Multiscale analysis and simulation with accounting for frictional contact and sliding

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Moscow, 30.10.2014

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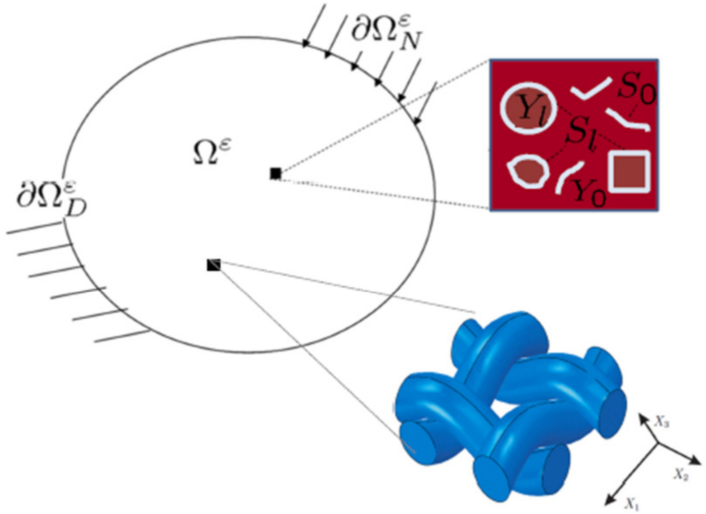
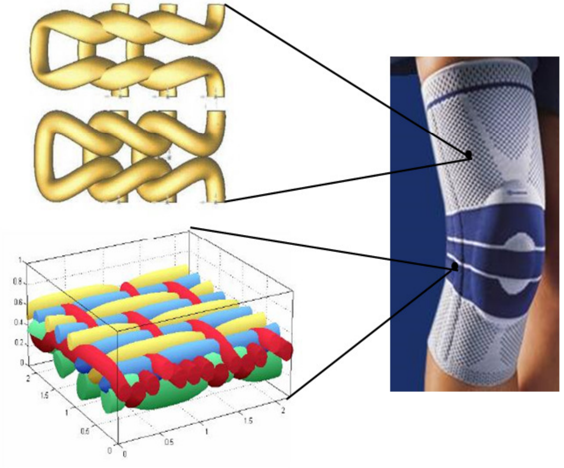
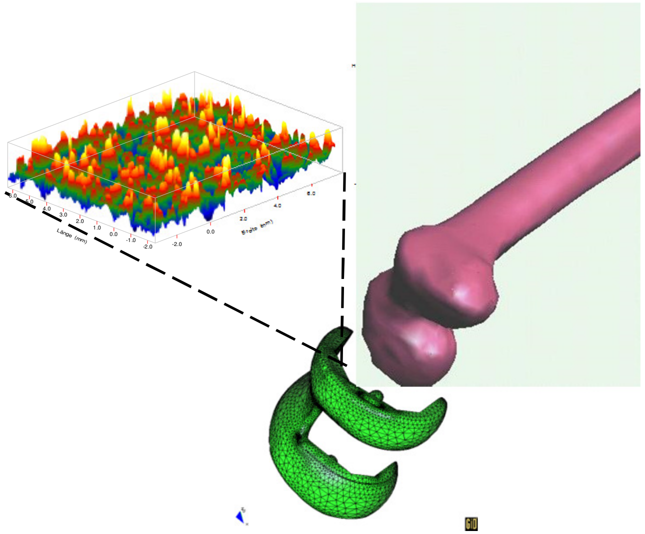


Motivation for Homogenization

Problem: it is difficult to treat PDE's in composites and porous materials numerically due to highly oscillating coefficients, related to the heterogeneities at the small scale.



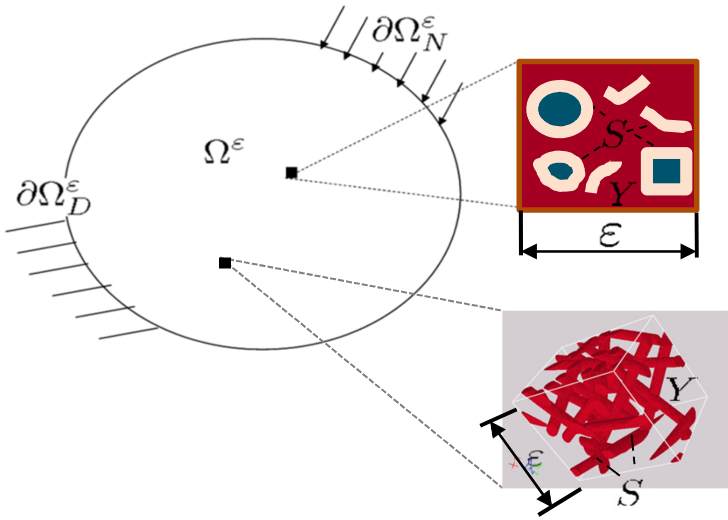


We are interested in the **asymptotic behavior** of the solutions of these equations at the scale of macroscopic size. The ratio between the period of the structure and the macroscopic length of the sample tends to zero.

Strong formulation of the contact problem

$$
\begin{cases}
-\operatorname{div}\sigma_\varepsilon = \bar{f}_\varepsilon \quad \text{in } \Omega_\varepsilon \qquad \text{with Hook's law} \quad \sigma^\varepsilon_{ij} = \sum_{k,l=1}^{3} a^\varepsilon_{ijkl} e(v)_{kl}, \\
\text{Non-penetration (Signorini) condition:} \\
[(\dot{u}_\varepsilon)_\nu]_{S_\varepsilon} - g_\varepsilon \leq 0, \quad \sigma_\varepsilon(\nu)_\nu \leq 0, \quad \sigma_\varepsilon(\nu)_\nu\big([(\dot{u}_\varepsilon)_\nu]_{S_\varepsilon} - g_\varepsilon\big) = 0 \quad \text{on } S_\varepsilon, \\
\text{Coulomb friction condition:} \\
|\sigma_\varepsilon(x)_\tau| \leq \mu|\sigma_{\varepsilon\ \nu}|, \quad \Rightarrow \quad [(\dot{u}_\varepsilon)_\tau]_{S_\varepsilon} = 0, \\
|\sigma_\varepsilon(x)_\tau| = \mu|\sigma_{\varepsilon\ \nu}| \quad \Rightarrow \quad \exists\lambda_\varepsilon > 0, \ \text{s.t.} \ [(\dot{u}_\varepsilon)_\tau]_{S_\varepsilon} = -\lambda_\varepsilon \sigma_\varepsilon(x)_\tau \quad \text{a.e. on } S_\varepsilon, \\
\text{Boundary conditions:} \\
\dot{u}_\varepsilon = \dot{g} \quad \text{on } \Gamma_{D\varepsilon}, \\
\sigma_\varepsilon \cdot \nu = 0 \quad \text{on } \Gamma_{N\varepsilon}, \\
\text{Initial condition:} \\
u_\varepsilon(0,x) = u_{0\varepsilon}(x), \qquad x \in \Omega_\varepsilon,
\end{cases}
$$

Weak problem formulation



Problem $\mathcal{P}'_\varepsilon$**:** Find $u_\varepsilon \in \mathcal{K}^\varepsilon$ such that for every $v \in \mathcal{K}^\varepsilon$,

$$\mathbf{a}^\varepsilon(e(u_\varepsilon), e(v - u_\varepsilon)) + \sum_{j=0}^{m} \left(\Psi_\varepsilon^j([v_\tau]_{S_\varepsilon^j}) - \Psi_\varepsilon^j([(u_\varepsilon)_\tau]_{S_\varepsilon^j})\right) \geq \int_{\Omega_\varepsilon^*} f_\varepsilon(v - u_\varepsilon)\, dx.$$

$$\Psi_\varepsilon^j(w) \doteq \int_{S_\varepsilon^j} G_\varepsilon^j(x)|w|\, d\sigma(x)$$

