numerical simulations

- Geometries :



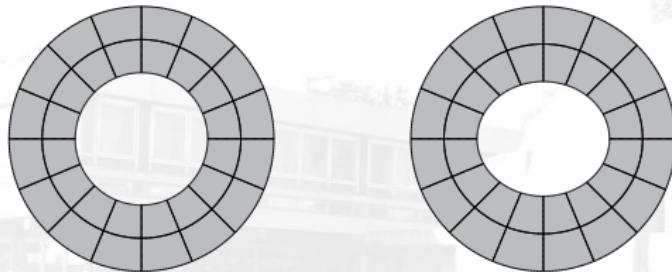
- Norm of inverse w.r.t. R



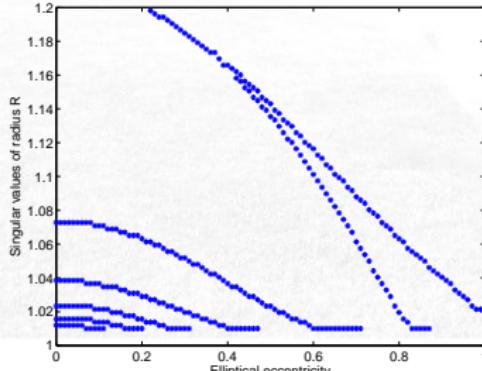
Ventcel problem for linear elasticity

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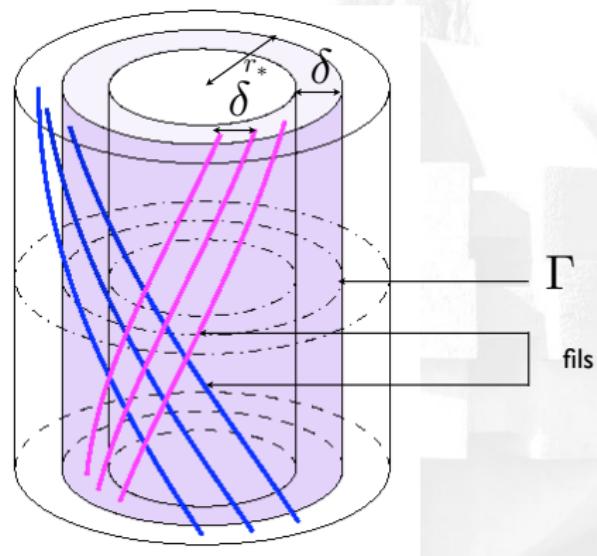
- Singular radii R w.r.t. eccentricity



Other non coercive Ventcel-type problems

Electromagnetics [PhD B. Delourme 2010]

- ▶ Thin 3D structure
- ▶ Periodic repartition
- ▶ Harmonic Maxwell equation
- ▶ Expansion in δ
(homog. and matched expansions)
- ▶ Approximate boundary conditions on Γ



Other non coercive Ventcel-type problems

Electromagnetics [PhD B. Delourme 2010]

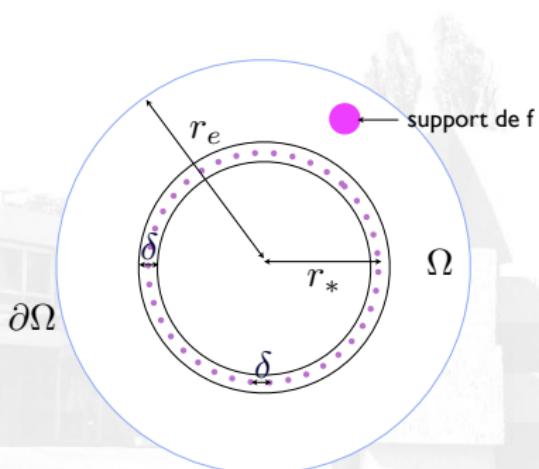
2D model

- ▶ Simplified geometry
- ▶ Helmholtz equation :

$$\operatorname{div}(\varepsilon_{\delta}^{-1} \nabla u) + \omega^2 \mu_{\delta} u = f.$$

- ▶ Radiation condition on $\partial\Omega$

$$\partial_r u + i\omega u = 0.$$



Asymptotic model of order 1

$$\begin{cases} \varepsilon_{\infty}^{-1} \Delta u + \omega \mu_{\infty} u = f & \text{in } \Omega_{\pm}, \\ \partial_r u + i\omega u = 0 & \text{on } \partial\Omega, \\ [u]_{\Gamma} = \delta A_0 \langle r \partial_r u \rangle_{\Gamma} & \text{on } \Gamma, \\ [r \partial_r u]_{\Gamma} = \delta \left(-\omega^2 B_0 \langle u \rangle_{\Gamma} - B_2 \partial_{\theta}^2 \langle u \rangle_{\Gamma} \right) & \text{on } \Gamma. \end{cases}$$

Other non coercive Ventcel-type problems

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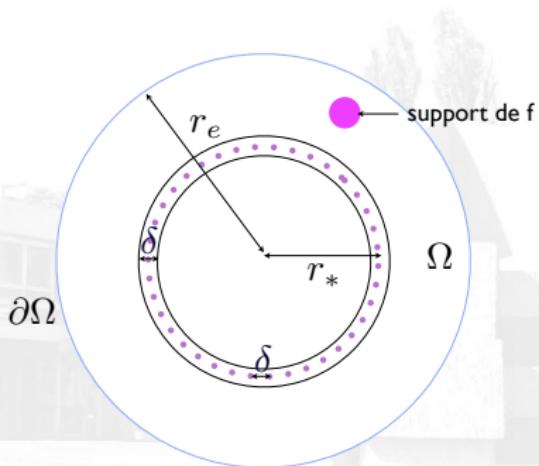
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Asymptotic model of order 1 $\langle \phi \rangle_{\Gamma} = \frac{1}{2} (\phi(x+0) + \phi(x-0))$