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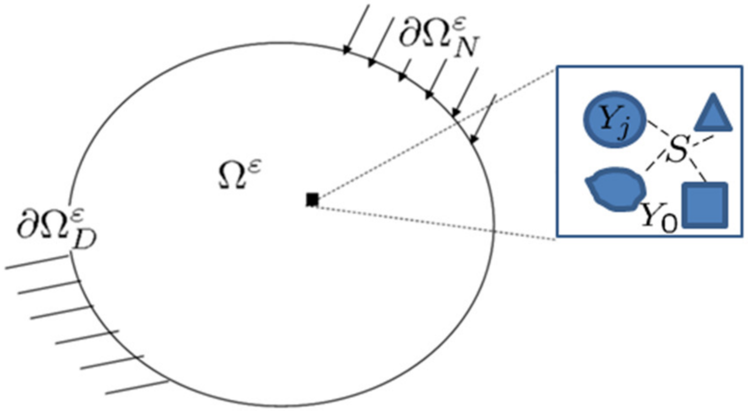
Korn-like inequality for non-locked ε-periodic inclusions



Proposition

There exists a constant C such that for all u in $H^1(\Omega_\varepsilon^j)$, $j = 1, \dots, m$,

$$\|u^j\|_{L^2(\Omega_\varepsilon^j)} + \varepsilon\|\nabla u^j\|_{L^2(\Omega_\varepsilon^j)} \leq C\big(\|e(u^0)\|_{L^2(\Omega_\varepsilon^0)} + \varepsilon\|e(u^j)\|_{L^2(\Omega_\varepsilon^j)} + \varepsilon^{1/2}\|g_j^\varepsilon\|_{L^1(S_\varepsilon^j))} + \varepsilon^{-1/2}\|[u]_\tau\|_{L^1(S_\varepsilon^j)}\big).$$

- D. Cioranescu, A. Damlamian and J. Orlik, Homogenization via unfolding in periodic elasticity with contact on closed and open cracks, Asymptotic Analysis, Vol. 82, Issue 3-4, 2013

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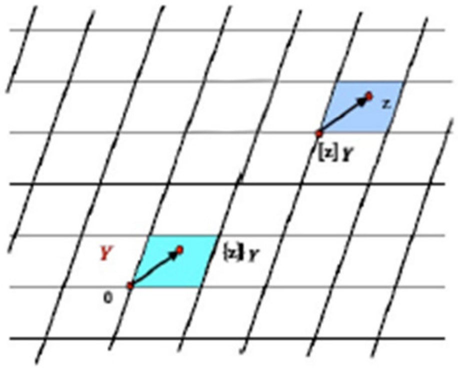
Unfolding (Griso, et. al. 2002)
Decomposition

Let $Y = \prod_{i=1}^{n} \ell_i$ be a reference cell and Ω an open subset of $\mathbb{R}^n$.

For $z \in \mathbb{R}^n$, $[z]_Y$ denotes the unique integer combination $\sum_{j=1}^{n} k_j \ell_j$ of the periods such that $z - [z]_Y \in Y$, and set

$$\{z\}_Y = z - [z]_Y \in Y \quad \text{a.e. for} \quad z \in \mathbb{R}^n.$$



Then for each $x \in \mathbb{R}^n$, one has

$$x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right) \quad \text{a.e. for} \quad x \in \mathbb{R}^n.$$

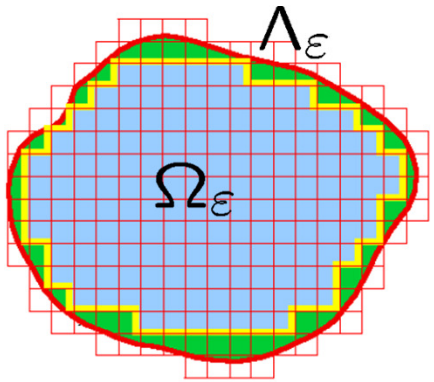
Unfolding operator

The unfolding operator $\mathcal{T}_\varepsilon$ for functions defined on periodic domain Ω_ε as follows:
For any function f Lebesgue-measurable on Ω_ε, the unfolding operator $\mathcal{T}_\varepsilon$ is defined by

$$\mathcal{T}_\varepsilon(f)(x,y) = \begin{cases} f\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } (x,y) \in \Omega_\varepsilon \times Y, \\ 0 & \text{a.e. for } (x,y) \in \Lambda_\varepsilon \times Y. \end{cases}$$

This operator maps functions defined on the oscillating domain Ω_ε to functions defined on the fixed domain $\Omega \times Y$.
For every $f \in L^1(\Omega_\varepsilon)$ one has the integration formula

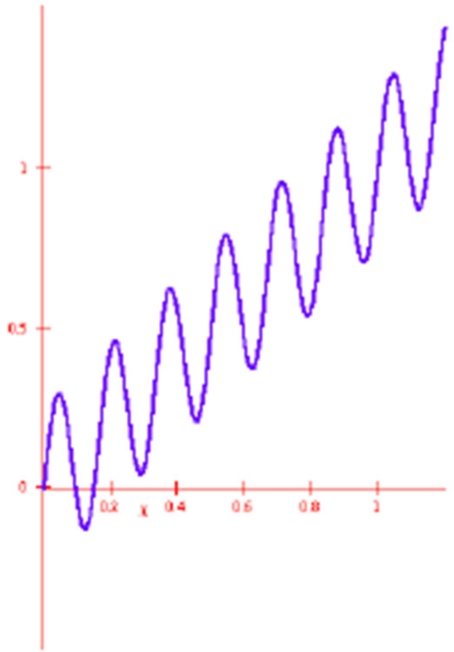
- $$\frac{1}{|Y|}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(f)(x,y)dxdy = \int_{\Omega_\varepsilon} f\, dx$$



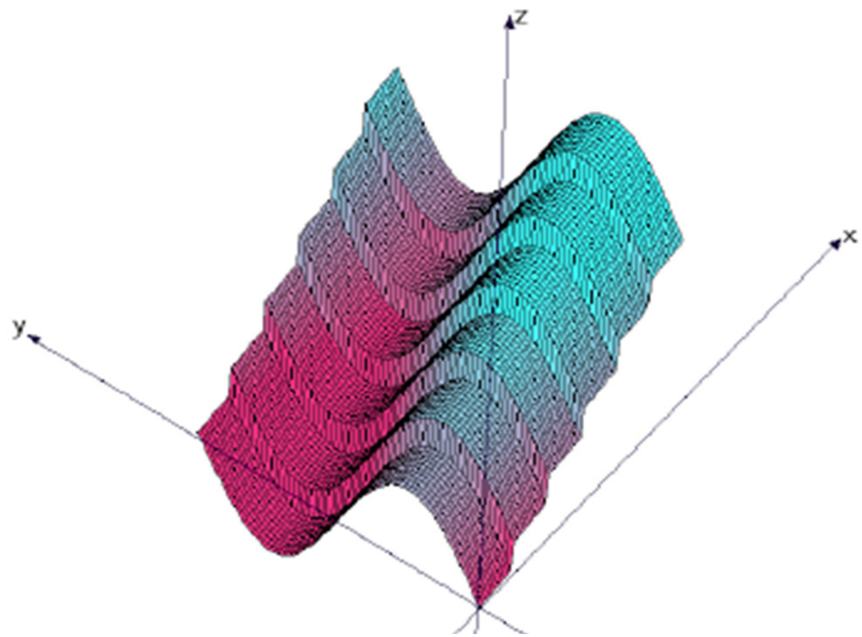
and

$$\|\mathcal{T}_\varepsilon(f)\|_{L^2(\Omega\times Y)} = (|Y|)^{1/2}\|f\|_{L^2(\Omega_\varepsilon)}.$$

Example 1: how does it look like in one dimension



$f_\varepsilon(x) = x + \frac{1}{4}\sin(2\pi x/\varepsilon)$, $\varepsilon = \frac{1}{6}$.



$\mathcal{T}^\varepsilon(f_\varepsilon)$

Note that $\mathcal{T}_\varepsilon(f_\varepsilon)$ is piece-wise constant with respect to x, so that even if f is very regular, this cannot be the case of $\mathcal{T}_\varepsilon(f_\varepsilon)$, at least with respect to the variable x.

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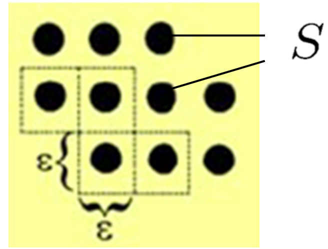
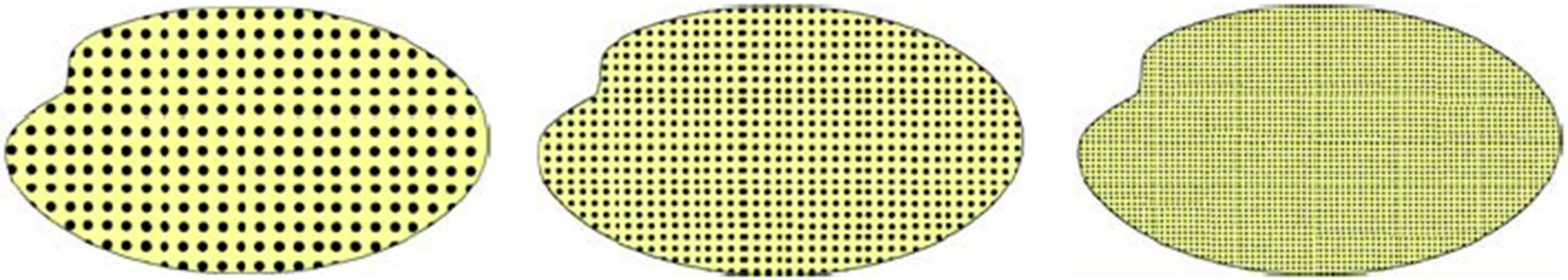
Interface measure.



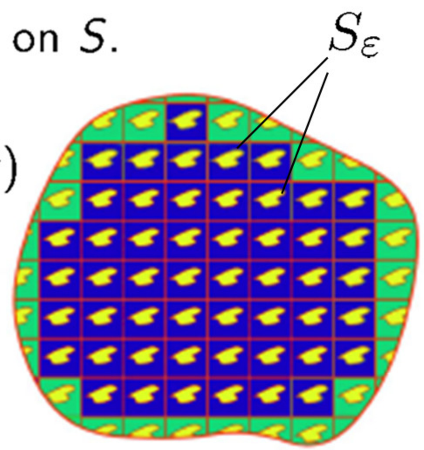


If $\Omega_\varepsilon \in \mathbb{R}^n$ and we let $\varepsilon \to 0$, the volume of each periodicity cell, $|\varepsilon Y|$, will be of the order ε^n.
Hence, the number of the cells, N, will increase as ε^{-n}, while the surface measure, $|S|$, in each periodicity cell will be of the order ε^{n-1}.
Then,

$$|S_\varepsilon| \approx N|S| \qquad \text{will be of the order} \qquad \frac{1}{\varepsilon}.$$

Interface unfolding operator

For $f \in H^1(\Omega_\varepsilon \setminus S_\varepsilon)$, the interface unfolding is the trace of $\mathcal{T}_\varepsilon(f)$ on S.



- $$\frac{1}{\varepsilon|Y|}\int_{\Omega\times S}\mathcal{T}_\varepsilon(f)(x,y)dx\,d\sigma(y) = \int_{S_\varepsilon} f\,d\sigma(x)$$

Theorem

If $u_\varepsilon \in H^1(\Omega \setminus S_\varepsilon, \partial\Omega_u)$ is bounded, then

$$\varepsilon^{-1}\mathcal{T}_\varepsilon([u_\varepsilon]) \rightharpoonup [u_1] \text{ in } L^2(\Omega, H^{1/2}(S))$$

up to a subsequence.

- Cioranescu, Damlamian, Donato, Griso, Zaki, *Unfolding in domains with holes, SIAM J. Math. Anal.,44(2), 718760, 2012*
- D. Cioranescu, A. Damlamian, J. Orlik, *Homogenization via unfolding in periodic elasticity with contact on closed and open cracks, Asymptotic Analysis, 2013*
- H.-K. Hummel, *Homogenization for heat transfer in polycrystals with inter- facial resistances, Appl. Anal. 75 (2000) (with two-scale convergence)*

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Boundedness.

Proposition 5.4. *Suppose that*

a) *either* $\frac{1}{\varepsilon^{N/2-1}M_\varepsilon^j}|f_\varepsilon^j|_{L^2(\Omega_\varepsilon^j)}$ *is small enough (compared to the diametr of* Ω_ε^j *),*

b) *or* f_ε^j *satisfies condition* $\int_{Y^j} T_\varepsilon(f_\varepsilon^j)\, r\, dy = 0$ *for every* $r \in \mathcal{R}$

Then, under the above hypotheses, there exits a constant C *independent of* ε *such that for* u *in a minimizing sequence*

$$|e(u^0)|^2_{L^2(\Omega_\varepsilon^0)} + \sum_{j=1}^m |e(u^j)|^2_{L^2(\Omega_\varepsilon^j)} + \sum_{j=0}^m \Psi_\varepsilon^j(u) \leq C\Big(\sum_{j=0}^m |f_\varepsilon^j|^2_{L^2(\Omega_\varepsilon^j)} + \sum_{j=1}^m (\varepsilon^2 |f_\varepsilon^j|^2_{L^2(\Omega_\varepsilon^j)} + |T_\varepsilon(g_j^\varepsilon)|^2_{L^2(\Omega;L^1(\partial Y^j))}\Big). \tag{5.13}$$

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Boundedness and convergence of the conormal derivatives on the oscillating interface

Theorem 3.3. *Let* $\mu_\varepsilon(x) = \mu(x)$, $C_{A\ \varepsilon} = C_A$, $\delta_\varepsilon = \varepsilon\delta$, $a_{ijkl} \in L^\infty(\Omega_\varepsilon)$, *furthermore,* $a_{ijkl} \in C^{0,\alpha}$, $0 < \alpha < 1/2$ *in a neighborhood of* S_ε, $\mu \in C^1(S_\varepsilon)$ *with compact support in* S_ε. *Furthermore, let coeficients* $||\mu||_{L^\infty(S_\varepsilon)} < \sqrt{\alpha/(2C_A)}$, $f_\varepsilon \in L^2(\Omega_\varepsilon)$, $g^\varepsilon \in L^2(\Gamma_D \cup S_\varepsilon)$, *then*

$$||\varphi(\sigma_\nu(u_\varepsilon))||_{H^{-1/2+\alpha}(S_\varepsilon)} \leq const(C_A, \mu, ||\bar{f}_e||_{L^2(\Omega)}). \quad (3.12)$$

Theorem 5.2. *Let assumptions of Theorem 3.12 hold, then* $\frac{\partial u_\varepsilon}{\partial \nu} \equiv A_\varepsilon \nabla u_\varepsilon = \sigma_\nu(u_\varepsilon)$

$$T_\varepsilon\left(\frac{\partial u_\varepsilon}{\partial \nu}\right) \to a^0(x,y)(\nabla u_0 + \nabla_y \hat{u}(x,y))\nu_y(x,y) \qquad \text{strongly in } L^2(\Omega, H^{-1/2}(S)). \quad (5.4)$$

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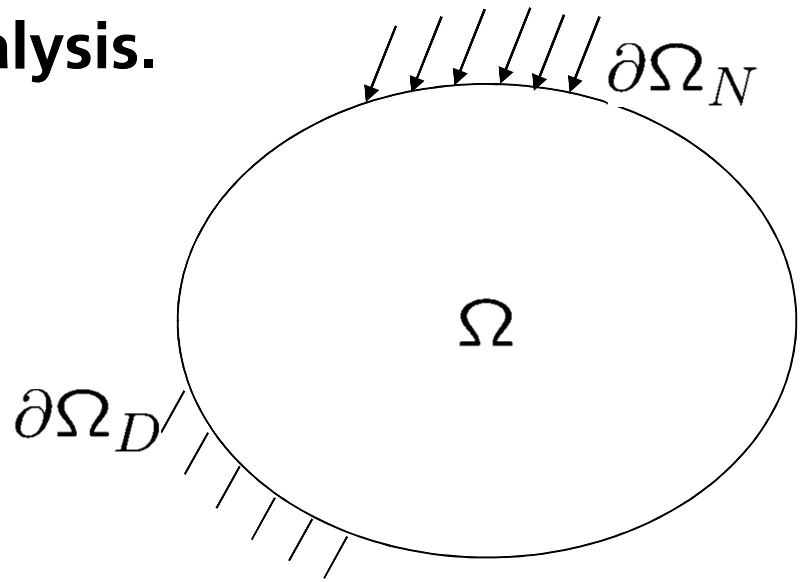
Result of homogenization. Analysis.