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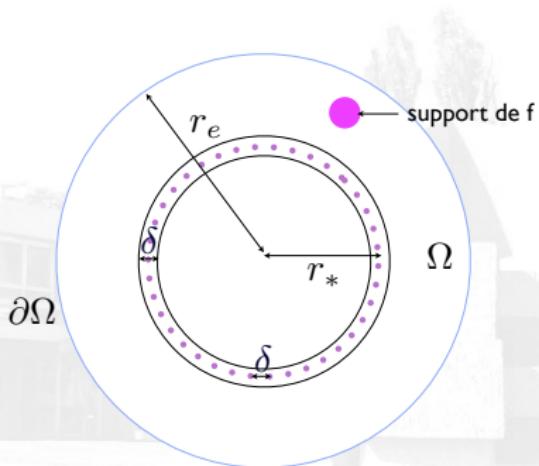
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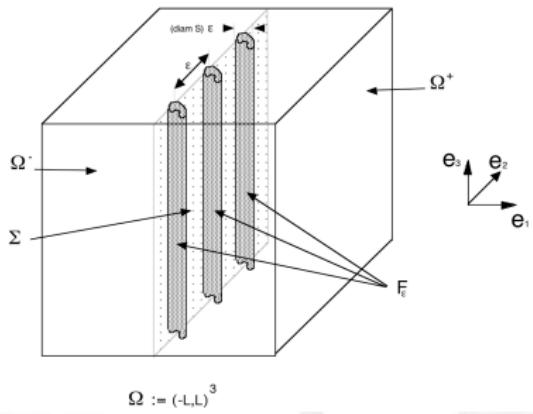
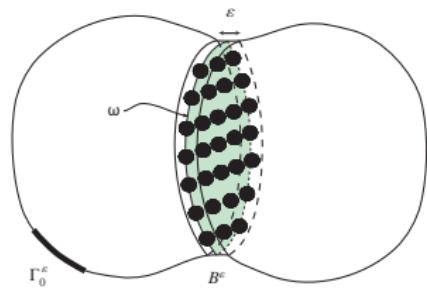
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Pb. $B_2 < 0$



Other non coercive Ventcel-type problems

In mechanics [Projet ANR Epsilon -2012]

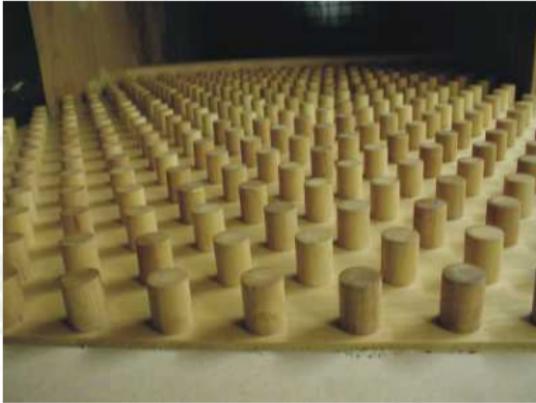


Heterogeneous thin layers in a material

- ▶ Approximate transmission condition of order 2.
- ▶ Ventcel with bad sign.

Other non coercive Ventcel-type problems

For flows [Bresch-Milisic 2008]

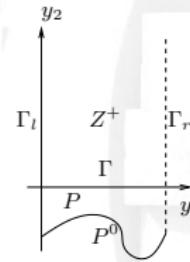
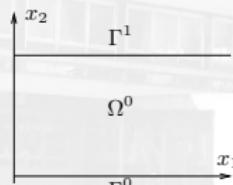
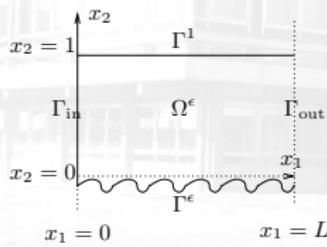


- ▶ Flows over rough boundary.
- ▶ Periodic geometry.
- ▶ Construction of approximate problems : *wall laws*

Other non coercive Ventcel-type problems

For flows [Bresch-Milisic 2008]

- ▶ 2D periodic model.
- ▶ Laplace Dirichlet/periodic equation.



Implicit wall law of order 2 :

$$\begin{cases} -\Delta \mathcal{V}_\varepsilon = C & \text{dans } \Omega^0, \\ \mathcal{V}_\varepsilon - \varepsilon \partial_n \mathcal{V}_\varepsilon \beta\left(\frac{x_1}{\varepsilon}, 0\right) + \frac{\varepsilon^2}{2} \partial_\tau^2 \mathcal{V}_\varepsilon \gamma\left(\frac{x_1}{\varepsilon}, 0\right) = 0 & \text{on } \Gamma^0. \end{cases}$$

(β, γ solutions of cell problems, of different sign).

Conclusions

- ▶ More general geometries.
- ▶ Explicit bound for forbidden radii ?
- ▶ Numerical analysis of the whole numerical method.
- ▶ Interaction between inclusions.