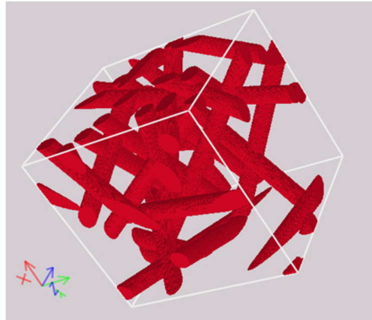
Macroscopic problem

$$\begin{cases} \boldsymbol{u}^0 \in \boldsymbol{H}^1(\Omega, \partial\Omega_D), \\ -\mathrm{div}\, \boldsymbol{\sigma}^{hom}(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{u}^0)) = \boldsymbol{f} \quad in\, \Omega, \\ \boldsymbol{\sigma}^{hom} \cdot \boldsymbol{n} = \boldsymbol{t} \quad on\ \partial\Omega_N, \end{cases}$$

$$\sigma_{ij}^{hom}(\boldsymbol{x}, \boldsymbol{e}(\boldsymbol{u}^0)) \doteq \frac{1}{|Y|} \int_Y a_{ijkl}(\boldsymbol{x}, \boldsymbol{y}) \left(\frac{\partial u_k^0(\boldsymbol{x})}{\partial x_l} + \frac{\partial u_k^1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}^0)}{\partial y_l} \right) dy.$$

Result of homogenization. Analysis.

$$y = \frac{x}{\varepsilon} \in Y$$



Periodic contact problem for the corrector on the periodicity cell Y

$$\int_Y a_{ijkl} \left(\frac{\partial u_k^0(x)}{\partial x_l} + \frac{\partial u_k^1(x,y)}{\partial y_l} \right) \frac{\partial (v_i^1(x,y) - u_i^1(x,y))}{\partial y_j} \, dy$$

$$+ \int_S G(y)(|[v^1]_\tau(x,y)| - |[u^1]_\tau(x,y)|) ds_y \geq 0,$$

$$v^1 \in L^2(\Omega, H^1_{per}(Y \setminus S)) | \; [v]^1_n(x,y) \leq \bar{g}(y), \;\; y \in S$$

- A. Damlamian, D. Cioranescu, J. Orlik , *Homogenization via unfolding in periodic elasticity with contact*, Asymptotic Analysis, Vol. 82, issue 3-4, 2013.

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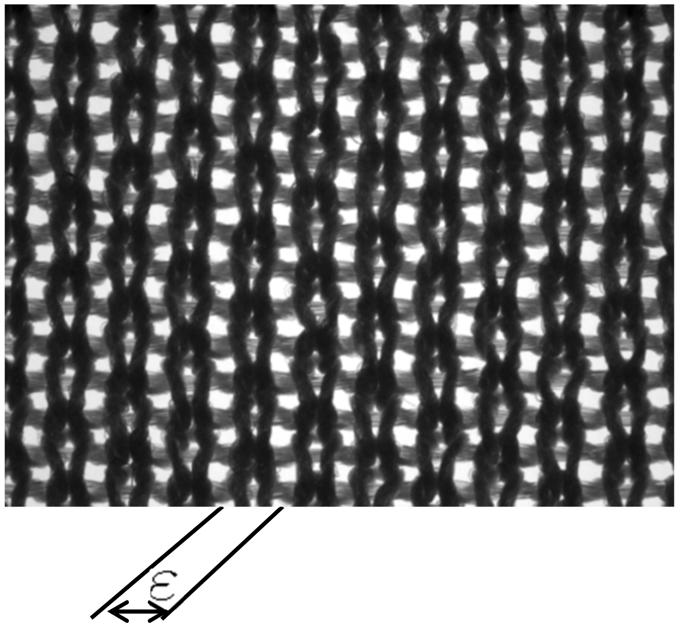
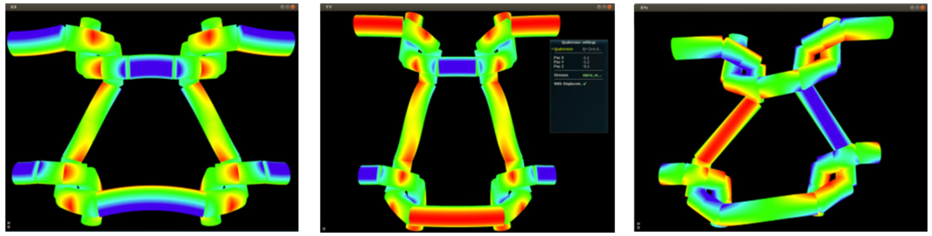
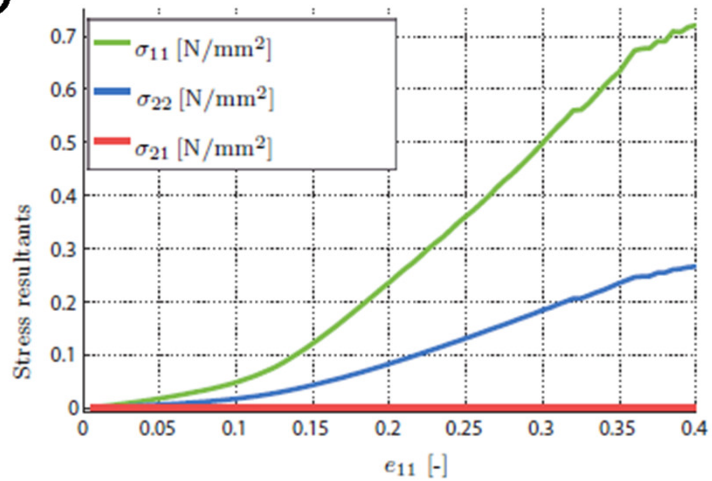
Mechanical interpretation of the result

- Micro-sliding results into macroscopic plasticity.





$\varepsilon \to 0$



- S. Fillep, J. Orlik, Z. Bare, P. Steinmann, *Homogenization of elasticity in periodically heterogeneous bodies with contact on the microstructural elements*, Mathematics and Mechanics of Solids, 2013.

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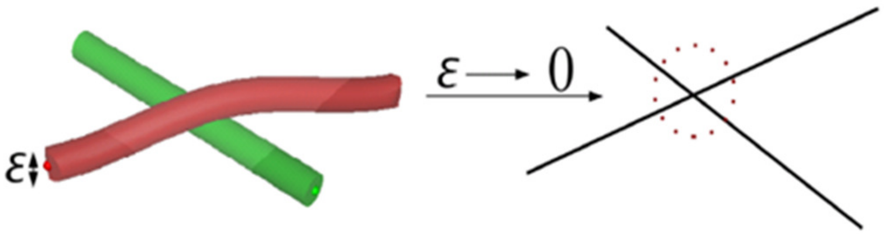
How the 3-D contact conditions enter the 1-D cell-problems?



$$-EA\frac{\partial^2 u_1(x_1)}{\partial x_1^2} = \hat{f}_1(x_1) \quad in\ (-1,0)\cup(0,1),$$
$$\left[EA\frac{\partial u_1(x_1)}{\partial x_1}\right] = -|\alpha|\frac{G}{\gamma}u_1(x_1) \quad on\ 0,$$

tension

$$EI_j\frac{\partial^4 u_j(x_1)}{\partial x_1^4} = \hat{f}_j(x_1) \quad x_1\in(-1,0)\cup(0,1)$$
$$\left[-EI_j\frac{\partial^3 u_j(x_1)}{\partial x_1^3}\right] = -\frac{Gs_j}{\gamma}\sum_{i=2}^4 a_{ji}(\omega,\alpha)u_i(x_1) \quad x_1=0$$
$$\left[EI_j\frac{\partial^2 u_j(x_1)}{\partial x_1^2}\right] = -\frac{G}{\gamma}\sum_{i=1}^4 b_{ji}(\omega,\alpha)\frac{\partial u_i(x_1)}{\partial x_1} \quad x_1=0$$

bending

- 1-D contact parameters are computed from those known for the 3-D Problem and the beam cross-sectional characteristics.
- Entrance of the bending moments into the 1-D contact problem
- Z.Bare, J.Orlik, G.Panasenko, Asymptotic dimesion reduction in thin networks with Robin conditions in nodes, Applicable Analysis, 2013

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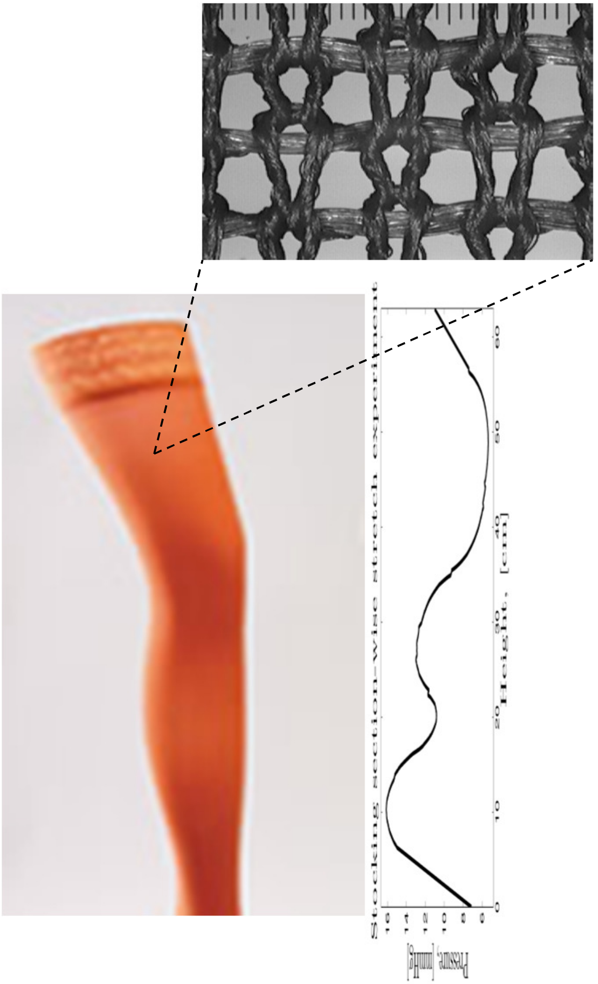
Simulation of the compressive stockings

Simulation of the behaviour of compressive stockings



Given

a textile structure made of thin yarns.

Aims:

- simulate the hysteretic behaviour of the textile membrane based on the structural, geometrical and mechanical properties of yarns
- optimize the knitting pattern along the stocking to reach a certain pressure profile along the axis of the leg.

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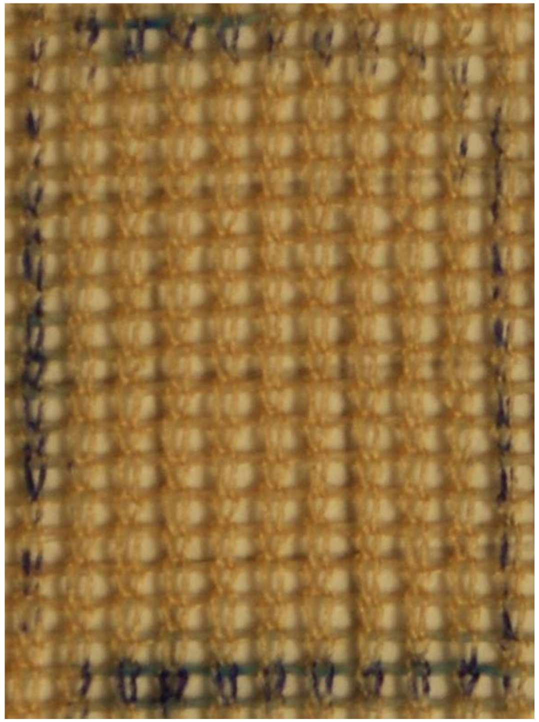
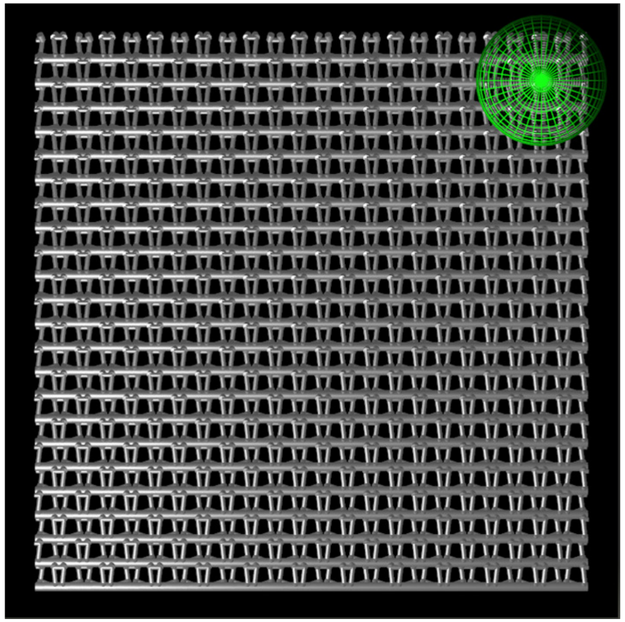
1-D truss model

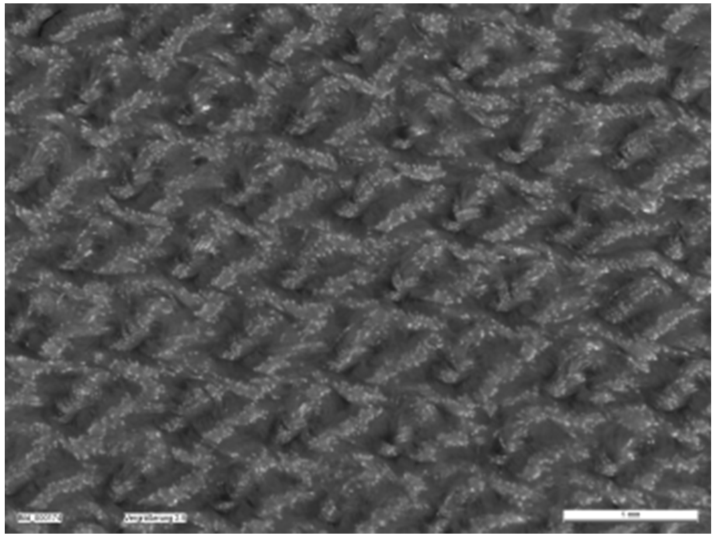




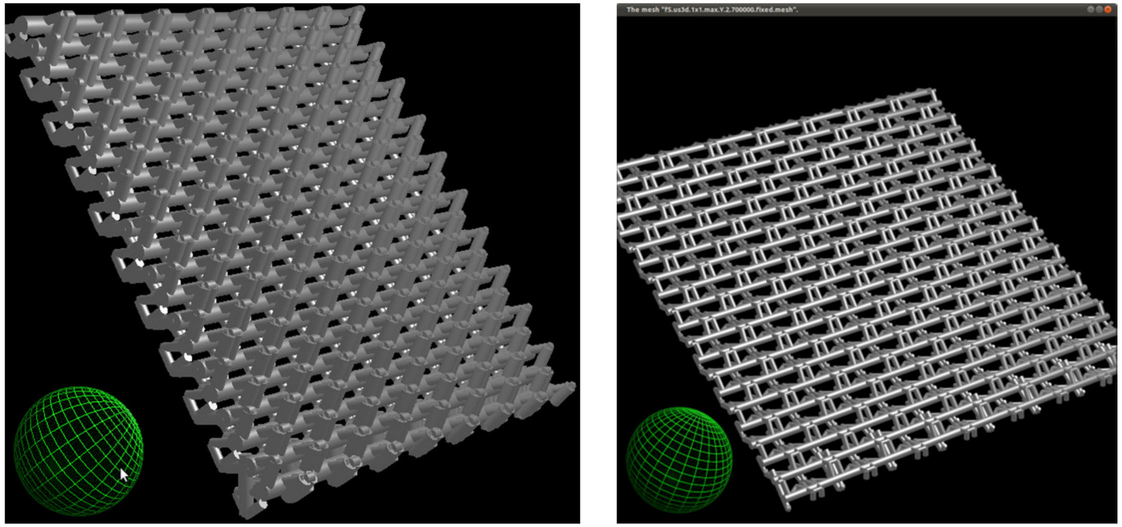
Assumptions:

- yarns are thin elastic strings and
- work only on tension (no bending and compression)

Parametrized yarn structures



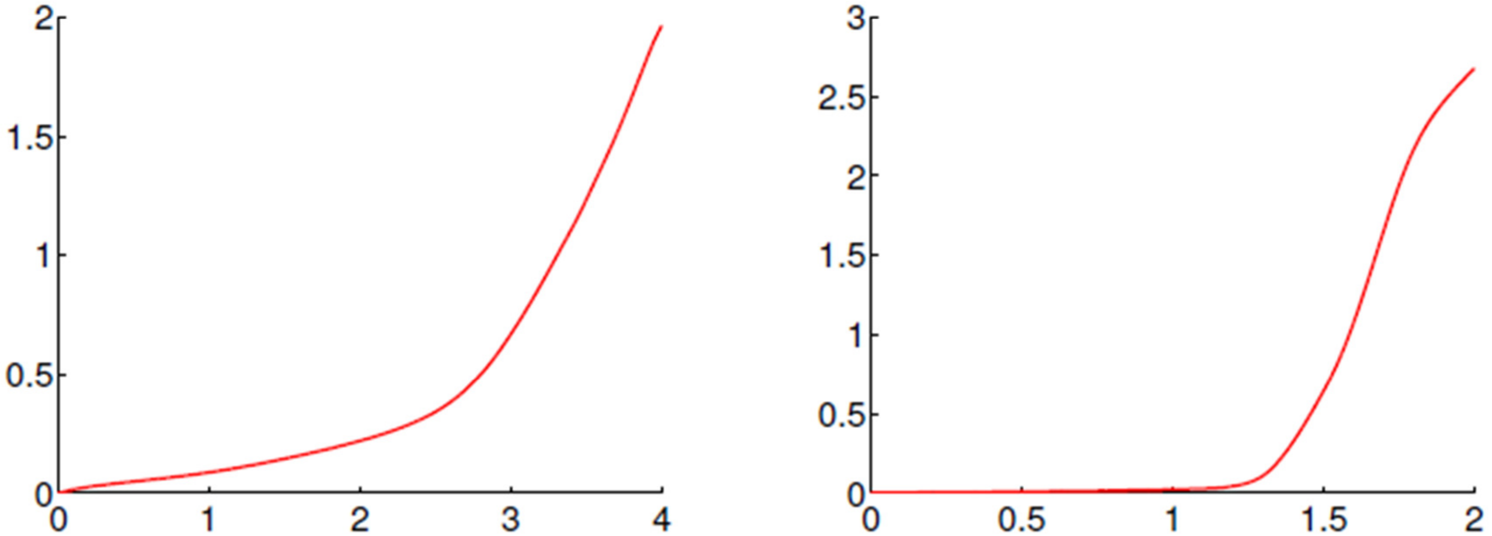
Micro-CT image



The model

Single yarn hyperelasticity model

From measured force-strain curves (force is given in Newtons, strain is dimensionless)

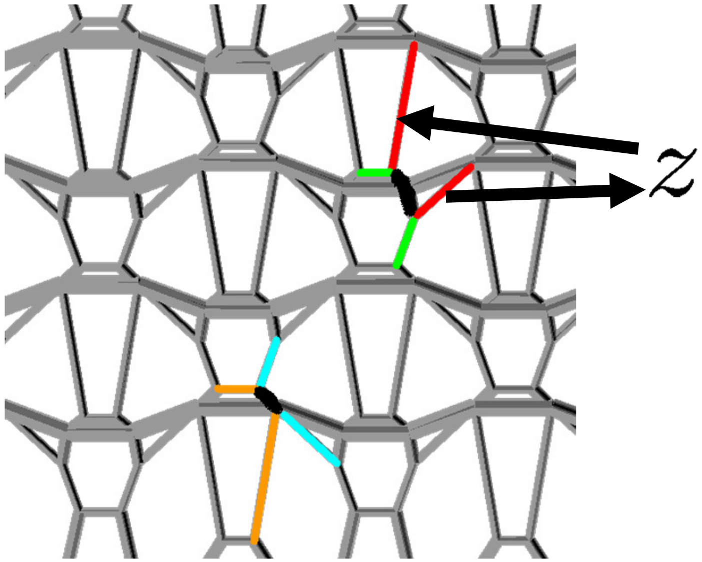


one can obtain hyperelastic energy

$$E(\varepsilon) = l_0 \int_0^{\varepsilon} f(\zeta)\, d\zeta,$$

where l_0 is the length of the undeformed yarn. If $\min f' > 0$, then $E(\varepsilon)$ is strictly convex w. r. t. ε.

The model of the contact sliding



Consider two elements e_1 and e_2 in frictional contact. Assume that length z is transferred from e_1 to e_2. Then

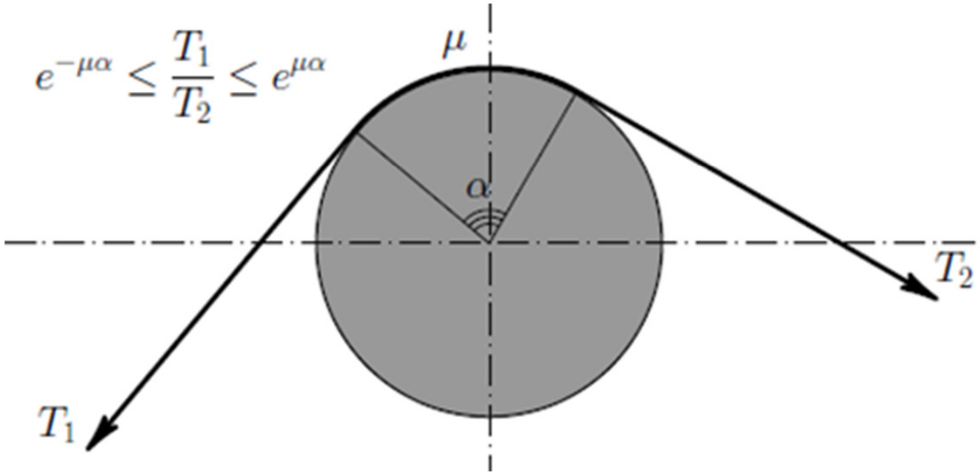
$$E_{e_i} = (l_{e_i} + (-1)^i z) \int_0^{\varepsilon(e_i, u_{e_i}, z)} f_{e_i}(\zeta)\, d\zeta, \quad \varepsilon(e_i, u_{e_i}, z) = \frac{\|e_i + u_{e_i}\|}{l_{e_i} + (-1)^i z}.$$

Euler-Eutelwein friction law

$$\Psi(u, z, \dot{z}) = \sum_{\{e_i, e_j\}} |\dot{z}(e_i, e_j)| \min(T_{e_i}(u), T_{e_j}(u)) \left(e^{\alpha(e_i, e_j)\mu} - 1\right)$$

$$T_{e_i}(u) = f_{e_i}\left(\frac{\|e_i + u\|}{l_{e_i} - Z_{e_i}} - 1\right),$$

$$e^{-\mu\alpha} \leq \frac{T_1}{T_2} \leq e^{\mu\alpha}$$



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Hyperelastic strings with Coulomb friction

$$\mathcal{E}(t,u,z) \leq \mathcal{E}(t,\hat{u},\hat{z}) + \Psi(u,z,\hat{u}-u,\hat{z}-z) \quad \forall \hat{u} \in U, \hat{z} \in Z,$$

The problem is solved

- by Newton method with continuation (or implicit time integration) [Deuflhard 2011],
- implemented on C++ with use of Intel MKL and
- PARDISO, [Schenk, Gärtner, 2006]) sparse solver for systems of linear equations.

- V. Shiryaev, J. Orlik, One-dimensional computational model for hyperelastic string structures with Coulomb Friction, submitted to M2AS

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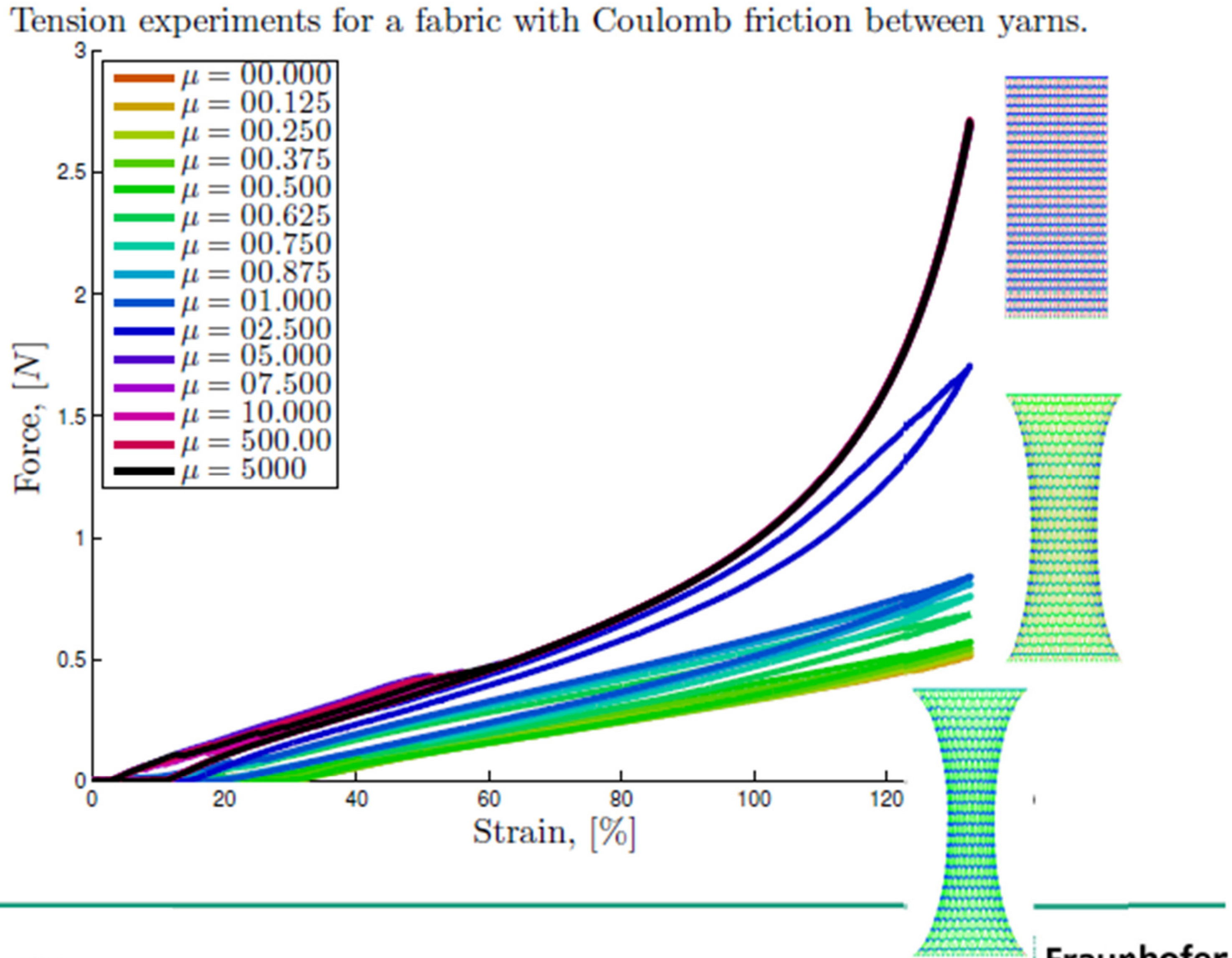
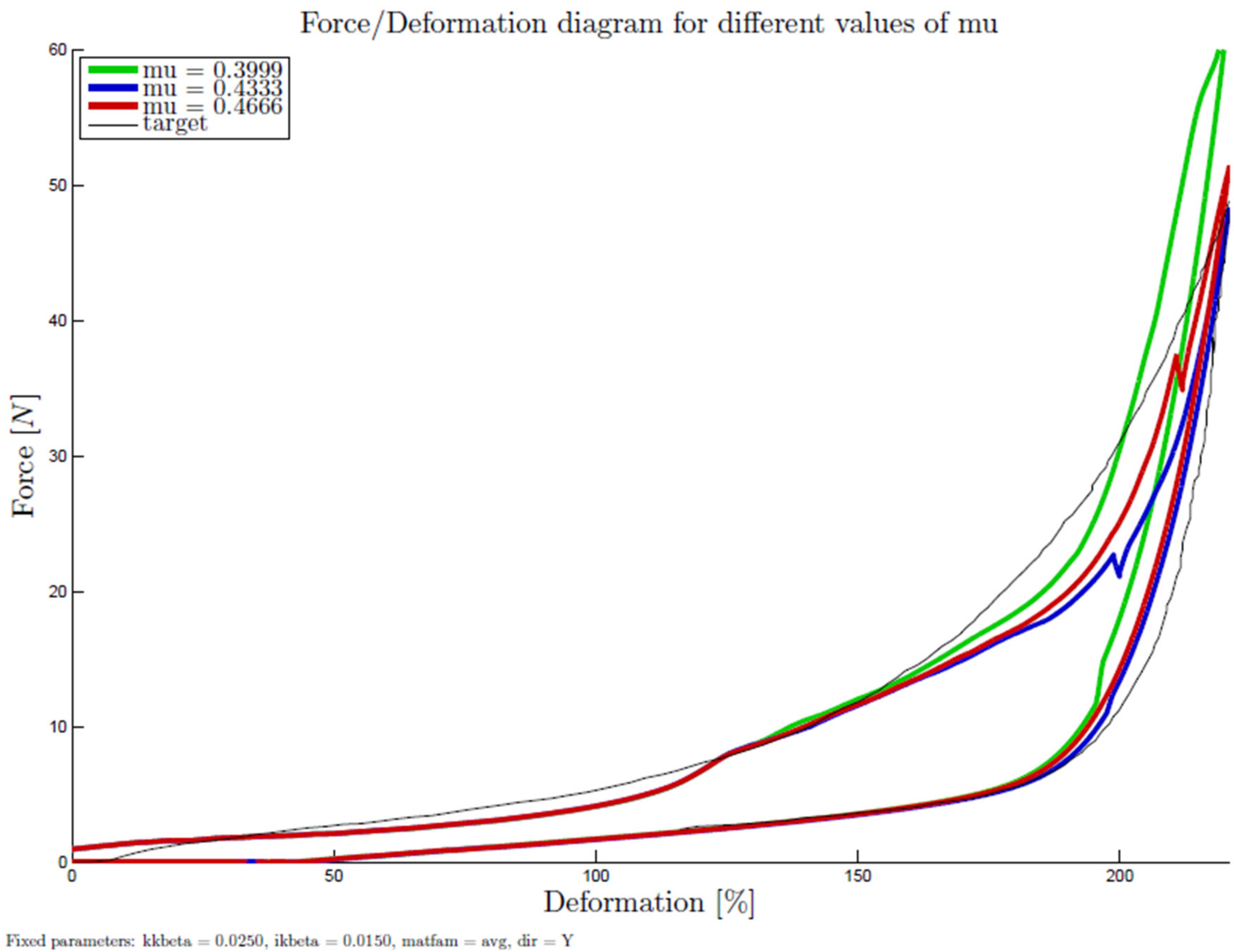
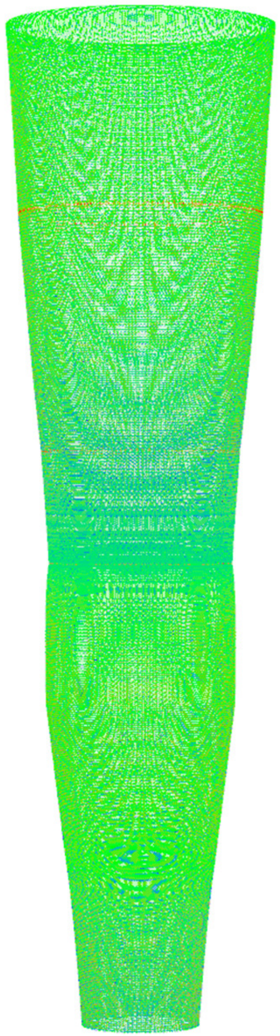
Influence of different friction coefficients. Hysteresis.

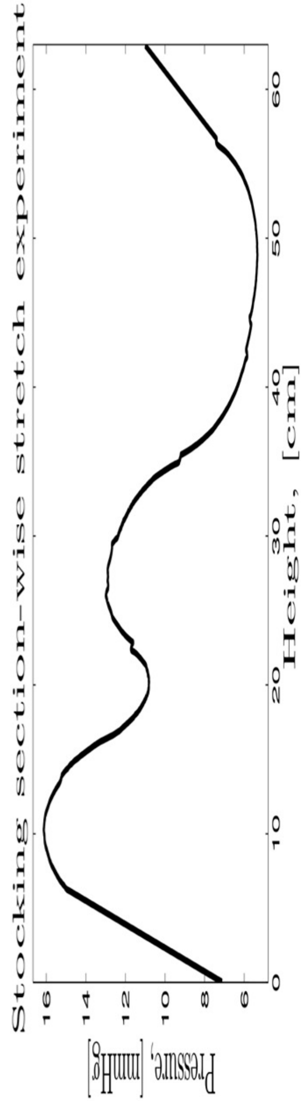


Experimental Validation



Pressure profile in the stocking.





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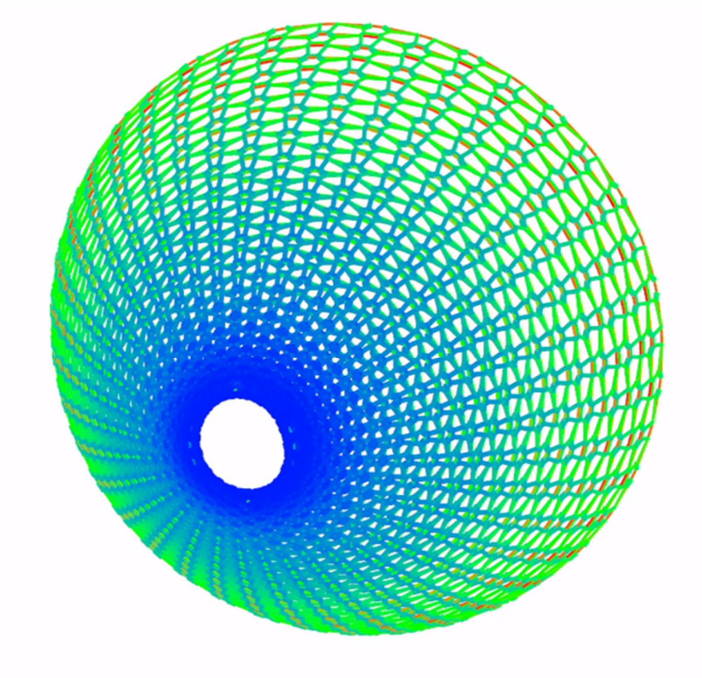
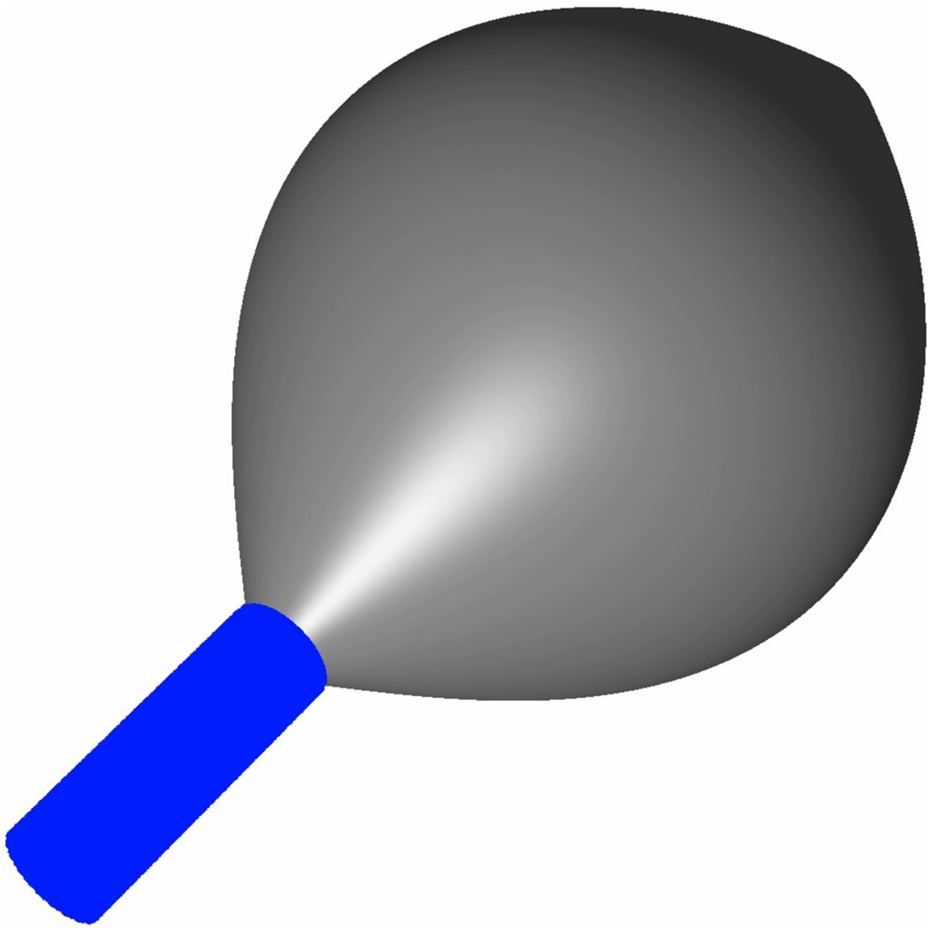
Simulation of putting on of knitted compressible wear





Fraunhofer
ITWM

Software KneeMech for preoperative planning

Aim:

to develop a fast FE based multi-scale contact analysis, which helps surgeons on basis of

- X-ray or CT images of the patient,
- effective time and space dependent load conditions

to make

- a correct virtual bone cut (w.r.t. its rotational axis) and positioning of the prosthesis in the bone
- a preoperative choice of appropriate shape, size, material, and coating for the knee prosthesis for each patient